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# *D*-Modules, Perverse Sheaves, and Representation Theory

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# Preface

*D-Modules, Perverse Sheaves, and Representation Theory* is a greatly expanded translation of the Japanese edition entitled *D kagun to daisugun* (*D-Modules and Algebraic Groups*) which was published by Springer-Verlag Tokyo, 1995. For the new English edition, the two authors of the original book, R. Hotta and T. Tanisaki, have added K. Takeuchi as a coauthor. Significant new material along with corrections and modifications have been made to this English edition.

In the summer of 1982, a symposium was held in Kinosaki in which the subject of $D$-modules and their applications to representation theory was introduced. At that time the theory of regular holonomic $D$-modules had just been completed and the Kazhdan–Lusztig conjecture had been settled by Brylinski–Kashiwara and Beilinson–Bernstein. The articles that appeared in the published proceedings of the symposium were not well presented and of course the subject was still in its infancy. Several monographs, however, did appear later on $D$-modules, for example, Björk [Bj2], Borel et al. [Bor3], Kashiwara–Schapira [KS2], Mebkhout [Me5] and others, all of which were taken into account and helped us make our Japanese book more comprehensive and readable. In particular, J. Bernstein's notes [Ber1] were extremely useful to understand the subject in the algebraic case; our treatment in many aspects follows the method used in the notes. Our plan was to present the combination of $D$-module theory and its typical applications to representation theory as we believe that this is a nice way to understand the whole subject.

Let us briefly explain the contents of this book. Part I is devoted to $D$-module theory, placing special emphasis on holonomic modules and constructible sheaves. The aim here is to present a proof of the Riemann–Hilbert correspondence. Part II is devoted to representation theory. In particular, we will explain how the Kazhdan–Lusztig conjecture was solved using the theory of $D$-modules. To a certain extent we assume the reader's familiarity with algebraic geometry, homological algebras, and sheaf theory. Although we include in the appendices brief introductions to algebraic varieties and derived categories, which are sufficient overall for dealing with the text, the reader should occasionally refer to appropriate references mentioned in the text.

The main difference from the original Japanese edition is that we made some new chapters and sections for analytic $D$-modules, meromorphic connections, perverse

sheaves, and so on. We thus emphasized the strong connections of $D$-modules with various other fields of mathematics.

We express our cordial thanks to A. D'Agnolo, C. Marastoni, Y. Matsui, P. Schapira, and J. Schürmann for reading very carefully the draft of the English version and giving us many valuable comments. Discussions with M. Kaneda, K. Kimura, S. Naito, J.-P. Schneiders, K. Vilonen, and others were also very helpful in completing the exposition. M. Nagura and Y. Sugiki greatly helped us in typing and correcting our manuscript. Thanks also go to many people for useful comments on our Japanese version, in particular to T. Ohsawa. Last but not least, we cannot exaggerate our gratitude to M. Kashiwara throughout the period since 1980 on various occasions.

2006 March

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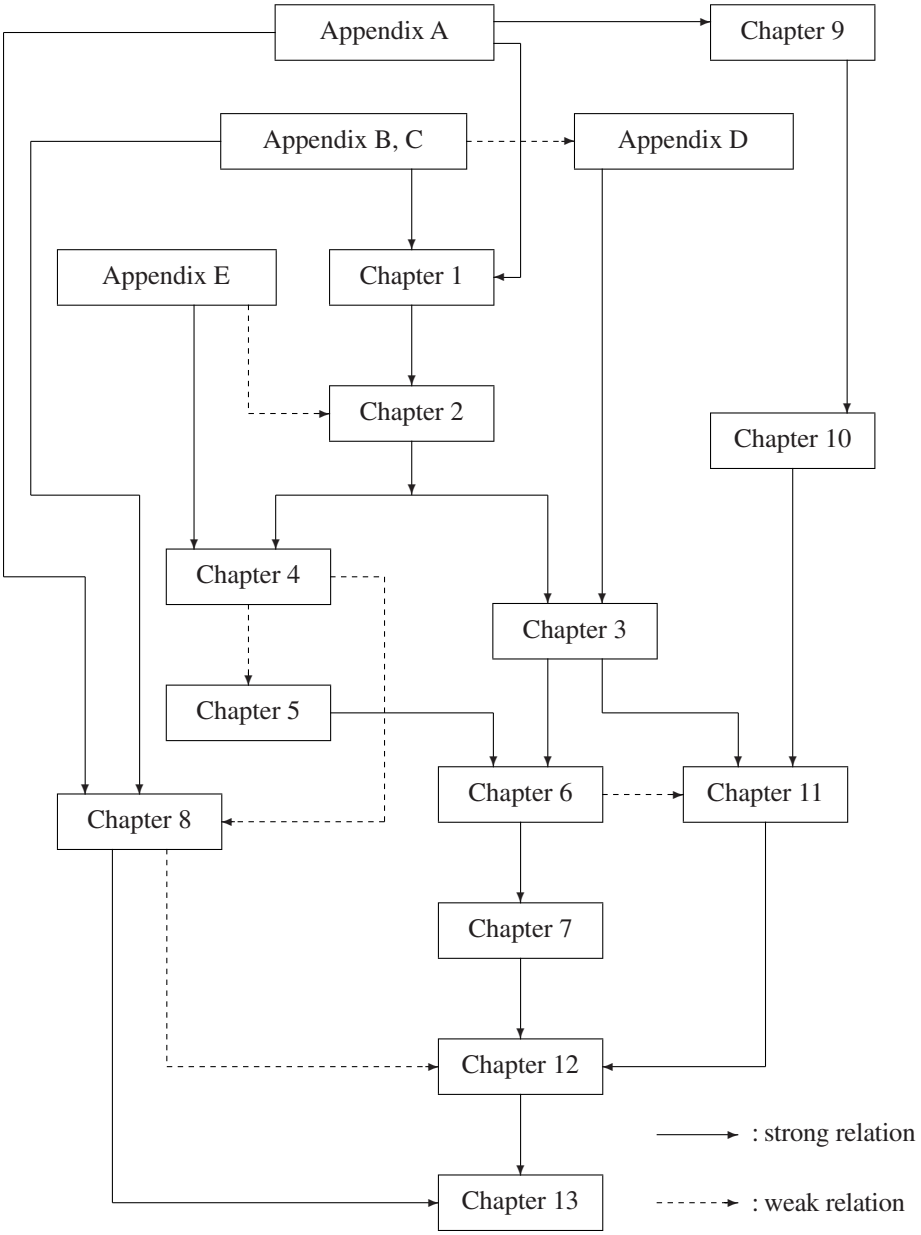
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# Relation Among Chapters



# Introduction

The theory of $D$-modules plays a key role in algebraic analysis. For the purposes of this text, by "algebraic analysis," we mean analysis using algebraic methods, such as ring theory and homological algebra. In addition to the contributions by French mathematicians, J. Bernstein, and others, this area of research has been extensively developed since the 1960s by Japanese mathematicians, notably in the important contributions of M. Sato, T. Kawai, and M. Kashiwara of the Kyoto school.

To this day, there continue to be outstanding results and significant theories coming from the Kyoto school, including Sato's hyperfunctions, microlocal analysis, $D$-modules and their applications to representation theory and mathematical physics. In particular, the theory of regular holonomic $D$-modules and their solution complexes (e.g., the theory of the Riemann–Hilbert correspondence which gave a sophisticated answer to Hilbert's 21st problem) was a most important and influential result. Indeed, it provided the germ for the theory of perverse sheaves, which was a natural development from intersection cohomologies. Moreover, M. Saito used this result effectively to construct his theory of Hodge modules, which largely extended the scope of Hodge theory. In representation theory, this result opened totally new perspectives, such as the resolution of the Kazhdan–Lusztig conjecture.

As stated above, in addition to the strong impact on analysis which was the initial main motivation, the theory of algebraic analysis, especially that of $D$-modules, continues to play a central role in various fields of contemporary mathematics. In fact, $D$-module theory is a source for creating new research areas from which new theories emerge. This striking feature of $D$-module theory has stimulated mathematicians in various other fields to become interested in the subject.

Our aim is to give a comprehensive introduction to $D$-modules. Until recently, in order to really learn it, we had to read and become familiar with many articles, which took long time and considerable effort. However, as we mentioned in the preface, thanks to some textbooks and monographs, the theory has become much more accessible nowadays, especially for those who have some basic knowledge of complex analysis or algebraic geometry. Still, to understand and appreciate the real significance of the subject on a deep level, it would be better to learn both the theory and its typical applications.

In Part I of this book we introduce $D$-modules principally in the context of presenting the theory of the Riemann–Hilbert correspondence. Part II is devoted to explaining applications to representation theory, especially to the solution to the Kazhdan–Lusztig conjecture. Since we mainly treat the theory of algebraic $D$-modules on smooth algebraic varieties rather than the (original) analytic theory on complex manifolds, we shall follow the unpublished notes [Ber3] of Bernstein (the book [Bor3] is also written along this line). The topics treated in Part II reveal how useful $D$-module theory is in other branches of mathematics. Among other things, the essential usefulness of this theory contributed heavily to resolving the Kazhdan–Lusztig conjecture, which was of course a great breakthrough in representation theory.

As we started Part II by giving a brief introduction to some basic notions of Lie algebras and algebraic groups using concrete examples, we expect that researchers in other fields can also read Part II without much difficulty.

Let us give a brief overview of the topics developed in this text. First, we explain how $D$-modules are related to systems of linear partial differential equations. Let $X$ be an open subset of $\mathbb{C}^n$ and denote by $\mathcal{O}$ the commutative ring of complex analytic functions globally defined on $X$. We denote by $D$ the set of linear partial differential operators with coefficients in $\mathcal{O}$. Namely, the set $D$ consists of the operators of the form

$$\sum_{i_1,i_2,\dots,i_n}^{\infty} f_{i_1,i_2,\dots,i_n} \left(\frac{\partial}{\partial x_1}\right)^{i_1} \left(\frac{\partial}{\partial x_2}\right)^{i_2} \cdots \left(\frac{\partial}{\partial x_n}\right)^{i_n} \quad (f_{i_1,i_2,\dots,i_n} \in \mathcal{O})$$

(each sum is a finite sum), where $(x_1, x_2, \dots, x_n)$ is a coordinate system of $\mathbb{C}^n$. Note that $D$ is a non-commutative ring by the composition of differential operators. Since the ring $D$ acts on $\mathcal{O}$ by differentiation, $\mathcal{O}$ is a left $D$-module. Now, for $P \in D$, let us consider the differential equation

$$Pu = 0 \tag{0.0.1}$$

for an unknown function $u$. According to Sato, we associate to this equation the left $D$-module $M = D/DP$. In this setting, if we consider the set $\mathrm{Hom}_D(M, \mathcal{O})$ of $D$-linear homomorphisms from $M$ to $\mathcal{O}$, we get the isomorphism

$$\begin{aligned}\mathrm{Hom}_D(M, \mathcal{O}) &= \mathrm{Hom}_D(D/DP, \mathcal{O}) \\ &\simeq \{\varphi \in \mathrm{Hom}_D(D, \mathcal{O}) \mid \varphi(P) = 0\}.\end{aligned}$$

Hence we see by $\mathrm{Hom}_D(D, \mathcal{O}) \simeq \mathcal{O}$ $(\varphi \mapsto \varphi(1))$ that

$$\mathrm{Hom}_D(M, \mathcal{O}) \simeq \{f \in \mathcal{O} \mid Pf = 0\}$$

$(Pf = P\varphi(1) = \varphi(P1) = \varphi(P) = 0)$. In other words, the (additive) group of the holomorphic solutions to the equation (0.0.1) is naturally isomorphic to $\mathrm{Hom}_D(M, \mathcal{O})$. If we replace $\mathcal{O}$ with another function space $\mathcal{F}$ admitting a natural action of $D$ (for example, the space of $C^\infty$-functions, Schwartz distributions,

Sato's hyperfunctions, etc.), then $\mathrm{Hom}_D(M, \mathcal{F})$ is the set of solutions to (0.0.1) in that function space.

More generally, a system of linear partial differential equations of $l$-unknown functions $u_1, u_2, \ldots, u_l$ can be written in the form

$$\sum_{j=1}^{l} P_{ij} u_j = 0 \quad (i = 1, 2, \ldots, k) \tag{0.0.2}$$

by using some $P_{ij} \in D$ $(1 \leq i \leq k,\ 1 \leq j \leq l)$. In this situation we have also a similar description of the space of solutions. Indeed if we define a left $D$-module $M$ by the exact sequence

$$D^k \xrightarrow{\varphi} D^l \longrightarrow M \longrightarrow 0 \tag{0.0.3}$$

$$\varphi(Q_1, Q_2, \ldots, Q_k) = \left( \sum_{i=1}^{k} Q_i P_{i1},\ \sum_{i=1}^{k} Q_i P_{i2},\ \ldots,\ \sum_{i=1}^{k} Q_i P_{il} \right),$$

then the space of the holomorphic solutions to (0.0.2) is isomorphic to $\mathrm{Hom}_D(M, \mathcal{O})$. Therefore, systems of linear partial differential equations can be identified with the $D$-modules having some finite presentations like (0.0.3), and the purpose of the theory of linear PDEs is to study the solution space $\mathrm{Hom}_D(M, \mathcal{O})$. Since the space $\mathrm{Hom}_D(M, \mathcal{O})$ does not depend on the concrete descriptions (0.0.2) and (0.0.3) of $M$ (it depends only on the $D$-linear isomorphism class of $M$), we can study these analytical problems through left $D$-modules admitting finite presentations. In the language of categories, the theory of linear PDEs is nothing but the investigation of the contravariant functor $\mathrm{Hom}_D(\bullet, \mathcal{O})$ from the category $M(D)$ of $D$-modules admitting finite presentations to the category $M(\mathbb{C})$ of $\mathbb{C}$-modules.

In order to develop this basic idea, we need to introduce sheaf theory and homological algebra. First, let us explain why sheaf theory is indispensable. It is sometimes important to consider solutions locally, rather than globally on $X$. For example, in the case of ordinary differential equations (or more generally, the case of integrable systems), the space of local solutions is always finite dimensional; however, it may happen that the analytic continuations (after turning around a closed path) of a solution are different from the original one. This phenomenon is called monodromy. Hence we also have to take into account how local solutions are connected to each other globally.

Sheaf theory is the most appropriate language for treating such problems. Therefore, sheafifying $\mathcal{O}$, $D$, let us now consider the sheaf $\mathcal{O}_X$ of holomorphic functions and the sheaf $D_X$ (of rings) of differential operators with holomorphic coefficients. We also consider sheaves of $D_X$-modules (in what follows, we simply call them $D_X$-modules) instead of $D$-modules. In this setting, the main objects to be studied are left $D_X$-modules admitting locally finite presentations (i.e., coherent $D_X$-modules). Sheafifying also the solution space, we get the sheaf $\mathcal{H}om_{D_X}(M, \mathcal{O}_X)$ of the holomorphic solutions to a $D_X$-module $M$. It follows that what we should investigate is the contravariant functor $\mathcal{H}om_{D_X}(\bullet, \mathcal{O}_X)$ from the category $\mathrm{Mod}_c(D_X)$ of coherent $D_X$-modules to the category $\mathrm{Mod}(\mathbb{C}_X)$ of (sheaves of) $\mathbb{C}_X$-modules.

Let us next explain the need for homological algebra. Although both $\mathrm{Mod}_c(D_X)$ and $\mathrm{Mod}(\mathbb{C}_X)$ are abelian categories, $\mathcal{H}om_{D_X}(\bullet, \mathcal{O}_X)$ is not an exact functor. Indeed, for a short exact sequence

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0 \tag{0.0.4}$$

in the category $\mathrm{Mod}_c(D_X)$ the sequence

$$0 \to \mathcal{H}om_{D_X}(M_3, \mathcal{O}_X) \to \mathcal{H}om_{D_X}(M_2, \mathcal{O}_X) \to \mathcal{H}om_{D_X}(M_1, \mathcal{O}_X) \tag{0.0.5}$$

associated to it is also exact; however, the final arrow $\mathcal{H}om_{D_X}(M_2, \mathcal{O}_X) \to \mathcal{H}om_{D_X}(M_1, \mathcal{O}_X)$ is not necessarily surjective. Hence we cannot recover information about the solutions of $M_2$ from those of $M_1, M_3$. A remedy for this is to consider also the "higher solutions" $\mathcal{E}xt^i_{D_X}(M, \mathcal{O}_X)$ $(i = 0, 1, 2, \dots)$ by introducing techniques in homological algebra. We have $\mathcal{E}xt^0_{D_X}(M, \mathcal{O}_X) = \mathcal{H}om_{D_X}(M, \mathcal{O}_X)$ and the exact sequence (0.0.5) is naturally extended to the long exact sequence

$$\begin{aligned}\cdots \to \mathcal{E}xt^i_{D_X}(M_3, \mathcal{O}_X) \to \mathcal{E}xt^i_{D_X}(M_2, \mathcal{O}_X) \to \mathcal{E}xt^i_{D_X}(M_1, \mathcal{O}_X)\\ \to \mathcal{E}xt^{i+1}_{D_X}(M_3, \mathcal{O}_X) \to \mathcal{E}xt^{i+1}_{D_X}(M_2, \mathcal{O}_X) \to \mathcal{E}xt^{i+1}_{D_X}(M_1, \mathcal{O}_X) \to \cdots .\end{aligned}$$

Hence the theory will be developed more smoothly by considering all higher solutions together.

Furthermore, in order to apply the methods of homological algebra in full generality, it is even more effective to consider the object $R\mathcal{H}om_{D_X}(M, \mathcal{O}_X)$ in the derived category (it is a certain complex of sheaves of $\mathbb{C}_X$-modules whose $i$-th cohomology sheaf is $\mathcal{E}xt^i_{D_X}(M, \mathcal{O}_X)$) instead of treating the sheaves $\mathcal{E}xt^i_{D_X}(M, \mathcal{O}_X)$ separately for various $i$'s. Among the many other advantages for introducing the methods of homological algebra, we point out here the fact that the sheaf of a hyperfunction solution can be obtained by taking the local cohomology of the complex $R\mathcal{H}om_{D_X}(M, \mathcal{O}_X)$ of holomorphic solutions. This is quite natural since hyperfunctions are determined by the boundary values (local cohomologies) of holomorphic functions.

Although we have assumed so far that $X$ is an open subset of $\mathbb{C}^n$, we may replace it with an arbitrary complex manifold. Moreover, also in the framework of smooth algebraic varieties over algebraically closed fields $k$ of characteristic zero, almost all arguments remain valid except when considering the solution complex $R\mathcal{H}om_{D_X}(\bullet, \mathcal{O}_X)$, in which case we need to assume again that $k = \mathbb{C}$ and return to the classical topology (not the Zariski topology) as a complex manifold. In this book we shall mainly treat $D$-modules on smooth algebraic varieties over $\mathbb{C}$; however, in this introduction, we will continue to explain everything on complex manifolds. Hence $X$ denotes a complex manifold in what follows.

There were some tentative approaches to $D$-modules by D. Quillen, Malgrange, and others in the 1960s; however, the real intensive investigation leading to later development was started by Kashiwara in his master thesis [Kas1] (we also note that this important contribution to $D$-module theory was also made independently by Bernstein [Ber1],[Ber2] around the same period). After this groundbreaking work, in collaboration with Kawai, Kashiwara developed the theory of (regular) holonomic

$D$-modules [KK3], which is a main theme in Part I of this book. Let us discuss this subject.

It is well known that the space of the holomorphic solutions to every ordinary differential equation is finite dimensional. However, when $X$ is higher dimensional, the dimensions of the spaces of holomorphic solutions can be infinite. This is because, in such cases, the solution contains parameters given by arbitrary functions unless the number of given equations is sufficiently large. Hence our task is to look for a suitable class of $D_X$-modules whose solution spaces are finite dimensional. That is, we want to find a generalization of the notion of ordinary differential equations in higher-dimensional cases.

For this purpose we consider the characteristic variety $\mathrm{Ch}(M)$ for a coherent $D_X$-module $M$, which is a closed analytic subset of the cotangent bundle $T^*X$ of $X$ (we sometimes call this the singular support of $M$ and denote it by $\mathrm{SS}(M)$). We know by a fundamental theorem of algebraic analysis due to Sato–Kawai–Kashiwara [SKK] that $\mathrm{Ch}(M)$ is an involutive subvariety in $T^*X$ with respect to the canonical symplectic structure of $T^*X$. In particular, we have $\dim \mathrm{Ch}(M) \geq \dim X$ for any coherent $D_X$-module $M \neq 0$.

Now we say that a coherent $D_X$-module $M$ is holonomic (a maximally overdetermined system) if it satisfies the equality $\dim \mathrm{Ch}(M) = \dim X$. Let us give the definition of characteristic varieties only in the simple case of $D_X$-modules

$$M = D_X/I, \quad I = D_X P_1 + D_X P_2 + \cdots + D_X P_k$$

associated to the systems

$$P_1 u = P_2 u = \cdots = P_k u = 0 \quad (P_i \in D_X) \tag{0.0.6}$$

for a single unknown function $u$. In this case, the characteristic variety $\mathrm{Ch}(M)$ of $M$ is the common zero set of the principal symbols $\sigma(Q)$ $(Q \in I)$ (recall that for $Q \in D_X$ its principal symbol $\sigma(Q)$ is a holomorphic function on $T^*X$). In many cases $\mathrm{Ch}(M)$ coincides with the common zero set of $\sigma(P_1), \sigma(P_2), \ldots, \sigma(P_k)$, but it sometimes happens to be smaller (we also see from this observation that the abstract $D_X$-module $M$ itself is more essential than its concrete expression (0.0.6)).

To make the solution space as small (finite dimensional) as possible we should consider as many equations as possible. That is, we should take the ideal $I \subset D_X$ as large as possible. This corresponds to making the ideal generated by the principal symbols $\sigma(P)$ $(P \in I)$ (in the ring of functions on $T^*X$) as large as possible, for which we have to take the characteristic variety $\mathrm{Ch}(M)$, i.e., the zero set of the $\sigma(P)$'s, as small as possible. On the other hand, a non-zero coherent $D_X$-module is holonomic if the dimension of its characteristic variety takes the smallest possible value $\dim X$. This philosophical observation suggests a possible connection between the holonomicity and the finite dimensionality of the solution spaces. Indeed such connections were established by Kashiwara as we explain below.

Let us point out here that the introduction of the notion of characteristic varieties is motivated by the ideas of microlocal analysis. In microlocal analysis, the sheaf $\mathcal{E}_X$ of microdifferential operators is employed instead of the sheaf $D_X$ of differential

operators. This is a sheaf of rings on the cotangent bundle $T^*X$ containing $\pi^{-1}D_X$ ($\pi : T^*X \to X$) as a subring. Originally, the characteristic variety $\mathrm{Ch}(M)$ of a coherent $D_X$-module $M$ was defined to be the support $\mathrm{supp}(\mathcal{E}_X \otimes_{\pi^{-1}D_X} \pi^{-1}M)$ of the corresponding coherent $\mathcal{E}_X$-module $\mathcal{E}_X \otimes_{\pi^{-1}D_X} \pi^{-1}M$. A guiding principle of Sato–Kawai–Kashiwara [SKK] was to develop the theory in the category of $\mathcal{E}_X$-modules even if one wants results for $D_X$-modules. In this process, they almost completely classified coherent $\mathcal{E}_X$-modules and proved the involutivity of $\mathrm{Ch}(M)$.

Let us return to holonomic $D$-modules. In his Ph.D. thesis [Kas3], Kashiwara proved for any holonomic $D_X$-module $M$ that all of its higher solution sheaves $\mathcal{E}xt^i_{D_X}(M, \mathcal{O}_X)$ are constructible sheaves (i.e., all its stalks are finite-dimensional vector spaces and for a stratification $X = \bigsqcup X_i$ of $X$ its restriction to each $X_i$ is a locally constant sheaf on $X_i$). From this result we can conclude that the notion of holonomic $D_X$-module is a natural generalization of that of linear ordinary differential equations to the case of higher-dimensional complex manifolds. We note that it is also proved in [Kas3] that the solution complex $R\mathcal{H}om_{D_X}(M, \mathcal{O}_X)$ satisfies the conditions of perversity (in language introduced later). The theory of perverse sheaves [BBD] must have been motivated (at least partially) by this result.

In the theory of linear ordinary differential equations, we have a good class of equations called equations with regular singularities, that is, equations admitting only mild singularities. We also have a successful generalization of this class to higher dimensions, that is, to regular holonomic $D_X$-modules. There are roughly two methods to define this class; the first (traditional) one will be to use higher-dimensional analogues of the properties characterizing ordinary differential equations with regular singularities, and the second (rather tactical) will be to define a holonomic $D_X$-module to be regular if its restriction to any algebraic curve is an ordinary differential equation with regular singularities. The two methods are known to be equivalent. We adopt here the latter as the definition. Moreover, we note that there is a conceptual difference between the complex analytic case and the algebraic case for the global meaning of regularity.

Next, let us explain the Riemann–Hilbert correspondence. By the monodromy of a linear differential equation we get a representation of the fundamental group of the base space. The original 21st problem of Hilbert asks for its converse: that is, for any representation of the fundamental group, is there an ordinary differential equation (with regular singularities) whose monodromy representation coincides with the given one? (there exist several points of view in formulating this problem more precisely, but we do not discuss them here. For example, see [AB], and others).

Let us consider the generalization in higher dimensions of this problem. A satisfactory answer in the case of integrable connections with regular singularities was given by P. Deligne [De1]. In this book, we deal with the problem for regular holonomic $D_X$-modules. As we have already seen, for any holonomic $D_X$-module $M$, its solutions $\mathcal{E}xt^i_{D_X}(M, \mathcal{O}_X)$ are constructible sheaves. Hence, if we denote by $D^b_c(\mathbb{C}_X)$ the derived category consisting of bounded complexes of $\mathbb{C}_X$-modules whose cohomology sheaves are constructible, the holomorphic solution complex $R\mathcal{H}om_{D_X}(M, \mathcal{O}_X)$ is an object of $D^b_c(\mathbb{C}_X)$. Therefore, denoting by $D^b_{rh}(D_X)$ the

derived category consisting of bounded complexes of $D_X$-modules whose cohomology sheaves are regular holonomic $D_X$-modules, we can define the contravariant functor

$$R\mathcal{H}om_{D_X}(\bullet, \mathcal{O}_X) : D^b_{rh}(D_X) \longrightarrow D^b_c(\mathbb{C}_X). \tag{0.0.7}$$

One of the most important results in the theory of $D$-modules is the (contravariant) equivalence of categories $D^b_{rh}(D_X) \simeq D^b_c(\mathbb{C}_X)$ via this functor. The crucial point of this equivalence (the Riemann–Hilbert correspondence, which we noted is the most sophisticated solution to Hilbert's 21st problem) lies in the concept of regularity and this problem was properly settled by Kashiwara–Kawai [KK3]. The correct formulation of the above equivalence of categories was already conjectured by Kashiwara in the middle 1970s and the proof was completed around 1980 (see [Kas6]). The full proof was published in [Kas10]. For this purpose, Kashiwara constructed the inverse functor of the correspondence (0.0.7). We should note that another proof of this correspondence was also obtained by Mebkhout [Me4]. For the more detailed historical comments, compare the foreword by Schapira in the English translation [Kas16] of Kashiwara's master thesis [Kas1]. As mentioned earlier we will mainly deal with algebraic $D$-modules in this book, and hence what we really consider is a version of the Riemann–Hilbert correspondence for algebraic $D$-modules. After the appearance of the theory of regular holonomic $D$-modules and the Riemann–Hilbert correspondence for analytic $D$-modules, A. Beilinson and J. Bernstein developed the corresponding theory for algebraic $D$-modules based on much simpler arguments. Some part of this book relies on their results.

The content of Part I is as follows. In Chapters 1–3 we develop the basic theory of algebraic $D$-modules. In Chapter 4 we give a survey of the theory of analytic $D$-modules and present some properties of the solution and the de Rham functors. Chapter 5 is concerned with results on regular meromorphic connections due to Deligne [De1]. As for the content of Chapter 5, we follow the notes of Malgrange in [Bor3], which will be a basis of the general theory of regular holonomic $D$-modules described in Chapters 6 and 7. In Chapter 6 we define the notion of regular holonomic algebraic $D$-modules and show its stability under various functors. In Chapter 7 we present a proof of an algebraic version the Riemann–Hilbert correspondence. The results in Chapters 6 and 7 are totally due to the unpublished notes of Bernstein [Ber3] explaining his work with Beilinson. In Chapter 8 we give a relatively self-contained account of the theory of intersection cohomology groups and perverse sheaves (M. Goresky–R. MacPherson [GM1], Beilinson–Bernstein–Deligne [BBD]) assuming basic facts about constructible sheaves. This part is independent of other parts of the book. We also include a brief survey of the theory of Hodge modules due to Morihiko Saito [Sa1], [Sa2] without proofs.

We finally note that the readers of this book who are only interested in algebraic $D$-module theory (and not in the analytic one) can skip Sections 4.4 and 4.6, and need not become involved with symplectic geometry.

In the rest of the introduction we shall give a brief account of the content of Part II which deals with applications of $D$-module theory to representation theory.

The history of Lie groups and Lie algebras dates back to the 19th century, the period of S. Lie and F. Klein. Fundamental results about semisimple Lie groups such as those concerning structure theorems, classification, and finite-dimensional representation theory were obtained by W. Killing, E. Cartan, H. Weyl, and others until the 1930s. Afterwards, the theory of infinite-dimensional (unitary) representations was initiated during the period of World War II by E. P. Wigner, V. Bargmann, I. M. Gelfand, M. A. Naimark, and others, and partly motivated by problems in physics. Since then and until today the subject has been intensively investigated from various points of view. Besides functional analysis, which was the main method at the first stage, various theories from differential equations, differential geometry, algebraic geometry, algebraic analysis, etc. were applied to the theory of infinite-dimensional representations. The theory of automorphic forms also exerted a significant influence. Nowadays infinite-dimensional representation theory is a place where many branches of mathematics come together. As contributors representing the development until the 1970s, we mention the names of Harish-Chandra, B. Kostant, R. P. Langlands.

On the other hand, the theory of algebraic groups was started by the fundamental works of C. Chevalley, A. Borel, and others [Ch] and became recognized widely by the textbook of Borel [Bor1]. Algebraic groups are obtained by replacing the underlying complex or real manifolds of Lie groups with algebraic varieties. Over the fields of complex or real numbers algebraic groups form only a special class of Lie groups; however, various new classes of groups are produced by taking other fields as the base field. In this book we will only be concerned with semisimple groups over the field of complex numbers, for which Lie groups and algebraic groups provide the same class of groups. We regard them as algebraic groups since we basically employ the language of algebraic geometry.

The application of algebraic analysis to representation theory was started by the resolution of the Helgason conjecture [six] due to Kashiwara, A. Kowata, K. Minemura, K. Okamoto, T. Oshima, and M. Tanaka. In this book, we focus however on the resolution of the Kazhdan–Lusztig conjecture which was the first achievement in representation theory obtained by applying $D$-module theory.

Let us explain the problem. It is well known that all finite-dimensional irreducible representations of complex semisimple Lie algebras are highest weight modules with dominant integral highest weights. For such representations the characters are described by Weyl's character formula. Inspired by the works of Harish-Chandra on infinite-dimensional representations of semisimple Lie groups, D. N. Verma proposed in the late 1960s the problem of determining the characters of (infinite-dimensional) irreducible highest weight modules with not necessarily dominant integral highest weights. Important contributions to this problem by a purely algebraic approach were made in the 1970s by Bernstein, I. M. Gelfand, S. I. Gelfand, and J. C. Jantzen, although the original problem was not solved.

A breakthrough using totally new methods was made around 1980. D. Kazhdan and G. Lusztig introduced a family of special polynomials (the Kazhdan–Lusztig polynomials) using Hecke algebras and proposed a conjecture giving the explicit form

of the characters of irreducible highest weight modules in terms of these polynomials [KL1]. They also gave a geometric meaning for Kazhdan–Lusztig polynomials using the intersection cohomology groups of Schubert varieties. Promptly responding to this, Beilinson–Bernstein [BB] and J.-L. Brylinski–Kashiwara independently solved the conjecture by establishing a correspondence between highest weight modules and the intersection cohomology complexes of Schubert varieties via $D$-modules on the flag manifold. This successful achievement, i.e., employing theories and methods, from other fields, was quite astonishing for the specialists who had been studying the problem using purely algebraic means. Since then $D$-module theory has brought numerous new developments in representation theory.

Let us explain more precisely the methods used to solve the Kazhdan–Lusztig conjecture. Let $G$ be an algebraic group (or a Lie group), $\mathfrak{g}$ its Lie algebra and $U(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$. If $X$ is a smooth $G$-variety and $\mathcal{V}$ is a $G$-equivariant vector bundle on $X$, the set $\Gamma(X,\ \mathcal{V})$ of global sections of $\mathcal{V}$ naturally has a $\mathfrak{g}$-module structure. The construction of the representation of $\mathfrak{g}$ (or of $G$) in this manner is a fundamental technique in representation theory.

Let us now try to generalize this construction. Denote by $D_X^{\mathcal{V}} \subset \mathcal{E}nd_{\mathbb{C}}(\mathcal{V})$ the sheaf of rings of differential operators acting on the sections of $\mathcal{V}$. Then $D_X^{\mathcal{V}}$ is isomorphic to $\mathcal{V} \otimes_{\mathcal{O}_X} D_X \otimes_{\mathcal{O}_X} \mathcal{V}^*$ which coincides with the usual $D_X$ when $\mathcal{V} = \mathcal{O}_X$. In terms of $D_X^{\mathcal{V}}$ the $\mathfrak{g}$-module structure on $\Gamma(X,\ \mathcal{V})$ can be described as follows. Note that we have a canonical ring homomorphism $U(\mathfrak{g}) \to \Gamma(X,\ D_X^{\mathcal{V}})$ induced by the $G$-action on $\mathcal{V}$. Since $\mathcal{V}$ is a $D_X^{\mathcal{V}}$-module, $\Gamma(X,\ \mathcal{V})$ is a $\Gamma(X,\ D_X^{\mathcal{V}})$-module, and hence a $\mathfrak{g}$-module through the ring homomorphism $U(\mathfrak{g}) \to \Gamma(X,\ D_X^{\mathcal{V}})$. From this observation, we see that we can replace $\mathcal{V}$ with other $D_X^{\mathcal{V}}$-modules. That is, for any $D_X^{\mathcal{V}}$-module $M$ the $\mathbb{C}$-vector space $\Gamma(X,\ M)$ is endowed with a $\mathfrak{g}$-module structure.

Let us give an example. Let $G = SL_2(\mathbb{C})$. Since $G$ acts on $X = \mathbb{P}^1 = \mathbb{C} \sqcup \{\infty\}$ by the linear fractional transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (x) = \left(\frac{ax+b}{cx+d}\right) \quad \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G,\ (x) \in X\right),$$

it follows from the above arguments that $\Gamma(X,\ M)$ is a $\mathfrak{g}$-module for any $D_X$-module $M$. Let us consider the $D_X$-module $M = D_X\delta$ given by Dirac's delta function $\delta$ at the point $x = \infty$. In the coordinate $z = \frac{1}{x}$ in a neighborhood of $x = \infty$, the equation satisfied by Dirac's delta function $\delta$ is

$$z\delta = 0,$$

so we get

$$M = D_X/D_Xz$$

in a neighborhood of $x = \infty$. Set $\delta_n = (\frac{d}{dz})^n\delta$. Then $\{\delta_n\}_{n=0}^{\infty}$ is the basis of $\Gamma(X,\ M)$ and we have $\frac{d}{dz}\delta_n = \delta_{n+1}$, $z\delta_n = -n\delta_{n-1}$.

Let us describe the action of $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ on $\Gamma(X,\ M)$. For this purpose consider the following elements in $\mathfrak{g}$:

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

(these elements $h$, $e$, $f$ form a basis of $\mathfrak{g}$). Then the ring homomorphism $U(\mathfrak{g}) \to \Gamma(X,\ D_X)$ is given by

$$h \longmapsto 2z\frac{d}{dz}, \quad e \longmapsto z^2\frac{d}{dz}, \quad f \longmapsto -\frac{d}{dz}.$$

For example, since

$$\exp(-te) \cdot \begin{pmatrix} 1 \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ z/(1-tz) \end{pmatrix},$$

for $\varphi(z) \in \mathcal{O}_X$ we get

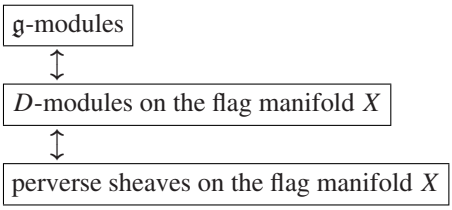
$$(e \cdot \varphi)(z) = \frac{d}{dt}\varphi\left(\frac{z}{1-tz}\right)\Big|_{t=0} = \left(z^2\frac{d}{dz}\varphi\right)(z)$$

and $e \mapsto z^2\frac{d}{dz}$. Therefore we obtain

$$h \cdot \delta_n = -2(n+1)\delta_n, \quad e \cdot \delta_n = n(n+1)\delta_{n-1}, \quad f \cdot \delta_n = -\delta_{n+1},$$

from which we see that $\Gamma(X,\ M)$ is the infinite-dimensional irreducible highest weight module with highest weight $-2$.

For the proof of the Kazhdan–Lusztig conjecture, we need to consider the case when $G$ is a semisimple algebraic group over the field of complex numbers and the $G$-variety $X$ is the flag variety of $G$. For each Schubert variety $Y$ in $X$ we consider a $D_X$-module $M$ satisfied by the delta function supported on $Y$. In our previous example, i.e., in the case of $G = SL_2(\mathbb{C})$, the flag variety is $X = \mathbb{P}^1$ and $Y = \{\infty\}$ is a Schubert variety. Since Schubert varieties $Y \subset X$ may have singularities for general algebraic groups $G$, we take the regular holonomic $D_X$-module $M$ characterized by the condition of having no subquotient whose support is contained in the boundary of $Y$. For this choice of $M$, $\Gamma(X,\ M)$ is an irreducible highest weight $\mathfrak{g}$-module and $R\mathcal{H}om_{D_X}(M,\ \mathcal{O}_X)$ is the intersection cohomology complex of $Y$. A link between highest weight $\mathfrak{g}$-modules and the intersection cohomology complexes of Schubert varieties $Y \subset X$ (perverse sheaves on the flag manifold $X$) is given in this manner. Diagrammatically the strategy of the proof of the Kazhdan–Lusztig conjecture can be explained as follows:



Here the first arrow is what we have briefly explained above, and the second one is the Riemann–Hilbert correspondence, a general theory of $D$-modules. The first arrow is called the Beilinson–Bernstein correspondence, which asserts that the category of $U(\mathfrak{g})$-modules with the trivial central character and that of $D_X$-modules are equivalent. By this correspondence, for a $D_X$-module $M$ on the flag manifold $X$, we associate to it the $U(\mathfrak{g})$-module $\Gamma(X, M)$. As a result, we can translate various problems for $\mathfrak{g}$-modules into those for regular holonomic $D$-modules (or through the Riemann–Hilbert correspondence, those for constructible sheaves).

The content of Part II is as follows. We review some preliminary results on algebraic groups in Chapters 9 and 10. In Chapters 11 and 12 we will explain how the Kazhdan–Lusztig conjecture was solved. Finally, in Chapter 13, a realization of Hecke algebras will be given by the theory of Hodge modules, and the relation between the intersection cohomology groups of Schubert varieties and Hecke algebras will be explained.

Let us briefly mention some developments of the theory, which could not be treated in this book. We can also formulate conjectures, similar to the Kazhdan–Lusztig conjecture, for Kac–Moody Lie algebras, i.e., natural generalizations of semisimple Lie algebras. In this case, we have to study two cases separately: (a) the case when the highest weight is conjugate to a dominant weight by the Weyl group, (b) the case when the highest weight is conjugate to an anti-dominant weight by the Weyl group. Moreover, Lusztig proposed certain Kazhdan–Lusztig type conjectures also for the following objects: (c) the representations of reductive algebraic groups in positive characteristics, (d) the representations of quantum groups in the case when the parameter $q$ is a root of unity. The conjecture of the case (a) was solved by Kashiwara (and Tanisaki) [Kas15], [KT2] and L. Casian [Ca1]. Following the so-called Lusztig program, the other conjectures were also solved:

(A) the equivalence of (c) and (d): H. H. Andersen, J. C. Jantzen, W. Soergel [AJS].
(B) the equivalence of (b) and (d) for affine Lie algebras: Kazhdan–Lusztig [KL3].
(C) the proof of (b) for affine Lie algebras: Kashiwara–Tanisaki [KT3] and Casian [Ca2].

# Part I

## *D*-Modules and Perverse Sheaves

# 1

# Preliminary Notions

In this chapter we introduce several standard operations for $D$-modules and present some fundamental results concerning them such as Kashiwara's equivalence theorem.

## 1.1 Differential operators

Let $X$ be a smooth (non-singular) algebraic variety over the complex number field $\mathbb{C}$ and $\mathcal{O}_X$ the sheaf of rings of regular functions (structure sheaf) on it. We denote by $\Theta_X$ the sheaf of vector fields (tangent sheaf, see Appendix A) on $X$:

$$\begin{aligned}\Theta_X &= \mathcal{D}er_{\mathbb{C}_X}(\mathcal{O}_X)\\ &= \{\theta \in \mathcal{E}nd_{\mathbb{C}_X}(\mathcal{O}_X) \mid \theta(fg) = \theta(f)g + f\theta(g)\ (f, g \in \mathcal{O}_X)\}.\end{aligned}$$

Hereafter, if there is no risk of confusion, we use the notation $f \in \mathcal{O}_X$ for a local section $f$ of $\mathcal{O}_X$. Since $X$ is smooth, the sheaf $\Theta_X$ is locally free of rank $n = \dim X$ over $\mathcal{O}_X$. We will identify $\mathcal{O}_X$ with a subsheaf of $\mathcal{E}nd_{\mathbb{C}_X}(\mathcal{O}_X)$ by identifying $f \in \mathcal{O}_X$ with $[\mathcal{O}_X \ni g \mapsto fg \in \mathcal{O}_X] \in \mathcal{E}nd_{\mathbb{C}_X}(\mathcal{O}_X)$. We define a sheaf $D_X$ as the $\mathbb{C}$-subalgebra of $\mathcal{E}nd_{\mathbb{C}_X}(\mathcal{O}_X)$ generated by $\mathcal{O}_X$ and $\Theta_X$. We call this sheaf $D_X$ the sheaf of *differential operators* on $X$. For any point of $X$ we can take its affine open neighborhood $U$ and a *local coordinate system* $\{x_i, \partial_i\}_{1 \le i \le n}$ on it satisfying

$$x_i \in \mathcal{O}_X(U), \quad \Theta_U = \bigoplus_{i=1}^{n} \mathcal{O}_U \partial_i, \quad [\partial_i, \partial_j] = 0, \quad [\partial_i, x_j] = \delta_{ij}$$

(see Appendix A). Hence we have

$$D_U = D_X|_U = \bigoplus_{\alpha \in \mathbb{N}^n} \mathcal{O}_U \partial_x^\alpha \qquad (\partial_x^\alpha := \partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_n^{\alpha_n}).$$

Here, $\mathbb{N}$ denotes the set of non-negative integers.

**Exercise 1.1.1.** Let $U$ be an affine open subset of $X$. Show that $D_X(U)$ is naturally isomorphic to the $\mathbb{C}$-algebra generated by elements $\{\tilde{f}, \tilde{\theta} \mid f \in \mathcal{O}_X(U), \theta \in \Theta_X(U)\}$ satisfying the following fundamental relations:

$$\begin{array}{lll}
(1) & \tilde{f}_1 + \tilde{f}_2 = \widetilde{f_1 + f_2}, \quad \tilde{f}_1\tilde{f}_2 = \widetilde{f_1 f_2} & (f_1, f_2 \in \mathcal{O}_X(U)), \\
(2) & \tilde{\theta}_1 + \tilde{\theta}_2 = \widetilde{\theta_1 + \theta_2}, \quad [\tilde{\theta}_1, \tilde{\theta}_2] = \widetilde{[\theta_1, \theta_2]} & (\theta_1, \theta_2 \in \Theta_X(U)), \\
(3) & \tilde{f}\tilde{\theta} = \widetilde{f\theta} & (f \in \mathcal{O}_X(U), \theta \in \Theta_X(U)), \\
(4) & [\tilde{\theta}, \tilde{f}] = \widetilde{\theta(f)} & (f \in \mathcal{O}_X(U), \theta \in \Theta_X(U)).
\end{array}$$

**Exercise 1.1.2.** Let $\{x_i, \partial_i\}_{1 \le i \le n}$ be a local coordinate system on an affine open subset $U$ of $X$. For $P = \sum_{\alpha \in \mathbb{N}^n} a_\alpha(x)\partial_x^\alpha \in D_X(U)$ we define its *total symbol* $\sigma(P)(x, \xi)$ by $\sigma(P)(x, \xi) := \sum_{\alpha \in \mathbb{N}^n} a_\alpha(x)\xi^\alpha$ ($\xi^\alpha := \xi_1^{\alpha_1}\xi_2^{\alpha_2}\cdots\xi_n^{\alpha_n}$). For $P, Q \in D_X(U)$ show that the total symbol $\sigma(R)(x, \xi)$ of the product $R = PQ \in D_X(U)$ is given by

$$\sigma(R)(x, \xi) = \sum_{\alpha \in \mathbb{N}^n} \frac{1}{\alpha!}\partial_\xi^\alpha \sigma(P)(x, \xi) \cdot \partial_x^\alpha \sigma(Q)(x, \xi),$$

where we set $\alpha! = \alpha_1!\alpha_2!\cdots\alpha_n!$ for each $\alpha \in \mathbb{N}^n$ (this is the "*Leibniz rule*").

Let $U$ be an affine open subset of $X$ with a local coordinate system $\{x_i, \partial_i\}$. We define the *order filtration* $F$ of $D_U$ by

$$F_l D_U = \sum_{|\alpha| \le l} \mathcal{O}_U \partial_x^\alpha \qquad \left(l \in \mathbb{N},\ |\alpha| = \sum_i \alpha_i\right).$$

More generally, for an arbitrary open subset $V$ of $X$ we can define the order filtration $F$ of $D_X$ over $V$ by

$$\begin{aligned}
&(F_l D_X)(V) \\
&\quad = \{P \in D_X(V) \mid \mathrm{res}_U^V P \in F_l D_X(U) \text{ for any affine open subset } U \text{ of } V\},
\end{aligned}$$

where $\mathrm{res}_U^V : D_X(V) \to D_X(U)$ is the restriction map (see also Exercise 1.1.4 below). For convenience we set $F_p D_X = 0$ for $p < 0$. The following result is obvious.

**Proposition 1.1.3.**

(i) *$\{F_l\}_{l \in \mathbb{N}}$ is an increasing filtration of $D_X$ such that $D_X = \bigcup_{l \in \mathbb{N}} F_l D_X$ and each $F_l D_X$ is a locally free module over $\mathcal{O}_X$.*

(ii) $F_0 D_X = \mathcal{O}_X$, $(F_l D_X)(F_m D_X) = F_{l+m} D_X$.

(iii) *If $P \in F_l D_X$ and $Q \in F_m D_X$, then $[P, Q] \in F_{l+m-1} D_X$.*

**Exercise 1.1.4.** Show that the formula

$$F_l D_X = \{P \in \mathcal{E}nd_{\mathbb{C}}(\mathcal{O}_X) \mid [P, f] \in F_{l-1} D_X \ (\forall f \in \mathcal{O}_X)\} \qquad (l \in \mathbb{N}).$$

(Note that this recursive expression of $F_l D_X$ together with $D_X = \bigcup_{l \in \mathbb{N}} F_l D_X$ gives an alternative intrinsic definition of $D_X$.)

**Principal symbols**

For the sheaf $(D_X, F)$ of *filtered rings* let us consider its *graded ring*

$$\operatorname{gr} D_X = \operatorname{gr}^F D_X = \bigoplus_{l=0}^{\infty} \operatorname{gr}_l D_X \quad (\operatorname{gr}_l D_X = F_l D_X / F_{l-1} D_X,\ F_{-1} D_X = 0).$$

Then by Proposition 1.1.3 $\operatorname{gr} D_X$ is a sheaf of *commutative* algebras finitely generated over $\mathcal{O}_X$. Take an affine chart $U$ with a coordinate system $\{x_i, \partial_i\}$ and set

$$\xi_i := \partial_i \mod F_0 D_U (= \mathcal{O}_U) \in \operatorname{gr}_1 D_U.$$

Then we have

$$\operatorname{gr}_l D_U = F_l D_U / F_{l-1} D_U = \bigoplus_{|\alpha|=l} \mathcal{O}_U \xi^\alpha,$$
$$\operatorname{gr} D_U = \mathcal{O}_U[\xi_1, \xi_2, \ldots, \xi_n].$$

For a differential operator $P \in F_l D_U \setminus F_{l-1} D_U$ the corresponding section $\sigma_l(P) \in \operatorname{gr}_l D_U \subset \mathcal{O}_U[\xi_1, \xi_2, \ldots, \xi_n]$ is called the *principal symbol* of $P$.

We can globalize this notion as follows. Let $T^*X$ be the *cotangent bundle* of $X$ and let $\pi : T^*X \to X$ be the projection. We may regard $\xi_1, \ldots, \xi_n$ as the coordinate system of the cotangent space $\bigoplus_{i=1}^n \mathbb{C} dx_i$, and hence $\mathcal{O}_U[\xi_1, \ldots, \xi_n]$ is canonically identified with the sheaf $\pi_* \mathcal{O}_{T^*X}|_U$ of algebras. Thus we obtain a canonical identification

$$\operatorname{gr} D_X \simeq \pi_* \mathcal{O}_{T^*X} (\simeq \operatorname{Symm} \Theta_X).$$

Therefore, for $P \in F_l D_X$ we can associate to it a regular function $\sigma_l(P)$ globally defined on the cotangent bundle $T^*X$.

## 1.2 $D$-modules—warming up

As we have already explained in the introduction, a system of differential equations can be regarded as a "coherent" left $D$-module.

Let $X$ be a smooth algebraic variety. We say that a sheaf $M$ on $X$ is a *left $D_X$-module* if $M(U)$ is endowed with a left $D_X(U)$-module structure for each open subset $U$ of $X$ and these actions are compatible with restriction morphisms.

Note that $\mathcal{O}_X$ is a left $D_X$-module via the canonical action of $D_X$.

We have the following very easy (but useful) interpretation of the notion of left $D_X$-modules.

**Lemma 1.2.1.** *Let $M$ be an $\mathcal{O}_X$-module. Giving a left $D_X$-module structure on $M$ extending the $\mathcal{O}_X$-module structure is equivalent to giving a $\mathbb{C}$-linear morphism*

$$\nabla : \Theta_X \to \mathcal{E}nd_{\mathbb{C}}(M) \quad (\theta \mapsto \nabla_\theta),$$

*satisfying the following conditions:*

(1) $\nabla_{f\theta}(s) = f\nabla_\theta(s)$ $\qquad (f \in \mathcal{O}_X,\ \theta \in \Theta_X,\ s \in M)$,

(2) $\nabla_\theta(fs) = \theta(f)\, s + f\nabla_\theta(s)$ $\qquad (f \in \mathcal{O}_X,\ \theta \in \Theta_X,\ s \in M)$,

(3) $\nabla_{[\theta_1,\theta_2]}(s) = [\nabla_{\theta_1}, \nabla_{\theta_2}](s)$ $\qquad (\theta_1, \theta_2 \in \Theta_X,\ s \in M)$.

*In terms of $\nabla$ the left $D_X$-module structure on $M$ is given by*

$$\theta s = \nabla_\theta(s) \qquad (\theta \in \Theta,\ s \in M).$$

*Proof.* The proof is immediate, because $D_X$ is generated by $\mathcal{O}_X, \Theta_X$ and satisfies the relation $[\theta,\ f] = \theta(f)$ (see Exercise 1.1.1). □

The condition (3) above is called the *integrability condition* on $M$.

For a locally free left $\mathcal{O}_X$-module $M$ of finite rank, a $\mathbb{C}$-linear morphism $\nabla : \Theta_X \to \mathcal{E}nd_{\mathbb{C}}(M)$ satisfying the conditions (1), (2) is usually called a *connection* (of the corresponding vector bundle). If it also satisfies the condition (3), it is called an *integrable (or flat) connection*. Hence we may regard a (left) $D_X$-module as an integrable connection of an $\mathcal{O}_X$-module which is not necessarily locally free of finite rank. In this book we use the following terminology.

**Definition 1.2.2.** We say that a $D_X$-module $M$ is an *integrable connection* if it is locally free of finite rank over $\mathcal{O}_X$.

**Notation 1.2.3.** We denote by $\mathrm{Conn}(X)$ the category of integrable connections on $X$.

Integrable connections are the most elementary left $D$-modules. Nevertheless, they are especially important because they generate (in a categorical sense) the category of holonomic systems, as we see later.

**Example 1.2.4 (ordinary differential equations).** Consider an ordinary differential operator $P = a_m(x)\partial^m + \cdots + a_0(x)$ $(\partial = \frac{d}{dx}, a_i \in \mathcal{O}_{\mathbb{C}})$ on $\mathbb{C}$ and the corresponding $D_{\mathbb{C}}$-module $M = D_{\mathbb{C}}/D_{\mathbb{C}}P = D_{\mathbb{C}}\, u$. Here $u \equiv 1 \mod D_{\mathbb{C}}P$, and hence $P\, u = 0$. Then on $U = \{x \in \mathbb{C} \mid a_m(x) \neq 0\}$ we have $M|_U \simeq \bigoplus_{i=0}^{m-1} \mathcal{O}_U u^{(i)}$ $(u^{(0)} = u$, $u^{(i)} = \partial^i u$ for $i = 1, 2, \ldots)$. Namely, it is an integrable connection of rank $m$ on $U$.

### Correspondence between left and right $D$-modules

Take a local coordinate system $\{x_i, \partial_i\}_{1 \le i \le n}$ on an affine open subset $U$ of $X$. For $P(x, \partial) = \sum_\alpha a_\alpha(x)\partial^\alpha \in D_U$ consider its *formal adjoint*

$${}^tP(x, \partial) := \sum_\alpha (-\partial)^\alpha a_\alpha(x) \in D_U.$$

Then we have ${}^t(PQ) = {}^tQ\,{}^tP$ and we get a ring anti-automorphism $P \mapsto {}^tP$ of $D_U$. Therefore, for a left $D_U$-module $M$ we can define a right action of $D_U$ on $M$ by $s\,P := {}^tP s$ for $s \in M,\ P \in D_U$, and obtain a *right* $D_U$-module ${}^tM$; however, this notion depends on the choice of a local coordinate. In order to globalize this

correspondence $M \leftrightarrow {}^tM$ to arbitrary smooth algebraic variety $X$ we need to use the *canonical sheaf*

$$\Omega_X := \bigwedge^n \Omega^1_X \qquad (n = \dim X)$$

since the formal adjoint of a differential operator naturally acts on $\Omega_X$. Recall that there exists a natural action of $\theta \in \Theta_X$ on $\Omega_X$ by the so-called *Lie derivative* $\mathrm{Lie}\,\theta$:

$$((\mathrm{Lie}\,\theta)\,\omega)(\theta_1, \theta_2, \ldots, \theta_n) := \theta(\omega(\theta_1, \theta_2, \ldots, \theta_n)) - \sum_{i=1}^{n} \omega(\theta_1, \ldots, [\theta, \theta_i], \ldots, \theta_n)$$
$$(\omega \in \Omega_X, \theta_1, \ldots, \theta_n \in \Theta_X).$$

Then we have

(1) $(\mathrm{Lie}[\theta_1, \theta_2])\omega = (\mathrm{Lie}\,\theta_1)((\mathrm{Lie}\,\theta_2)\omega) - (\mathrm{Lie}\,\theta_2)((\mathrm{Lie}\,\theta_1)\omega)$,
(2) $(\mathrm{Lie}\,\theta)(f\omega) = f((\mathrm{Lie}\,\theta)\omega) + \theta(f)\omega$,
(3) $(\mathrm{Lie}(f\theta))\omega = (\mathrm{Lie}\,\theta)(f\omega)$

for $\theta, \theta_1, \theta_2 \in \Theta_X$, $f \in \mathcal{O}_X$, $\omega \in \Omega_X$ ((1) and (2) hold even when $\omega$ is a differential form of any degree; however, (3) holds only for the highest degree case). Hence we can define a structure of a *right* $D_X$-module on $\Omega_X$ by

$$\omega\theta := -(\mathrm{Lie}\,\theta)\omega \qquad (\omega \in \Omega_X, \theta \in \Theta_X).$$

Here we have used the following analogue of Lemma 1.2.1.

**Lemma 1.2.5.** *Let $M$ be an $\mathcal{O}_X$-module. Giving a right $D_X$-module structure on $M$ extending the $\mathcal{O}_X$-module structure is equivalent to giving a $\mathbb{C}$-linear morphism*

$$\nabla' : \Theta_X \to \mathcal{E}nd_{\mathbb{C}}(M) \quad (\theta \mapsto \nabla'_\theta),$$

*satisfying the following conditions:*

(1) $\nabla'_{f\theta}(s) = \nabla'_\theta(fs)$ $\qquad (f \in \mathcal{O}_X,\ \theta \in \Theta_X,\ s \in M)$,
(2) $\nabla'_\theta(fs) = \theta(f)\,s + f\nabla'_\theta(s)$ $\qquad (f \in \mathcal{O}_X,\ \theta \in \Theta_X,\ s \in M)$,
(3) $\nabla'_{[\theta_1,\theta_2]}(s) = [\nabla'_{\theta_1}, \nabla'_{\theta_2}](s)$ $\qquad (\theta_1, \theta_2 \in \Theta_X,\ s \in M)$.

*In terms of $\nabla'$ the right $D_X$-module structure on $M$ is given by*

$$s\theta = -\nabla'_\theta(s) \qquad (\theta \in \Theta,\ s \in M).$$

The following is obvious from the definition.

**Lemma 1.2.6.** *In terms of a local coordinate system $\{x_i, \partial_i\}$, we have*

$$(f\,dx_1 \wedge \cdots \wedge dx_n)P(x, \partial) = ({}^tP(x, \partial)f)dx_1 \wedge \cdots \wedge dx_n \quad (f \in \mathcal{O}_X).$$

For a ring $R$ we denote by $R^{\mathrm{op}}$ the ring opposite to $R$. We have an identification $R \ni a \leftrightarrow a^\circ \in R^{\mathrm{op}}$ as an abelian group and the multiplication of $R^{\mathrm{op}}$ is defined by $a^\circ b^\circ = (ba)^\circ$.

For an invertible $\mathcal{O}_X$-module $\mathcal{L}$ we denote by $\mathcal{L}^{\otimes -1}$ its dual $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$.

The right $D_X$-module structure on $\Omega_X$ gives a homomorphism $D_X^{\mathrm{op}} \to \mathcal{E}nd_{\mathbb{C}}(\Omega_X)$ of $\mathbb{C}$-algebras. Note that we have an isomorphism

$$\mathcal{E}nd_{\mathbb{C}}(\Omega_X) \simeq \Omega_X \otimes_{\mathcal{O}_X} \mathcal{E}nd_{\mathbb{C}}(\mathcal{O}_X) \otimes_{\mathcal{O}_X} \Omega_X^{\otimes -1}$$

of sheaves of rings, where the left and the right $\mathcal{O}_X$-module structure on $\mathcal{E}nd_{\mathbb{C}}(\mathcal{O}_X)$ are given by the left- and right-multiplication of $\mathcal{O}_X$ (regarded as a subring of $\mathcal{E}nd_{\mathbb{C}}(\mathcal{O}_X)$) inside the (non-commutative) ring $\mathcal{E}nd_{\mathbb{C}}(\mathcal{O}_X)$, and the above isomorphism is given by associating $\omega \otimes F \otimes \eta \in \Omega_X \otimes_{\mathcal{O}_X} \mathcal{E}nd_{\mathbb{C}}(\mathcal{O}_X) \otimes_{\mathcal{O}_X} \Omega_X^{\otimes -1}$ ($\omega \in \Omega_X$, $F \in \mathcal{E}nd_{\mathbb{C}}(\mathcal{O}_X)$, $\eta \in \Omega_X^{\otimes -1}$) to the section of $\mathcal{E}nd_{\mathbb{C}}(\Omega_X)$ given by $\omega' \mapsto F(\langle \eta, \omega' \rangle)\omega$. By Lemma 1.2.6 (or by Exercise 1.1.4) we have the following.

**Lemma 1.2.7.** *We have a canonical isomorphism*

$$D_X^{\mathrm{op}} \simeq \Omega_X \otimes_{\mathcal{O}_X} D_X \otimes_{\mathcal{O}_X} \Omega_X^{\otimes -1}$$

*of $\mathbb{C}$-algebras.*

In terms of a local coordinate $\{x_i, \partial_i\}$ the correspondence in Lemma 1.2.7 is given by associating $P^\circ \in D_X^{\mathrm{op}}$ ($P \in D_X$) to $dx \otimes {}^tP \otimes dx^{\otimes -1}$, where $dx = dx_1 \wedge \cdots \wedge dx_n \in \Omega_X$ and $dx^{\otimes -1} \in \Omega_X^{\otimes -1}$ is given by $\langle dx, dx^{\otimes -1} \rangle = 1$.

**Notation 1.2.8.** For a ring (or a sheaf of rings on a topological space) $R$ we denote by $\mathrm{Mod}(R)$ the abelian category of left $R$-modules.

We will identify $\mathrm{Mod}(R^{\mathrm{op}})$ with the category of right $R$-modules. We easily see from Lemmas 1.2.1 and 1.2.5 the following.

**Proposition 1.2.9.** *Let $M, N \in \mathrm{Mod}(D_X)$ and $M', N' \in \mathrm{Mod}(D_X^{\mathrm{op}})$. Then we have*

| | | |
|---|---|---|
| (i) | $M \otimes_{\mathcal{O}_X} N \in \mathrm{Mod}(D_X)$; | $\theta(s \otimes t) = \theta(s) \otimes t + s \otimes \theta(t)$, |
| (ii) | $M' \otimes_{\mathcal{O}_X} N \in \mathrm{Mod}(D_X^{\mathrm{op}})$; | $(s' \otimes t)\theta = s'\theta \otimes t - s' \otimes \theta t$, |
| (iii) | $\mathcal{H}om_{\mathcal{O}_X}(M, N) \in \mathrm{Mod}(D_X)$; | $(\theta\psi)(s) = \theta(\psi(s)) - \psi(\theta(s))$, |
| (iv) | $\mathcal{H}om_{\mathcal{O}_X}(M', N') \in \mathrm{Mod}(D_X)$; | $(\theta\psi)(s) = -\psi(s)\theta + \psi(s\theta)$, |
| (v) | $\mathcal{H}om_{\mathcal{O}_X}(M, N') \in \mathrm{Mod}(D_X^{\mathrm{op}})$; | $(\psi\theta)(s) = \psi(s)\theta + \psi(\theta(s))$. |

*Here $\theta \in \Theta_X$.*

**Remark 1.2.10.** Let $X$ be a smooth algebraic curve $X$ of genus $g$. Note that $\deg \mathcal{O}_X = 0$ and $\deg \Omega_X = 2g - 2$. More generally, it is known that an invertible $\mathcal{O}_X$-module $\mathcal{L}$ is equipped with a left (resp. right) $D_X$-module structure if and only if $\deg \mathcal{L} = 0$ (resp. $2g - 2$). This gives an easy way to memorize all of the consequences of Proposition 1.2.9 (*Oda's rule* [O]). It also explains the reason why $M' \otimes_{\mathcal{O}_X} N'$ for $M', N' \in \mathrm{Mod}(D_X^{\mathrm{op}})$ is excluded from Proposition 1.2.9. Even if we do not know that $\deg \Omega_X = 2g - 2$, we can get the right answer by using the correspondences "left" $\leftrightarrow 0$, "right" $\leftrightarrow 1$ and $\otimes \leftrightarrow +$, $\mathcal{H}om(\bullet, \bigstar) = -\bullet + \bigstar$.

By Proposition 1.2.9 we easily see the following results.

**Lemma 1.2.11.** *Let $M, N \in \mathrm{Mod}(D_X)$ and $M' \in \mathrm{Mod}(D_X^{\mathrm{op}})$. Then we have isomorphisms*

$$(M' \otimes_{\mathcal{O}_X} N) \otimes_{D_X} M \simeq M' \otimes_{D_X} (M \otimes_{\mathcal{O}_X} N) \simeq (M' \otimes_{\mathcal{O}_X} M) \otimes_{D_X} N$$
$$((s' \otimes t) \otimes s \longleftrightarrow s' \otimes (s \otimes t) \longleftrightarrow (s' \otimes s) \otimes t)$$

*of $\mathbb{C}$-modules.*

**Proposition 1.2.12.** *The correspondence*

$$\Omega_X \otimes_{\mathcal{O}_X} (\bullet) : \mathrm{Mod}(D_X) \longrightarrow \mathrm{Mod}(D_X^{\mathrm{op}})$$

*gives an equivalence of categories. Its quasi-inverse is given by*

$$\Omega_X^{\otimes -1} \otimes_{\mathcal{O}_X} (\bullet) = \mathcal{H}om_{\mathcal{O}_X}(\Omega_X, \bullet) : \mathrm{Mod}(D_X^{\mathrm{op}}) \longrightarrow \mathrm{Mod}(D_X).$$

The operations $\Omega_X \otimes_{\mathcal{O}_X} (\bullet)$ and $\Omega_X^{\otimes -1} \otimes_{\mathcal{O}_X} (\bullet)$ exchanging the left and right $D$-module structures are called *side-changing* operations in this book.

## 1.3 Inverse and direct images I

For a morphism $f : X \to Y$ of smooth algebraic varieties, we introduce two operations on $D$-modules; the *inverse image* and the *direct image*.

### Inverse images

Let $M$ be a (left) $D_Y$-module and consider its inverse image

$$f^*M = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}M$$

of $M$ in the category of $\mathcal{O}$-modules. We can endow $f^*M$ with a (left) $D_X$-module structure as follows. First, note that we have a canonical $\mathcal{O}_X$-linear homomorphism

$$\Theta_X \to f^*\Theta_Y = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\Theta_Y \qquad (\theta \mapsto \tilde{\theta})$$

obtained by taking the $\mathcal{O}_X$-dual of $\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\Omega_Y^1 \to \Omega_X^1$. Then we can define a left $D_X$-module structure on $f^*M$ by

$$\theta(\psi \otimes s) = \theta(\psi) \otimes s + \psi\tilde{\theta}(s) \qquad (\theta \in \Theta_X,\ \psi \in \mathcal{O}_X,\ s \in M).$$

Here, if we write $\tilde{\theta} = \sum_j \varphi_j \otimes \theta_j$ ($\varphi_j \in \mathcal{O}_X, \theta_j \in \Theta_Y$), we set $\psi\tilde{\theta}(s) = \sum_j \psi\varphi_j \otimes \theta_j(s)$. This is the inverse image of $M$ in the category of $D$-modules. If we are given a local coordinate system $\{y_i, \partial_i\}$ of $Y$, then the action of $\theta \in \Theta_X$ can be written more explicitly as

$$\theta(\psi \otimes s) = \theta(\psi) \otimes s + \psi \sum_{i=1}^{n} \theta(y_i \circ f) \otimes \partial_i s \quad (\psi \otimes s \in \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}M)$$

(check it!).

Regarding $D_Y$ as a left $D_Y$-module by the left multiplication we obtain a left $D_X$-module $f^*D_Y = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}D_Y$. Then the right multiplication of $D_Y$ on $D_Y$ induces a right $f^{-1}D_Y$-module structure on $f^*D_Y$:

$$(\varphi \otimes P)Q = \varphi \otimes PQ \qquad (\varphi \in \mathcal{O}_X,\ P, Q \in D_Y),$$

and $f^*D_Y$ turns out to be a $(D_X, f^{-1}D_Y)$-bimodule.

**Definition 1.3.1.** The $(D_X, f^{-1}D_Y)$-bimodule $\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}D_Y$ is denoted by $D_{X\to Y}$.

It follows from the associativity of tensor products that we have an isomorphism

$$f^*M \simeq D_{X\to Y} \otimes_{f^{-1}D_Y} f^{-1}M$$

of left $D_X$-modules. We have obtained a right exact functor

$$D_{X\to Y} \otimes_{f^{-1}D_Y} f^{-1}(\bullet) : \mathrm{Mod}(D_Y) \to \mathrm{Mod}(D_X).$$

**Example 1.3.2.** Assume that $i : X \to Y$ is a closed embedding of smooth algebraic varieties. At any point of $X$ we can choose a local coordinate $\{y_k, \partial_{y_k}\}_{k=1,\dots,n}$ on an affine open subset of $Y$ such that $y_{r+1} = \cdots = y_n = 0$, gives a defining equation of $X$. Set $x_k = y_k \circ i$ for $k = 1, \dots, r$. This gives a local coordinate $\{x_k, \partial_{x_k}\}_{k=1,\dots,r}$ of an affine open subset of $X$. Moreover, the canonical morphism $\Theta_X \to \mathcal{O}_X \otimes_{i^{-1}\mathcal{O}_Y} i^{-1}\Theta_Y$ is given by $\partial_{x_k} \mapsto \partial_{y_k}$ $(k = 1, \dots, r)$. Set $D' = \bigoplus_{m_1,\dots,m_r} \mathcal{O}_Y \partial_{y_1}^{m_1} \cdots \partial_{y_r}^{m_r} \subset D_Y$. By $[\partial_{y_k}, \partial_{y_l}] = 0$ it is a subring of $D_Y$ and we have $D_Y \simeq D' \otimes_{\mathbb{C}} \mathbb{C}[\partial_{y_{r+1}}, \dots, \partial_{y_n}]$ as a left $D'$-module. Hence we have $D_{X\to Y} \simeq (\mathcal{O}_X \otimes_{i^{-1}\mathcal{O}_Y} i^{-1}D') \otimes_{\mathbb{C}} \mathbb{C}[\partial_{y_{r+1}}, \dots, \partial_{y_n}]$. It is easily seen that $\mathcal{O}_X \otimes_{i^{-1}\mathcal{O}_Y} i^{-1}D'$ is a $D_X$-submodule of $D_{X\to Y}$ isomorphic to $D_X$. We conclude that

$$D_{X\to Y} \simeq D_X \otimes_{\mathbb{C}} \mathbb{C}[\partial_{y_{r+1}}, \dots, \partial_{y_n}] \tag{1.3.1}$$

as a left $D_X$-module. In particular, $D_{X\to Y}$ is a locally free $D_X$-module of infinite rank (unless $r = n$).

### Direct images

Direct images of $D$-modules are more easily defined for right $D$-modules than for left $D$-modules. Let $M$ be a right $D_X$-module. Applying the sheaf-theoretical direct image functor $f_*$ to the right $f^{-1}D_Y$-module $M \otimes_{D_X} D_{X\to Y}$, we obtain a right $D_Y$-module $f_*(M \otimes_{D_X} D_{X\to Y})$. This gives an additive functor

$$f_*((\bullet) \otimes_{D_X} D_{X\to Y}) : \mathrm{Mod}(D_X^{\mathrm{op}}) \to \mathrm{Mod}(D_Y^{\mathrm{op}}),$$

which may be considered as a candidate for the direct image for right $D$-modules; however, unlike the case of the inverse image, this candidate does not suit the homological arguments since the right exact functor $\otimes$ and the left exact functor $f_*$ are both

involved. The right definition in the language of derived categories will be given later. Here we consider how to construct direct images for left $D$-modules. They can be defined by the correspondence (side-changing) of left and right $D$-modules explained in Proposition 1.2.12. Namely, a candidate for the direct image $\mathrm{Mod}(D_X) \to \mathrm{Mod}(D_Y)$ is obtained by the commutativity of

$$\begin{array}{ccc} \mathrm{Mod}(D_X) & \longrightarrow & \mathrm{Mod}(D_Y) \\ {\scriptstyle \Omega_X \otimes_{\mathcal{O}_X} (\bullet)} \big\downarrow \wr & & \wr \big\downarrow {\scriptstyle \Omega_Y \otimes_{\mathcal{O}_Y} (\bullet)} \\ \mathrm{Mod}(D_X^{\mathrm{op}}) & \longrightarrow & \mathrm{Mod}(D_Y^{\mathrm{op}}) \end{array}$$

where the lower horizontal arrow is given by $f_*((\bullet) \otimes_{D_X} D_{X\to Y})$. Thus, to a a left $D_X$-module $M$ we can associate a left $D_Y$-module

$$\Omega_Y^{\otimes -1} \otimes_{\mathcal{O}_Y} f_*((\Omega_X \otimes_{\mathcal{O}_X} M) \otimes_{D_X} D_{X\to Y}).$$

By Lemma 1.2.11 we have an isomorphism

$$(\Omega_X \otimes_{\mathcal{O}_X} M) \otimes_{D_X} D_{X\to Y} \simeq (\Omega_X \otimes_{\mathcal{O}_X} D_{X\to Y}) \otimes_{D_X} M$$

of right $f^{-1}D_Y$-modules, where $f^{-1}D_Y$ acts on $(\Omega_X \otimes_{\mathcal{O}_X} D_{X\to Y}) \otimes_{D_X} M$ by

$$((\omega \otimes R) \otimes s)P = (\omega \otimes RP) \otimes s \quad (\omega \in \Omega_X, R \in D_{X\to Y}, s \in M, P \in D_Y).$$

Hence we have

$$\begin{aligned} &\Omega_Y^{\otimes -1} \otimes_{\mathcal{O}_Y} f_*((\Omega_X \otimes_{\mathcal{O}_X} M) \otimes_{D_X} D_{X\to Y}) \\ &\quad \simeq \Omega_Y^{\otimes -1} \otimes_{\mathcal{O}_Y} f_*((\Omega_X \otimes_{\mathcal{O}_X} D_{X\to Y}) \otimes_{D_X} M) \\ &\quad \simeq f_*((\Omega_X \otimes_{\mathcal{O}_X} D_{X\to Y} \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\Omega_Y^{\otimes -1}) \otimes_{D_X} M). \end{aligned}$$

**Definition 1.3.3.** We define a $(f^{-1}D_Y, D_X)$-bimodule $D_{Y\leftarrow X}$ by

$$D_{Y\leftarrow X} := \Omega_X \otimes_{\mathcal{O}_X} D_{X\to Y} \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\Omega_Y^{\otimes -1}.$$

We call $D_{X\to Y}$ and $D_{Y\leftarrow X}$ the *transfer bimodules* for $f : X \to Y$.

In terms of $D_{Y\leftarrow X}$ our tentative definition of the direct image for left $D$-modules is given by

$$f_*(D_{Y\leftarrow X} \otimes_{D_X} (\bullet)) : \mathrm{Mod}(D_X) \to \mathrm{Mod}(D_Y).$$

By $D_Y^{\mathrm{op}} \simeq \Omega_Y \otimes_{\mathcal{O}_Y} D_Y \otimes_{\mathcal{O}_Y} \Omega_Y^{\otimes -1}$ we have

$$\begin{aligned} D_{Y\leftarrow X} &= \Omega_X \otimes_{\mathcal{O}_X} (\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}D_Y) \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\Omega_Y^{\otimes -1}, \\ &= \Omega_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}(D_Y \otimes_{\mathcal{O}_Y} \Omega_Y^{\otimes -1}), \\ &\simeq \Omega_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}(\Omega_Y^{\otimes -1} \otimes_{\mathcal{O}_Y} D_Y^{\mathrm{op}}) \\ &\simeq f^{-1}(D_Y \otimes_{\mathcal{O}_Y} \Omega_Y^{\otimes -1}) \otimes_{f^{-1}\mathcal{O}_Y} \Omega_X, \end{aligned}$$

where the last isomorphism is given by

$$\omega \otimes \eta \otimes P^\circ \leftrightarrow P \otimes \eta \otimes \omega \qquad (\omega \in \Omega_X, \eta \in \Omega_Y^{\otimes -1}, P \in D_Y).$$

Hence we obtain the following different description of $D_{Y\leftarrow X}$.

**Lemma 1.3.4.** *As a $(f^{-1}D_Y, D_X)$-bimodule we have*

$$D_{Y\leftarrow X} \simeq f^{-1}(D_Y \otimes_{\mathcal{O}_Y} \Omega_Y^{\otimes -1}) \otimes_{f^{-1}\mathcal{O}_Y} \Omega_X.$$

*Here the right-hand side is endowed with a left $f^{-1}D_Y$-module structure induced from the left multiplication of $D_Y$ on $D_Y$. The right $D_X$-module structure on it is given as follows. The right multiplication of $D_Y$ on $D_Y$ gives a right $D_Y$-modules structure on $D_Y$. By the side-changing operation, $D_Y \otimes_{\mathcal{O}_Y} \Omega_Y^{\otimes -1}$ is a left $D_Y$-module. Applying the inverse image functor for left $D$-modules we get a left $D_X$-module $f^{-1}(D_Y \otimes_{\mathcal{O}_Y} \Omega_Y^{\otimes -1}) \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$. Finally, by the side-changing operation we obtain a right $D_X$-module*

$$f^{-1}(D_Y \otimes_{\mathcal{O}_Y} \Omega_Y^{\otimes -1}) \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X \otimes_{\mathcal{O}_X} \Omega_X = f^{-1}(D_Y \otimes_{\mathcal{O}_Y} \Omega_Y^{\otimes -1}) \otimes_{f^{-1}\mathcal{O}_Y} \Omega_X.$$

**Example 1.3.5.** We give a local description of $D_{Y\leftarrow X}$ for a closed embedding $i : X \to Y$ of smooth algebraic varieties. Take a local coordinate $\{y_k, \partial_{y_k}\}_{1\le k\le n}$ of $Y$ as in Example 1.3.2, and set $x_k = y_k \circ i$ for $k = 1, \dots, r$. Note that

$$D_{Y\leftarrow X} = (i^{-1}D_Y \otimes_{i^{-1}\mathcal{O}_Y} \mathcal{O}_X) \otimes_{\mathcal{O}_X} (i^{-1}\Omega_Y^{\otimes -1} \otimes_{i^{-1}\mathcal{O}_Y} \Omega_X).$$

We (locally) identify $i^{-1}\Omega_Y^{\otimes -1} \otimes_{i^{-1}\mathcal{O}_Y} \Omega_X$ with $\mathcal{O}_X$ via the section

$$(dy_1 \wedge \cdots \wedge dy_n)^{\otimes -1} \otimes (dx_1 \wedge \cdots \wedge dx_r).$$

Set $D' = \bigoplus_{m_1,\dots,m_r} \partial_{y_1}^{m_1} \cdots \partial_{y_r}^{m_r} \mathcal{O}_Y \subset D_Y$. It is a subring of $D_Y$ and we have $D_Y \simeq \mathbb{C}[\partial_{y_{r+1}}, \dots, \partial_{y_n}] \otimes_{\mathbb{C}} D'$ as a right $D'$-module. Hence we have

$$D_{Y\leftarrow X} \simeq \mathbb{C}[\partial_{y_{r+1}}, \dots, \partial_{y_n}] \otimes_{\mathbb{C}} (i^{-1}D' \otimes_{i^{-1}\mathcal{O}_Y} \mathcal{O}_X). \tag{1.3.2}$$

The right $D_X$-action on the right-hand side is induced from the right $D_X$-action on $i^{-1}D' \otimes_{i^{-1}\mathcal{O}_Y} \mathcal{O}_X$ given by

$$(P \otimes 1)\partial_{x_k} = (P\partial_{y_k}) \otimes 1, \ (P \otimes 1)\varphi = P \otimes \varphi = P\tilde{\varphi} \otimes 1 \quad (P \in D', \ \varphi \in \mathcal{O}_X),$$

where $\tilde{\varphi} \in \mathcal{O}_Y$ is such that $\tilde{\varphi}|_X = \varphi$. Hence we have $i^{-1}D' \otimes_{i^{-1}\mathcal{O}_Y} \mathcal{O}_X \simeq D_X$ and we obtain a local isomorphism

$$D_{Y\leftarrow X} \simeq \mathbb{C}[\partial_{y_{r+1}}, \dots, \partial_{y_n}] \otimes_{\mathbb{C}} D_X.$$

The left $i^{-1}D_Y$-module structure on the right-hand side can be described as follows. Note that $D_Y \simeq \mathbb{C}[\partial_{y_{r+1}}, \dots, \partial_{y_n}] \otimes_{\mathbb{C}} D'$. Hence we have $i^{-1}D_Y \simeq \mathbb{C}[\partial_{y_{r+1}}, \dots, \partial_{y_n}] \otimes_{\mathbb{C}} i^{-1}D'$. Therefore, it is sufficient to give the actions of $\mathbb{C}[\partial_{y_{r+1}}, \dots, \partial_{y_n}]$ and $i^{-1}D'$ on $\mathbb{C}[\partial_{y_{r+1}}, \dots, \partial_{y_n}] \otimes_{\mathbb{C}} D_X$.

The action of $\mathbb{C}[\partial_{y_{r+1}}, \dots, \partial_{y_n}]$ is given by the multiplication on the first factor $\mathbb{C}[\partial_{y_{r+1}}, \dots, \partial_{y_n}]$ of the tensor product. Let $Q \in i^{-1}D'$, $F \in \mathbb{C}[\partial_{y_{r+1}}, \dots, \partial_{y_n}]$ and $R \in D_X$. If we have $QF = \sum_k F_k Q_k$ ($F_k \in \mathbb{C}[\partial_{y_{r+1}}, \dots, \partial_{y_n}]$, $Q_k \in i^{-1}D'$) in the ring $i^{-1}D_Y$, then the action of $Q \in i^{-1}D'$ on $F \otimes R \in \mathbb{C}[\partial_{y_{r+1}}, \dots, \partial_{y_n}] \otimes_{\mathbb{C}} D_X$ is given by $Q(F \otimes R) = \sum_k F_k \otimes Q_k R$, where the left $i^{-1}D'$-module structure on $D_X \simeq i^{-1}D' \otimes_{i^{-1}\mathcal{O}_Y} \mathcal{O}_X$ is given by

$$Q(P \otimes 1) = QP \otimes 1 \quad (P, Q \in D').$$

## 1.4 Some categories of $D$-modules

On algebraic varieties, the category of *quasi-coherent sheaves* (over $\mathcal{O}$) is sufficiently wide and suitable for various algebraic operations (see Appendix A for the notion of quasi-coherent sheaves). Since our sheaf $D_X$ is locally free over $\mathcal{O}_X$, it is quasi-coherent over $\mathcal{O}_X$. We mainly deal with $D_X$-modules which are quasi-coherent over $\mathcal{O}_X$.

**Notation 1.4.1.** For an algebraic variety $X$ we denote the category of quasi-coherent $\mathcal{O}_X$-modules by $\mathrm{Mod}_{qc}(\mathcal{O}_X)$. For a smooth algebraic variety $X$ we denote by $\mathrm{Mod}_{qc}(D_X)$ the category of $\mathcal{O}_X$-quasi-coherent $D_X$-modules.

The category $\mathrm{Mod}_{qc}(D_X)$ is an abelian category.
It is well known that for affine algebraic varieties $X$,

(a) the global section functor $\Gamma(X, \bullet) : \mathrm{Mod}_{qc}(\mathcal{O}_X) \to \mathrm{Mod}(\Gamma(X, \mathcal{O}_X))$ is exact,
(b) if $\Gamma(X, M) = 0$ for $M \in \mathrm{Mod}_{qc}(\mathcal{O}_X)$, then $M = 0$.

In fact, an algebraic variety is affine if and only if the condition (a) is satisfied. Replacing $\mathcal{O}_X$ by $D_X$ we come to the following notion.

**Definition 1.4.2.** A smooth algebraic variety $X$ is called *$D$-affine* if the following conditions are satisfied:

(a) the global section functor $\Gamma(X, \bullet) : \mathrm{Mod}_{qc}(D_X) \to \mathrm{Mod}(\Gamma(X, D_X))$ is exact,
(b) if $\Gamma(X, M) = 0$ for $M \in \mathrm{Mod}_{qc}(D_X)$, then $M = 0$.

The following is obvious.

**Proposition 1.4.3.** *Any smooth affine algebraic variety is $D$-affine.*

As in the case of quasi-coherent $\mathcal{O}$-modules on affine varieties we have the following.

**Proposition 1.4.4.** *Assume that $X$ is $D$-affine.*

(i) *Any $M \in \mathrm{Mod}_{qc}(D_X)$ is generated over $D_X$ by its global sections.*
(ii) *The functor*

$$\Gamma(X, \bullet) : \mathrm{Mod}_{qc}(D_X) \to \mathrm{Mod}(\Gamma(X, D_X))$$

*gives an equivalence of categories.*

*Proof.* (i) For $M \in \mathrm{Mod}_{qc}(D_X)$ let $M_0$ be the image of the natural morphism $D_X \otimes_{\mathbb{C}} \Gamma(X, M) \to M$ in $M$ (the submodule of $M$ generated by global sections). Since $X$ is $D$-affine, we obtain an exact sequence

$$0 \to \Gamma(X, M_0) \overset{i}{\to} \Gamma(X, M) \to \Gamma(X, M/M_0) \to 0.$$

Since $i$ is an isomorphism by the definition of $M_0$, we have $\Gamma(X, M/M_0) = 0$. Since $X$ is $D$-affine, we get $M/M_0 = 0$, i.e., $M = M_0$.

(ii) We will show that the functor $D_X \otimes_{\Gamma(X,D_X)} (\bullet) : \mathrm{Mod}(\Gamma(X, D_X)) \to \mathrm{Mod}_{qc}(D_X)$ is quasi-inverse to $\Gamma(X, \bullet)$. Since $D_X \otimes_{\Gamma(X,D_X)} (\bullet)$ is left adjoint to $\Gamma(X, \bullet)$, it is sufficient to show that the canonical homomorphisms

$$\alpha_M : D_X \otimes_{\Gamma(X,D_X)} \Gamma(X, M) \to M, \qquad \beta_V : V \to \Gamma(X, D_X \otimes_{\Gamma(X,D_X)} V)$$

are isomorphisms for $M \in \mathrm{Mod}_{qc}(D_X)$, $V \in \mathrm{Mod}(\Gamma(X, D_X))$.

Choose an exact sequence

$$\Gamma(X, D_X)^{\oplus I} \longrightarrow \Gamma(X, D_X)^{\oplus J} \longrightarrow V \longrightarrow 0.$$

Since $X$ is $D$-affine, we have that the functor $\Gamma(X, D_X \otimes_{\Gamma(X,D_X)} (\bullet))$ is right exact on $\mathrm{Mod}(\Gamma(X, D_X))$. Hence we obtain a commutative diagram

$$\begin{array}{ccccccc}
\Gamma(X, D_X)^{\oplus I} & \longrightarrow & \Gamma(X, D_X)^{\oplus J} & \longrightarrow & V & \longrightarrow & 0 \\
\| & & \| & & \downarrow{\scriptstyle \beta_V} & & \\
\Gamma(X, D_X)^{\oplus I} & \longrightarrow & \Gamma(X, D_X)^{\oplus J} & \longrightarrow & \Gamma(X, D_X \otimes_{\Gamma(X,D_X)} V) & \longrightarrow & 0
\end{array}$$

whose rows are exact. Hence $\beta_V$ is an isomorphism.

By (i) $\alpha_M$ is surjective. Hence we have an exact sequence

$$0 \longrightarrow K \longrightarrow D_X \otimes_{\Gamma(X,D_X)} \Gamma(X, M) \longrightarrow M \longrightarrow 0$$

for some $K \in \mathrm{Mod}_{qc}(D_X)$. Applying the exact functor $\Gamma(X, \bullet)$ we obtain

$$0 \longrightarrow \Gamma(X, K) \longrightarrow \Gamma(X, M) \longrightarrow \Gamma(X, M) \longrightarrow 0.$$

Here we have used the fact that $\beta_{\Gamma(X,M)}$ is an isomorphism. Hence we have $\Gamma(X, K) = 0$. This implies that $K = 0$ since $X$ is $D$-affine. Hence $\alpha_M$ is an isomorphism. □

**Remark 1.4.5.**

(i) The $D$-affinity holds also for certain non-affine varieties. In Section 1.6 we will show that projective spaces are $D$-affine. (Theorem 1.6.5). We will also show in Part II that flag manifolds for semisimple algebraic groups are $D$-affine. This fact was one of the key points in the settlement of the Kazhdan–Lusztig conjecture.

(ii) If $X$ is affine, we can replace $D_X$ with $D_X^{\mathrm{op}}$ in the above argument. In other words, smooth affine varieties are also $D^{\mathrm{op}}$-affine. Note that $D$-affine varieties are not necessarily $D^{\mathrm{op}}$-affine in general. For example, $\mathbb{P}^1$ is not $D^{\mathrm{op}}$-affine by $\Gamma(\mathbb{P}^1, \Omega_{\mathbb{P}^1}) = 0$.

The order filtration $F$ of $D_X$ induces filtrations (denoted also by $F$) of the rings $D_X(U)$ and $D_{X,x}$, where $U$ is an affine open subset of $X$ and $x \in X$. By this filtration we have filtered rings $(D_X(U), F)$ and $(D_{X,x}, F)$ in the sense of Appendix D. Let $\{x_i, \partial_i\}_{1 \le i \le n}$ be a coordinate system on $U$. Then we have

$$\mathrm{gr}^F D_X(U) = \mathcal{O}_X(U)[\xi_1, \ldots, \xi_n], \qquad \mathrm{gr}^F D_{X,x} = \mathcal{O}_{X,x}[\xi_1, \ldots, \xi_n] \quad (x \in X),$$

where $\xi_i = \sigma_1(\partial_i)$. In particular, $\mathrm{gr}^F D_X(U)$ and $\mathrm{gr}^F D_{X,x}$ are noetherian rings with global dimension $2 \dim X$. Hence we obtain from Proposition D.1.4 and Theorem D.2.6 the following.

**Proposition 1.4.6.** *Assume that $A = D_X(U)$ for some affine open subset $U$ of $X$ or $A = D_{X,x}$ for some $x \in X$.*

(i) *$A$ is a left (and right) noetherian ring.*
(ii) *The left and right global dimensions of $A$ are smaller than or equal to* $2 \dim X$.

**Remark 1.4.7.** It will be shown later in Section 2.6 that the left and right global dimensions of the ring $A$ in Proposition 1.4.6 are exactly $\dim X$ (see Theorem 2.6.11).

We recall the notion of coherent sheaves.

**Definition 1.4.8.** Let $R$ be a sheaf of rings on a topological space $X$.

(i) An $R$-module $M$ is called *coherent* if $M$ is locally finitely generated and if for any open subset $U$ of $X$ any locally finitely generated submodule of $M|_U$ is locally finitely presented.
(ii) $R$ is called a *coherent sheaf of rings* if $R$ is coherent as an $R$-module.

It is well known that if $R$ is a coherent sheaf of rings, an $R$-module is coherent if and only if it is locally finitely presented.

**Proposition 1.4.9.**
(i) *$D_X$ is a coherent sheaf of rings.*
(ii) *A $D_X$-module is coherent if and only if it is quasi-coherent over $\mathcal{O}_X$ and locally finitely generated over $D_X$.*

*Proof.* The statement (i) follows from (ii), and hence we only need to show (ii). Assume that $M$ is a coherent $D_X$-module. By definition $M$ is locally finitely generated over $D_X$. Moreover, $M$ is quasi-coherent over $\mathcal{O}_X$ since it is locally finitely presented as a $D_X$-module and $D_X$ is quasi-coherent over $\mathcal{O}_X$. Assume that $M$ is a locally finitely generated $D_X$-module quasi-coherent over $\mathcal{O}_X$. To show that $M$ is coherent over $D_X$ it is sufficient to show that for any affine open subset $U$ of $X$ the kernel of any homomorphism $\alpha : D_U^p \to M|_U$ ($p \in \mathbb{N}$) of $D_U$-modules is finitely generated over $D_U$. Since $D_U(U)$ is a left noetherian ring, the kernel of $D_U(U)^p \to M(U)$ is a finitely generated $D_U(U)$-module, and hence we obtain an exact sequence $D_U(U)^q \to D_U(U)^p \to M(U)$ for some $q \in \mathbb{N}$. By Proposition 1.4.3 this induces an exact sequence $D_U^q \to D_U^p \to M|_U$ in $\mathrm{Mod}_{qc}(D_U)$. In other words $\mathrm{Ker}\,\alpha$ is finitely generated. □

**Theorem 1.4.10.** *A $D_X$-module is coherent over $\mathcal{O}_X$ if and only if it is an integrable connection.*

*Proof.* It is sufficient to show that any $D_X$-module which is coherent over $\mathcal{O}_X$ is a locally free $\mathcal{O}_X$-module. Let $\mathcal{M}$ be a $D_X$-module coherent over $\mathcal{O}_X$. By a standard property of coherent $\mathcal{O}_X$-modules, it suffices to prove that for each $x \in X$ the stalk $\mathcal{M}_x$ is free over $\mathcal{O}_{X,x}$. Let us take a local coordinate system $\{x_i, \partial_i\}$ of $X$ such that the unique maximal ideal $\mathfrak{m}$ of the local ring $\mathcal{O}_{X,x}$ is generated by $x_1, x_2, \ldots, x_n$ ($n = \dim X$). Then it follows from Nakayama's lemma that there exist $s_1, s_2, \ldots, s_m \in$

$\mathcal{M}_x$ such that $\mathcal{M}_x = \sum_{i=1}^m \mathcal{O}_{X,x} s_i$ and $\overline{s}_1, \overline{s}_2, \ldots, \overline{s}_m \in \mathcal{M}_x \big/ \sum_{i=1}^n x_i \mathcal{M}_x$ are free generators of the vector space $\mathcal{M}_x \big/ \sum_{i=1}^n x_i \mathcal{M}_x$ over $\mathbb{C} = \mathcal{O}_{X,x}/\mathfrak{m}$. We will show that $\{s_1, s_2, \ldots, s_m\}$ is a free generator of the $\mathcal{O}_{X,x}$-module $\mathcal{M}_x$. Assume that there exists a non-trivial relation

$$\sum_{i=1}^m f_i s_i = 0 \quad (f_i \in \mathcal{O}_{X,x}).$$

Now we define the order of each $f_i \in \mathcal{O}_{X,x}$ at $x \in X$ by $\mathrm{ord}(f_i) = \max\{l \mid f_i \in \mathfrak{m}^l\}$. If we apply the differential operator $\partial_j$ to the above relation, we obtain the new relation

$$0 = \sum_{i=1}^m (\partial_j f_i) s_i + f_i (\partial_j s_i) = \sum_{i=1}^m g_i s_i \quad (g_i \in \mathcal{O}_{X,x}).$$

Since each $\partial_j s_i$ is a linear combination of $s_1, s_2, \ldots, s_m$ over $\mathcal{O}_{X,x}$, we can take a suitable index $j$ so that we have the inequality

$$\min\{\mathrm{ord}(f_i) \mid i = 1, 2, \ldots, m\} > \min\{\mathrm{ord}(g_i) \mid i = 1, 2, \ldots, m\}.$$

By repeating this argument, we finally get the non-trivial relation

$$\sum_{i=1}^m \overline{h}_i \overline{s}_i = 0 \quad (\overline{h}_i \in \mathcal{O}_{X,x}/\mathfrak{m} \simeq \mathbb{C})$$

in $\mathcal{M}_x \big/ \sum_{i=1}^n x_i \mathcal{M}_x$, which contradicts the choice of $s_1, \ldots, s_m$. □

**Corollary 1.4.11.** *The category* $\mathrm{Conn}(X)$ *is an abelian category.*

**Notation 1.4.12.**
(i) We denote by $\mathrm{Mod}_c(D_X)$ the category consisting of coherent $D_X$-modules.
(ii) For a ring $R$ we denote the category of finitely generated $R$-modules by $\mathrm{Mod}_f(R)$.

The category $\mathrm{Mod}_c(D_X)$ is an abelian category. If $R$ is a noetherian ring, $\mathrm{Mod}_f(R)$ is an abelian category.

**Proposition 1.4.13.** *Assume that $X$ is $D$-affine. The equivalence* $\mathrm{Mod}_{qc}(D_X) \simeq \mathrm{Mod}(\Gamma(X, D_X))$ *in Proposition* 1.4.4 *induces the equivalence*

$$\mathrm{Mod}_c(D_X) \simeq \mathrm{Mod}_f(\Gamma(X, D_X)).$$

*Proof.* For $V \in \mathrm{Mod}_f(\Gamma(X, D_X))$ the $D_X$-module $D_X \otimes_{\Gamma(X,D_X)} V$ is clearly finitely generated and belongs to $\mathrm{Mod}_c(D_X)$. Let $M \in \mathrm{Mod}_c(D_X)$. By definition $M$ is locally generated by finitely many sections. Moreover, the surjectivity of the morphism $D_X \otimes_{\Gamma(X,D_X)} \Gamma(X, M) \to M$ (see Proposition 1.4.4 (i)) implies that we can take the local finite generators from $\Gamma(X, M)$. Since $X$ is quasi-compact, we see that $M$ is globally generated by finitely many elements of $\Gamma(X, M)$. This means that we have a surjective homomorphism $D_X^p \to M$ for some $p \in \mathbb{N}$. From this we obtain a surjective homomorphism $\Gamma(X, D_X)^p \to \Gamma(X, M)$, and hence $\Gamma(X, M)$ belongs to $\mathrm{Mod}_f(\Gamma(X, D_X))$. □

**Proposition 1.4.14.** *Any $M \in \mathrm{Mod}_{qc}(D_X)$ is embedded into an injective object $I$ of $\mathrm{Mod}_{qc}(D_X)$ which is flabby (we do not claim that $I$ is injective in $\mathrm{Mod}(D_X)$).*

*Proof.* Take a finite affine open covering $X = \bigcup_i U_i$. Let $j_i : U_i \to X$ be the open embedding. By Proposition 1.4.3 we can embed $j_i^* M$ into an injective object $I_i$ of $\mathrm{Mod}_{qc}(D_{U_i})$. Set $I = \bigoplus_i j_{i*} I_i$. Then $I$ is an injective object of $\mathrm{Mod}_{qc}(D_X)$. Moreover, the canonical morphism $M \to I$ is a monomorphism. It remains to show that $I$ is flabby. For this it is sufficient to show that $I_i$ is flabby for each $i$. For any $M \in \mathrm{Mod}_{qc}(\mathcal{O}_{U_i})$ we have

$$\mathrm{Hom}_{\mathcal{O}_{U_i}}(M, I_i) \simeq \mathrm{Hom}_{D_{U_i}}(D_{U_i} \otimes_{\mathcal{O}_{U_i}} M, I_i),$$

and hence $I_i$ is an injective object of $\mathrm{Mod}_{qc}(\mathcal{O}_{U_i})$. Hence it is flabby (see, e.g., [Ha2, III, Proposition 3.4]). □

**Corollary 1.4.15.** *Assume that $X$ is $D$-affine. Then for any $M \in \mathrm{Mod}_{qc}(D_X)$ and $i > 0$ we have $H^i(X, M) = 0$.*

*Proof.* By Proposition 1.4.14 we can take a resolution

$$0 \to M \to I_0 \to I_1 \to \cdots,$$

where $I_j$ are injective objects of $\mathrm{Mod}_{qc}(D_X)$ which are flabby. Since $I_j$ are flabby, $H^i(X, M)$ is the $i$th cohomology group of the complex $\Gamma(X, I^\cdot)$. On the other hand since $X$ is $D$-affine, the functor $\Gamma(X, \bullet)$ is exact on $\mathrm{Mod}_{qc}(D_X)$, and hence $H^i(X, M) = 0$ for $i > 0$. □

The following facts are well known in algebraic geometry (see, e.g., [Ha2, p. 126]).

**Proposition 1.4.16.**

(i) *Let $F$ be a quasi-coherent $\mathcal{O}_X$-module. For an open subset $U \subset X$ consider a coherent $\mathcal{O}_U$-submodule $G \subset F|_U$ of the restriction $F|_U$ of $F$ to $U$. Then $G$ can be extended to a coherent $\mathcal{O}_X$-submodule $\widetilde{G} \subset F$ of $F$ (i.e., $\widetilde{G}|_U = G$).*

(ii) *A quasi-coherent $\mathcal{O}_X$-module is a union of coherent $\mathcal{O}_X$-submodules.*

**Corollary 1.4.17.**

(i) *A coherent $D_X$-module is (globally) generated by a coherent $\mathcal{O}_X$-submodule.*

(ii) *Let $M \in \mathrm{Mod}_{qc}(D_X)$ and let $U$ be an open subset of $X$. Then any coherent $D_U$-submodule $N$ of $M|_U$ is extended to a coherent $D_X$-submodule $\widetilde{N}$ of $M$ (i.e., $\widetilde{N}|_U = N$).*

(iii) *Any $M \in \mathrm{Mod}_{qc}(D_X)$ is a union of coherent $D_X$-submodules.*

*Proof.* (i) Let $M$ be a coherent $D_X$-module. Take a finite affine open covering $X = \bigcup_i U_i$ of $X$ such that $M|_{U_i}$ is finitely generated over $D_{U_i}$. Then $M|_{U_i}$ is generated by a coherent $\mathcal{O}_{U_i}$-submodule $F_i \subset M|_{U_i}$. By Proposition 1.4.16 (i) we can take an extension $\widetilde{F}_i \subset M$ of $F_i$, which is coherent over $\mathcal{O}_X$. Then the sum $\sum_i \widetilde{F}_i$ is coherent over $\mathcal{O}_X$ and generates $M$ over $D_X$. The proofs for (ii) and (iii) are similar and omitted. □

**Proposition 1.4.18.** *Assume that $X$ is* **quasi-projective** (*that is, isomorphic to a locally closed subvariety of a projective space*).

(i) *Any $M \in \mathrm{Mod}_{qc}(D_X)$ is a quotient of a locally free ($\Rightarrow$ locally projective $\Rightarrow$ flat) $D_X$-module.*
(ii) *Any $M \in \mathrm{Mod}_{c}(D_X)$ is a quotient of a locally free $D_X$-module of finite rank.*

*Proof.* (i) Let $M \in \mathrm{Mod}_{qc}(D_X)$. Take a quasi-coherent $\mathcal{O}_X$-submodule $F$ of $M$ satisfying $M = D_X F$ (we can take, e.g., $F = M$). It is sufficient to show that $F$ is a quotient of a locally free $\mathcal{O}_X$-module $F_0$. In fact, from such $F$, we obtain a sequence

$$D_X \otimes_{\mathcal{O}_X} F_0 \longrightarrow\!\!\!\!\rightarrow D_X \otimes_{\mathcal{O}_X} F \longrightarrow\!\!\!\!\rightarrow M = D_X F$$

of surjective $D_X$-linear morphisms, and $D_X \otimes_{\mathcal{O}_X} F_0$ is clearly a locally free $D_X$-module. Choose a locally closed embedding $i : X \to Y = \mathbb{P}^n$. We have only to show that the $\mathcal{O}_Y$-module $i_* F$ is a quotient of a locally free $\mathcal{O}_Y$-module. Since $i_* F$ is a quasi-coherent $\mathcal{O}_Y$-module, it is a sum of coherent $\mathcal{O}_Y$-submodules. Hence by Serre's theorem $i_* F$ is a quotient of a sum of invertible $\mathcal{O}_Y$-modules of the form $\mathcal{O}(m)$ for some $m$. Let $Y = \bigcup_{k=0}^{n} U_k$ be the standard affine open covering of $Y = \mathbb{P}^n$ ($U_k \simeq \mathbb{A}^n$). Then $\mathcal{O}(m)|_{U_k}$ is free for any $m$ and $k$. Hence $i_* F$ is a quotient of a locally free $\mathcal{O}_Y$-module.

(ii) If $M$ is coherent, one can take $F$ in the proof of (i) to be a coherent $\mathcal{O}_X$-module. Hence the assertion can be proved using the argument in (i). □

**Assumption 1.4.19.** Hereafter, all algebraic varieties are assumed to be quasi-projective.

**Corollary 1.4.20.** *Let $M \in \mathrm{Mod}_{qc}(D_X)$.*

(i) *There exists a resolution*

$$\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

*of $M$ by locally free $D_X$-modules.*
(ii) *There exists a finite resolution*

$$0 \longrightarrow P_m \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

*of $M$ by locally projective $D_X$-modules.*

*If $M \in \mathrm{Mod}_{c}(D_X)$, we can take all $P_i$'s as in* (i) *and* (ii) *to be of finite rank.*

*Proof.* (i) follows from Proposition 1.4.18.

(ii) Take a resolution as in (i) and set $Q = \mathrm{Coker}(P_{2\dim X+1} \to P_{2\dim X})$. It is sufficient to show that $Q$ is locally projective. Let $U$ be an affine open subset. By Proposition 1.4.3 we have a resolution

$$\begin{aligned} 0 \longrightarrow\ & Q(U) \longrightarrow P_{2\dim X-1}(U) \\ & \longrightarrow \cdots \longrightarrow P_1(U) \longrightarrow P_0(U) \longrightarrow M(U) \longrightarrow 0, \end{aligned}$$

where $P_i(U)$'s are projective $D_X(U)$-modules. Since the global dimension of $D_X(U)$ is smaller than or equal to $2 \dim X$, we easily see that $Q(U)$ is also projective by a standard argument in homological algebra. Hence $Q|_U$ is a projective object in $\mathrm{Mod}_{qc}(D_U)$ (a direct summand of a free $D_U$-module), in particular a projective $D_X$-module. □

## 1.5 Inverse images and direct images II

In this section we shall define several functors on derived categories of $D$-modules and prove fundamental properties concerning them.

### Derived categories

**Notation 1.5.1.** For a ring $R$ (or a sheaf $R$ of rings on a topological space) the derived categories $D(\mathrm{Mod}(R))$, $D^+(\mathrm{Mod}(R))$, $D^-(\mathrm{Mod}(R))$, $D^b(\mathrm{Mod}(R))$ of the abelian category $\mathrm{Mod}(R)$ are simply denoted by $D(R)$, $D^+(R)$, $D^-(R)$, $D^b(R)$, respectively.

The following well-known fact is fundamental.

**Lemma 1.5.2.** *Let $R$ be a sheaf of rings on a topological space $X$.*

(i) *For any $M \in \mathrm{Mod}(R)$ there exists a monomorphism $M \to I$ where $I$ is an injective object of* $\mathrm{Mod}(R)$.
(ii) *For any $M \in \mathrm{Mod}(R)$ there exists a epimorphism $F \to M$ where $F$ is a flat $R$-module.*

In particular, any object $M^\cdot$ of $D^+(R)$ (resp. $D^-(R)$) is quasi-isomorphic to a complex $I^\cdot$ (resp. $F^\cdot$) of injective (resp. flat) $R$-modules belonging to $D^+(R)$ (resp. $D^-(R)$).

Let $f : X \to Y$ be a continuous map between topological spaces, and let $R$ be a sheaf of rings on $Y$. The direct image functor $f_* : \mathrm{Mod}(f^{-1}R) \to \mathrm{Mod}(R)$ is left exact and we can define its right derived functor

$$Rf_* : D^+(f^{-1}R) \to D^+(R)$$

by using an injective resolution of $M^\cdot$. In the case $R = \mathbb{Z}_Y$ this gives a functor

$$Rf_* : D^+(\mathrm{Sh}(X)) \to D^+(\mathrm{Sh}(Y)),$$

where $\mathrm{Sh}(Z)$ denotes the category of abelian sheaves on a topological space $Z$.

Since any injective $f^{-1}R$-module is a flabby sheaf (see, e.g., [KS2, Proposition 2.4.6]), we have the following.

**Proposition 1.5.3.** *Let $f : X \to Y$ be a continuous map between topological spaces, and let $R$ be a sheaf of rings on $Y$. Then we have a commutative diagram*

$$\begin{array}{ccc} D^+(f^{-1}R) & \xrightarrow{Rf_*} & D^+(R) \\ \downarrow & & \downarrow \\ D^+(\mathrm{Sh}(X)) & \xrightarrow[Rf_*]{} & D^+(\mathrm{Sh}(Y)), \end{array}$$

*where the vertical arrows are the forgetful functors.*

**Proposition 1.5.4.** *Let $f : X \to Y$ be a morphism of algebraic varieties, and let $R$ be a sheaf of rings on $Y$. Then $Rf_*$ sends $D^b(f^{-1}R)$ to $D^b(R)$ and commutes with arbitrary direct sums.*

*Proof.* By Proposition 1.5.3 we may assume that $R = \mathbb{Z}_Y$. Then the assertion follows from the well-known corresponding fact concerning $Rf_*$ for abelian sheaves on noetherian topological spaces (see, e.g., [Ha2, III, Theorem 2.7, Lemma 2.8]). □

In the rest of this section $X$ denotes a smooth algebraic variety.

**Notation 1.5.5.** For $\sharp = +, -, b$ we denote by $D^\sharp_{qc}(D_X)$ (resp. $D^\sharp_c(D_X)$) the full subcategory of $D^\sharp(D_X)$ consisting of complexes whose cohomology sheaves belong to $\mathrm{Mod}_{qc}(D_X)$ (resp. $\mathrm{Mod}_c(D_X)$).

The categories $D^\sharp_{qc}(D_X)$ and $D^\sharp_c(D_X)$ are triangulated categories. Those triangulated categories and certain full triangulated subcategories of them will play major roles in the rest of this book.

By Proposition 1.4.6 and Corollary 1.4.20 we have the following (see also [KS2, Proposition 2.4.12]).

**Proposition 1.5.6.** *Any object of $D^b(D_X)$ (resp. $D^b_{qc}(D_X)$) is represented by a bounded complex of flat $D_X$-modules (resp. locally projective $D_X$-modules belonging to $\mathrm{Mod}_{qc}(D_X)$).*

We note the following result due to Bernstein (see [Bor3]). It will not be used in what then follows and the proof is omitted.

**Theorem 1.5.7.** *The natural functors*

$$D^b(\mathrm{Mod}_{qc}(D_X)) \longrightarrow D^b_{qc}(D_X),$$
$$D^b(\mathrm{Mod}_c(D_X)) \longrightarrow D^b_c(D_X)$$

*give equivalences of categories.*

### Inverse images

Let $f : X \to Y$ be a morphism of smooth algebraic varieties. We can define a functor

$$Lf^* : D^b(D_Y) \to D^b(D_X) \qquad (M^\cdot \longmapsto D_{X\to Y} \otimes^L_{f^{-1}D_Y} f^{-1}M^\cdot)$$

by using a flat resolution of $M^\cdot$.

**Proposition 1.5.8.** *$Lf^*$ sends $D^b_{qc}(D_Y)$ to $D^b_{qc}(D_X)$.*

*Proof.* As a complex of $\mathcal{O}_X$-module (that is, applying the forgetful functor $D^b(D_X) \to D^b(\mathcal{O}_X)$) we have

$$\begin{aligned} D_{X\to Y} \otimes^L_{f^{-1}D_Y} f^{-1}M^\cdot &= (\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}D_Y) \otimes^L_{f^{-1}D_Y} f^{-1}M^\cdot \\ &= (\mathcal{O}_X \otimes^L_{f^{-1}\mathcal{O}_Y} f^{-1}D_Y) \otimes^L_{f^{-1}D_Y} f^{-1}M^\cdot \\ &= \mathcal{O}_X \otimes^L_{f^{-1}\mathcal{O}_Y} f^{-1}M^\cdot, \end{aligned}$$

and hence the assertion follows from Proposition 1.5.9 below. □

For an algebraic variety $Z$ and $\sharp = +, -, b, \emptyset$ we denote by $D^\sharp_{qc}(\mathcal{O}_Z)$ (resp. $D^\sharp_c(\mathcal{O}_Z)$) the full subcategory of $D^\sharp(\mathcal{O}_Z)$ consisting of $F^\cdot \in D^\sharp(\mathcal{O}_Z)$ such that $H^i(F^\cdot) \in \mathrm{Mod}_{qc}(\mathcal{O}_Z)$ (resp. $\mathrm{Mod}_c(\mathcal{O}_Z)$) for any $i$.

**Proposition 1.5.9.** *The functor $\mathcal{O}_X \otimes^L_{f^{-1}\mathcal{O}_Y} f^{-1}(\bullet) : D^-(\mathcal{O}_Y) \to D^-(\mathcal{O}_X)$ sends $D^-_{qc}(\mathcal{O}_Y)$ (resp. $D^-_c(\mathcal{O}_Y)$) to $D^-_{qc}(\mathcal{O}_X)$ (resp. $D^-_c(\mathcal{O}_X)$).*

*Proof.* Let $F^\cdot \in D^-_{qc}(\mathcal{O}_Y)$ (resp. $D^-_c(\mathcal{O}_Y)$). Using the arguments in Proposition 1.4.18 we see that $F^\cdot$ is represented by a complex of locally free $\mathcal{O}_Y$-modules (resp. locally free $\mathcal{O}_Y$-modules of finite ranks). Hence the assertion is obvious. □

**Remark 1.5.10.** Note that $Lf^*D_Y = D_{X\to Y} \otimes^L_{f^{-1}D_Y} f^{-1}D_Y = D_{X\to Y}$. If $f$ is a closed embedding with $\dim X < \dim Y$, then the $D_X$-module $D_{X\to Y}$ is locally free of infinite rank (see Example 1.3.2). We see from this that the functor $Lf^*$ for $f : X \to Y$ does not necessarily send $D^b_c(D_Y)$ to $D^b_c(D_X)$.

We call $Lf^*$ the *inverse image* functor on derived categories of $D$-modules. We will also use the *shifted inverse image* functor

$$f^\dagger = Lf^*[\dim X - \dim Y] : D^b(D_Y) \to D^b(D_X).$$

defined by $f^\dagger M^\cdot = Lf^*M^\cdot[\dim X - \dim Y]$. The shifted one will be more practical in considering the Riemann–Hilbert correspondence.

**Proposition 1.5.11.** *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of smooth algebraic varieties. Then we have*

$$L(g \circ f)^* \simeq Lf^* \circ Lg^*, \qquad (g \circ f)^\dagger \simeq f^\dagger \circ g^\dagger.$$

*Proof.* We have

$$\begin{aligned} &D_{X\to Y} \otimes^L_{f^{-1}D_Y} f^{-1}D_{Y\to Z} \\ &\quad = (\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}D_Y) \otimes^L_{f^{-1}D_Y} f^{-1}(\mathcal{O}_Y \otimes_{g^{-1}\mathcal{O}_Z} g^{-1}D_Z) \\ &\quad = (\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}D_Y) \otimes^L_{f^{-1}D_Y} (f^{-1}\mathcal{O}_Y \otimes_{(g\circ f)^{-1}\mathcal{O}_Z} (g \circ f)^{-1}D_Z) \end{aligned}$$

$$
\begin{aligned}
&= (\mathcal{O}_X \otimes^L_{f^{-1}\mathcal{O}_Y} f^{-1}D_Y) \otimes^L_{f^{-1}D_Y} (f^{-1}\mathcal{O}_Y \otimes^L_{(g\circ f)^{-1}\mathcal{O}_Z} (g\circ f)^{-1}D_Z)\\
&\simeq \mathcal{O}_X \otimes^L_{(g\circ f)^{-1}\mathcal{O}_Z} (g\circ f)^{-1}D_Z\\
&= \mathcal{O}_X \otimes_{(g\circ f)^{-1}\mathcal{O}_Z} (g\circ f)^{-1}D_Z\\
&= D_{X\to Z}.
\end{aligned}
$$

Here we have used the fact the $D$ is a locally free $\mathcal{O}$-module. Hence we obtain isomorphisms

$$D_{X\to Z} \simeq D_{X\to Y} \otimes_{f^{-1}D_Y} f^{-1}D_{Y\to Z} \simeq D_{X\to Y} \otimes^L_{f^{-1}D_Y} f^{-1}D_{Y\to Z}$$

of (complexes of) $(D_X, (g\circ f)^{-1}D_Z)$-bimodules. Therefore, we have

$$
\begin{aligned}
L(g\circ f)^*(M^\cdot) &= D_{X\to Z} \otimes^L_{(g\circ f)^{-1}D_Y} (g\circ f)^{-1}M^\cdot\\
&\simeq (D_{X\to Y} \otimes^L_{f^{-1}D_Y} f^{-1}D_{Y\to Z}) \otimes^L_{f^{-1}g^{-1}D_Y} f^{-1}g^{-1}M^\cdot\\
&= D_{X\to Y} \otimes^L_{f^{-1}D_Y} f^{-1}(D_{Y\to Z} \otimes^L_{g^{-1}D_Y} g^{-1}M^\cdot)\\
&= Lf^*(Lg^*(M^\cdot)).
\end{aligned}
$$

This completes the proof. □

**Example 1.5.12.** Assume that $U$ is an open subset of a smooth algebraic variety $X$. Let $j : U \hookrightarrow X$ be the embedding. Then we have $D_{U\to X} = j^{-1}D_X = D_U$ and hence

$$j^\dagger = Lj^* = j^{-1} \quad \text{(the restriction to } U\text{)}.$$

**Proposition 1.5.13.** *Assume that $f : X \to Y$ is a smooth morphism between smooth algebraic varieties.*

(i) *For $M \in \mathrm{Mod}(D_Y)$ we have $H^i(Lf^*M) = 0$ for $i \neq 0$ (hence we write $f^*$ for $Lf^*$ in this case).*
(ii) *For $M \in \mathrm{Mod}_c(D_Y)$ we have $f^*M \in \mathrm{Mod}_c(D_X)$.*

*Proof.* (i) The assertion follows from the flatness of $\mathcal{O}_X$ over $f^{-1}\mathcal{O}_Y$ and $Lf^*M \simeq \mathcal{O}_X \otimes^L_{f^{-1}\mathcal{O}_Y} f^{-1}M$.

(ii) It is sufficient to show that the canonical morphism $D_X \to D_{X\to Y} = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}D_Y$ $(P \mapsto P(1\otimes 1))$ is surjective. Since the question is local, we may assume that $X$ and $Y$ are affine and admit local coordinates $\{x_i, \partial_{x_i}\}_{i=1,\dots,n}$ and $\{y_i, \partial_{y_i}\}_{i=1,\dots,m}$, respectively, such that

$$\partial_{x_i} \mapsto \begin{cases} 1\otimes\partial_{y_i} & (1 \le i \le m)\\ 0 & (m+1 \le i \le n)\end{cases}$$

under the canonical morphism $\Theta_X \to f^*\Theta_Y = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\Theta_Y$. Then we have

$$D_{X\to Y} = \bigoplus_{r_1,\dots,r_m} \mathcal{O}_X \partial_{y_1}^{r_1} \cdots \partial_{y_m}^{r_m},$$

and the canonical homomorphism $D_X \to D_{X\to Y} = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}D_Y$ is given by $\partial_{x_1}^{r_1}\cdots\partial_{x_n}^{r_n} \mapsto \delta_{r_{m+1}+\cdots+r_n,0}\partial_{y_1}^{r_1}\cdots\partial_{y_m}^{r_m}$, from which the assertion follows. □

Let us give a description of the cohomology sheaves of $Li^*M$ for $M \in \mathrm{Mod}_{qc}(D_X)$. We have a locally free resolution

$$0 \to K_{n-r} \to \cdots \to K_1 \to K_0 \to \mathcal{O}_X \to 0 \tag{1.5.1}$$

of the $i^{-1}\mathcal{O}_Y$-module $\mathcal{O}_X$ (*Koszul resolution*, see, e.g., [Matm, Theorem 43]). In terms of a local coordinate $\{y_i, \partial_{y_i}\}$ in Example 1.3.2, we have

$$K_j = \bigwedge^j \left( \bigoplus_{k=r+1}^{n} i^{-1}\mathcal{O}_Y dy_k \right).$$

The morphism $K_0(= i^{-1}\mathcal{O}_Y) \to \mathcal{O}_X$ is the canonical one, and $K_j \to K_{j-1}$ is given by

$$f dy_{k_1} \wedge \cdots \wedge dy_{k_j} \longmapsto \sum_{p=1}^{j} (-1)^{p+1} y_{k_p} f dy_{k_1} \wedge \cdots \wedge \widehat{dy_{k_p}} \wedge \cdots \wedge dy_{k_j}.$$

If we take another coordinate $\{z_l, \partial_{z_l}\}$ such that $z_{r+1} = \cdots = z_n = 0$ gives a defining equation of $X$, then we can write $y_k = \sum_{l=r+1}^{n} c_{kl} z_l$ $(r+1 \le k \le n)$, and the correspondence $dy_k \mapsto \sum_{l=r+1}^{n} c_{kl} dz_l$ $(r+1 \le k \le n)$ gives the canonical identification. From this resolution we obtain a locally free resolution of the right $i^{-1}D_Y$-module $D_{X\to Y}$:

$$0 \to K_{n-r} \otimes_{i^{-1}\mathcal{O}_Y} i^{-1}D_Y \to \cdots \to K_0 \otimes_{i^{-1}\mathcal{O}_Y} i^{-1}D_Y \to D_{X\to Y} \to 0.$$

Hence $Li^*M$ is represented by the complex

$$\cdots \to 0 \to K_{n-r} \otimes_{i^{-1}\mathcal{O}_Y} i^{-1}M \to \cdots \to K_0 \otimes_{i^{-1}\mathcal{O}_Y} i^{-1}M \to 0 \to \cdots.$$

In terms of the local coordinate the action of $D_X$ on the cohomology sheaf $H^j(K. \otimes_{i^{-1}\mathcal{O}_Y} i^{-1}M)$ is described as follows. Let $D'$ be the subalgebra of $D_Y$ generated by $\mathcal{O}_Y$ and $\partial_{y_1}, \ldots, \partial_{y_r}$. Note that the homomorphism $i^{-1}D' \to D_X$ induced by the identification $\mathcal{O}_X \otimes_{i^{-1}\mathcal{O}_Y} i^{-1}D' \simeq D_X$ is a ring homomorphism. Hence we can regard $D_{X\to Y}$ as a $(i^{-1}D', i^{-1}D_Y)$-bimodule. Moreover, our resolution of $D_{X\to Y}$ is that of $(i^{-1}D', i^{-1}D_Y)$-bimodules, where $i^{-1}D'$ acts on $K_j \otimes_{i^{-1}\mathcal{O}_Y} i^{-1}D_Y$ via the left multiplication on $i^{-1}D_Y$. Hence $K. \otimes_{i^{-1}\mathcal{O}_Y} i^{-1}M$ is a complex of $i^{-1}D'$-modules. Then the actions of $i^{-1}D'$ on $H^j(K. \otimes_{i^{-1}\mathcal{O}_Y} i^{-1}M)$ factors through $D_X$ and this gives the desired $D_X$-module structure.

**Proposition 1.5.14.** *Let $i : X \to Y$ be a closed embedding of smooth algebraic varieties. Set $d = \mathrm{codim}_Y(X)$.*

(i) *For $M \in \mathrm{Mod}(D_Y)$ we have $H^j(Li^*M) = 0$ unless $-d \le j \le 0$.*

(ii) *For $M^{\cdot} \in D^{+}(D_Y)$ we have a canonical isomorphism*

$$Li^{*}M^{\cdot} \simeq R\mathcal{H}om_{i^{-1}D_Y}(D_{Y\leftarrow X}, i^{-1}M^{\cdot})[d]$$

*in $D^{b}(D_X)$, where the left $D_X$-module structure on the right-hand side is given by the right action of $D_X$ on $D_{Y\leftarrow X}$.*

*Proof.* The statement (i) is already shown. In order to show (ii) it is sufficient to show the isomorphism

$$R\mathcal{H}om_{i^{-1}D_Y}(D_{Y\leftarrow X}, i^{-1}D_Y) \simeq D_{X\to Y}[-d]. \tag{1.5.2}$$

Indeed, from (1.5.2) we obtain

$$\begin{aligned} Li^{*}M &= D_{X\to Y} \otimes^{L}_{i^{-1}D_Y} i^{-1}M \\ &\simeq R\mathcal{H}om_{i^{-1}D_Y}(D_{Y\leftarrow X}, i^{-1}D_Y) \otimes^{L}_{i^{-1}D_Y} i^{-1}M[d] \\ &\simeq R\mathcal{H}om_{i^{-1}D_Y}(D_{Y\leftarrow X}, i^{-1}M)[d]. \end{aligned}$$

Here the last equality is shown similarly to Lemma 2.6.13 below. Note that (1.5.2) is equivalent to

$$R\mathcal{H}om_{i^{-1}D_Y^{\mathrm{op}}}(D_{X\to Y}, i^{-1}D_Y) \simeq D_{Y\leftarrow X}[-d] \tag{1.5.3}$$

by the side-changing operation. Let us show (1.5.3). We have

$$\begin{aligned} &R\mathcal{H}om_{i^{-1}D_Y^{\mathrm{op}}}(D_{X\to Y}, i^{-1}D_Y) \\ &\quad\simeq R\mathcal{H}om_{i^{-1}D_Y^{\mathrm{op}}}(\mathcal{O}_X \otimes_{i^{-1}\mathcal{O}_Y} i^{-1}D_Y, i^{-1}D_Y) \\ &\quad\simeq R\mathcal{H}om_{i^{-1}\mathcal{O}_Y}(\mathcal{O}_X, i^{-1}D_Y) \\ &\quad\simeq i^{-1}D_Y \otimes_{i^{-1}\mathcal{O}_Y} R\mathcal{H}om_{i^{-1}\mathcal{O}_Y}(\mathcal{O}_X, i^{-1}\mathcal{O}_Y). \end{aligned}$$

By using the Koszul resolution (1.5.1) of the $i^{-1}\mathcal{O}_Y$-module $\mathcal{O}_X$ we see that $R\mathcal{H}om_{i^{-1}\mathcal{O}_Y}(\mathcal{O}_X, i^{-1}\mathcal{O}_Y)$ is represented by the complex

$$[K_0^{*} \to K_1^{*} \to \cdots \to K_d^{*}],$$

where $K_j^{*} = \mathcal{H}om_{i^{-1}\mathcal{O}_Y}(K_j, i^{-1}\mathcal{O}_Y)$. Note that $K_d$ is a locally free $i^{-1}\mathcal{O}_Y$-module of rank one and we have a canonical perfect paring $K_j \otimes_{i^{-1}\mathcal{O}_Y} K_{d-j} \to K_d$ for each $j$. Hence we have $K_j^{*} \simeq K_{d-j} \otimes_{i^{-1}\mathcal{O}_Y} K_d^{*}$. Then we obtain

$$\begin{aligned} [K_0^{*} \to K_1^{*} \to \cdots \to K_d^{*}] &\simeq [K_d \to K_{d-1} \to \cdots \to K_0] \otimes_{i^{-1}\mathcal{O}_Y} K_d^{*} \\ &\simeq \mathcal{O}_X \otimes_{i^{-1}\mathcal{O}_Y} K_d^{*}[-d] \simeq i^{-1}\Omega_Y^{\otimes -1} \otimes_{i^{-1}\mathcal{O}_Y} \Omega_X[-d]. \end{aligned}$$

Therefore, we have

$$\begin{aligned} R\mathcal{H}om_{i^{-1}D_Y^{\mathrm{op}}}(D_{X\to Y}, i^{-1}D_Y) &\simeq i^{-1}D_Y \otimes_{i^{-1}\mathcal{O}_Y} i^{-1}\Omega_Y^{\otimes -1} \otimes_{i^{-1}\mathcal{O}_Y} \Omega_X[-d] \\ &\simeq D_{Y\leftarrow X}[-d] \end{aligned}$$

by Lemma 1.3.4. □

**Definition 1.5.15.** For a closed embedding $i : X \to Y$ of smooth algebraic varieties we define a left exact functor

$$i^\natural : \mathrm{Mod}(D_Y) \to \mathrm{Mod}(D_X)$$

by $i^\natural M = \mathcal{H}om_{i^{-1}D_Y}(D_{Y\leftarrow X}, i^{-1}M)$.

**Proposition 1.5.16.** *Let $i : X \to Y$ be a closed embedding. Then we have*

$$i^\dagger M^\cdot \simeq R\mathcal{H}om_{i^{-1}D_Y}(D_{Y\leftarrow X}, i^{-1}M^\cdot) \simeq Ri^\natural M^\cdot$$

*for any $M^\cdot \in D^+(D_Y)$.*

*Proof.* The first equality is already shown in Proposition 1.5.14. Let us show the second one.

We first show that

$$i^\natural M \simeq \mathcal{H}om_{i^{-1}D_Y}(D_{Y\leftarrow X}, i^{-1}\Gamma_X(M)) \tag{1.5.4}$$

for $M \in \mathrm{Mod}(D_Y)$, where $\Gamma_X(M)$ denotes the subsheaf of $M$ consisting of sections whose support is contained in $X$. For this it is sufficient to show that $\psi(s) \in i^{-1}\Gamma_X(M)$ for any $\psi \in \mathcal{H}om_{i^{-1}D_Y}(D_{Y\leftarrow X}, i^{-1}M)$ and $s \in D_{Y\leftarrow X}$. Since the question is local, we can take a local coordinate as in Example 1.3.5. Then we have $D_{Y\leftarrow X} \simeq \mathbb{C}[\partial_{y_{r+1}}, \dots, \partial_{y_n}] \otimes_{\mathbb{C}} D_X$. Since the $i^{-1}D_Y$-module $\mathbb{C}[\partial_{y_{r+1}}, \dots, \partial_{y_n}] \otimes_{\mathbb{C}} D_X$ is generated by $1 \otimes 1$, we may assume that $s = 1 \otimes 1$. Let $\mathcal{J} \subset \mathcal{O}_Y$ be the defining ideal of $X$. By $(i^{-1}\mathcal{J})s = 0$ we have $(i^{-1}\mathcal{J})\psi(s) = 0$. It implies that $\psi(s) \in i^{-1}\Gamma_X(M)$. The assertion (1.5.4) is shown.

We next show that

$$Ri^\natural M^\cdot \simeq R\mathcal{H}om_{i^{-1}D_Y}(D_{Y\leftarrow X}, i^{-1}R\Gamma_X(M^\cdot)) \tag{1.5.5}$$

for $M^\cdot \in D^+(D_Y)$. For this it is sufficient to show that if $I$ is an injective $D_Y$-module, then $i^{-1}\Gamma_X(I)$ is an injective $i^{-1}D_Y$-module. This follows from

$$\begin{aligned}\mathrm{Hom}_{i^{-1}D_Y}(K, i^{-1}\Gamma_X(I)) &\simeq \mathrm{Hom}_{i^{-1}D_Y}(i^{-1}i_*K, i^{-1}\Gamma_X(I))\\ &\simeq \mathrm{Hom}_{D_Y}(i_*K, i_*i^{-1}\Gamma_X(I))\\ &\simeq \mathrm{Hom}_{D_Y}(i_*K, \Gamma_X(I)) \simeq \mathrm{Hom}_{D_Y}(i_*K, I)\end{aligned}$$

for any $i^{-1}D_Y$-module $K$.

It remains to show that the canonical morphism

$$R\mathcal{H}om_{i^{-1}D_Y}(D_{Y\leftarrow X}, i^{-1}R\Gamma_X(M^\cdot)) \to R\mathcal{H}om_{i^{-1}D_Y}(D_{Y\leftarrow X}, i^{-1}M^\cdot)$$

is an isomorphism. Let $j : Y \setminus X \to Y$ be the complementary open embedding. By the distinguished triangle

$$R\Gamma_X(M^\cdot) \longrightarrow M^\cdot \longrightarrow Rj_*j^{-1}M^\cdot \xrightarrow{+1}$$

(see Proposition 1.7.1 below) it is sufficient to show that

$$R\mathcal{H}om_{i^{-1}D_Y}(D_{Y\leftarrow X}, i^{-1}j_*j^{-1}M^\cdot)(\simeq i^\dagger Rj_*j^{-1}M^\cdot) = 0.$$

This follows from Lemma 1.5.17 below. □

**Lemma 1.5.17.** *Let $i : X \to Y$ be a closed embedding of algebraic varieties. Set $U = Y \setminus X$ and denote by $j : U \to X$ the complementary open embedding. Then for any $K^{\cdot} \in D^b(\mathcal{O}_U)$ we have $\mathcal{O}_X \otimes^L_{i^{-1}\mathcal{O}_Y} i^{-1}Rj_*K^{\cdot} = 0$.*

*Proof.* We have

$$i_*(\mathcal{O}_X \otimes^L_{i^{-1}\mathcal{O}_Y} i^{-1}Rj_*K) = i_*\mathcal{O}_X \otimes^L_{\mathcal{O}_Y} Rj_*K = Rj_*(j^{-1}i_*\mathcal{O}_X \otimes^L_{\mathcal{O}_{Y\setminus X}} K) = 0.$$

Here we have used the projection formula for $\mathcal{O}$-modules (see, e.g., [Ha1, II, Proposition 5.6]). □

**Tensor products**

The bifunctor

$$(\bullet) \otimes_{\mathcal{O}_X} (\bullet) : \mathrm{Mod}(D_X) \times \mathrm{Mod}(D_X) \to \mathrm{Mod}(D_X) \quad ((M, N) \mapsto M \otimes_{\mathcal{O}_X} N)$$

is right exact with respect to both factors, and we can define its left derived functor as

$$(\bullet) \otimes^L_{\mathcal{O}_X} (\bullet) : D^b(D_X) \times D^b(D_X) \to D^b(D_X) \quad ((M^{\cdot}, N^{\cdot}) \mapsto M^{\cdot} \otimes^L_{\mathcal{O}_X} N^{\cdot})$$

by using a flat resolution of $M^{\cdot}$ or $N^{\cdot}$. Since a flat $D_X$-module is flat over $\mathcal{O}_X$, we have a commutative diagram

$$\begin{array}{ccc} D^b(D_X) \times D^b(D_X) & \xrightarrow{(\bullet)\otimes^L_{\mathcal{O}_X}(\bullet)} & D^b(D_X) \\ \downarrow & & \downarrow \\ D^b(\mathcal{O}_X) \times D^b(\mathcal{O}_X) & \xrightarrow{(\bullet)\otimes^L_{\mathcal{O}_X}(\bullet)} & D^b(\mathcal{O}_X), \end{array}$$

where vertical arrows are forgetful functors. In particular, the functor $(\bullet) \otimes^L_{\mathcal{O}_X} (\bullet)$ sends $D^b_{qc}(D_X) \times D^b_{qc}(D_X)$ to $D^b_{qc}(D_X)$.

Let $X$ and $Y$ be smooth algebraic varieties and let $p_1 : X \times Y \to X$, $p_2 : X \times Y \to Y$ be the first and the second projections, respectively. For $M \in \mathrm{Mod}(\mathcal{O}_X)$ and $N \in \mathrm{Mod}(\mathcal{O}_Y)$ we set

$$M \boxtimes N := \mathcal{O}_{X\times Y} \otimes_{p_1^{-1}\mathcal{O}_X \otimes_{\mathbb{C}} p_2^{-1}\mathcal{O}_Y} (p_1^{-1}M \otimes_{\mathbb{C}} p_2^{-1}N) \in \mathrm{Mod}(\mathcal{O}_{X\times Y}).$$

This gives a bifunctor

$$(\bullet) \boxtimes (\bullet) : \mathrm{Mod}(\mathcal{O}_X) \times \mathrm{Mod}(\mathcal{O}_Y) \longrightarrow \mathrm{Mod}(\mathcal{O}_{X\times Y}).$$

Since the functor $(\bullet)\boxtimes(\bullet)$ is exact with respect to both factors, it extends immediately to a functor

$$(\bullet) \boxtimes (\bullet) : D^b(\mathcal{O}_X) \times D^b(\mathcal{O}_Y) \longrightarrow D^b(\mathcal{O}_{X\times Y})$$

for derived categories.

Let $M \in \mathrm{Mod}(D_X)$ and $N \in \mathrm{Mod}(D_Y)$. Then the $D_{X\times Y}$-module

$$D_{X\times Y} \otimes_{p_1^{-1}D_X \otimes_{\mathbb{C}} p_2^{-1}D_Y} (p_1^{-1}M \otimes_{\mathbb{C}} p_2^{-1}N)$$

is isomorphic as an $\mathcal{O}_{X\times Y}$-module to $M \boxtimes N$ by

$$D_{X\times Y} \simeq \mathcal{O}_{X\times Y} \otimes_{p_1^{-1}\mathcal{O}_X \otimes_{\mathbb{C}} p_2^{-1}\mathcal{O}_Y} p_1^{-1}D_X \otimes_{\mathbb{C}} p_2^{-1}D_Y.$$

This $D_{X\times Y}$-module is again denoted by $M \boxtimes N$, and is called the *exterior tensor product* of $M$ and $N$. The bifunctor

$$(\bullet) \boxtimes (\bullet) : \mathrm{Mod}(D_X) \times \mathrm{Mod}(D_Y) \longrightarrow \mathrm{Mod}(D_{X\times Y})$$

is exact with respect to both factors, and it extends to a functor

$$(\bullet) \boxtimes (\bullet) : D^b(D_X) \times D^b(D_Y) \longrightarrow D^b(D_{X\times Y})$$

for derived categories such that the following diagram is commutative:

$$\begin{array}{ccc} D^b(D_X) \otimes D^b(D_Y) & \xrightarrow{(\bullet)\boxtimes(\bullet)} & D^b(D_{X\times Y}) \\ \downarrow & & \downarrow \\ D^b(\mathcal{O}_X) \otimes D^b(\mathcal{O}_Y) & \xrightarrow{(\bullet)\boxtimes(\bullet)} & D^b(\mathcal{O}_{X\times Y}), \end{array}$$

where vertical arrows are forgetful functors. It is easily seen that the functor $(\bullet) \boxtimes (\bullet)$ sends $D^b_{qc}(D_X) \times D^b_{qc}(D_Y)$ (resp. $D^b_c(D_X) \times D^b_c(D_Y)$) to $D^b_{qc}(D_{X\times Y})$ (resp. $D^b_c(D_{X\times Y})$). We also note that

$$p_1^*M \simeq M \boxtimes \mathcal{O}_Y, \qquad p_2^*N \simeq \mathcal{O}_X \boxtimes N.$$

Let $X$ be a smooth algebraic variety and let $\Delta_X : X \to X \times X$ $(x \mapsto (x, x))$ be the *diagonal embedding*. For $M, N \in \mathrm{Mod}(D_X)$ we easily see that $M \otimes_{\mathcal{O}_X} N$ is isomorphic to $\Delta_X^*(M \boxtimes N)$ as a $D_X$-module. Moreover, if $P_1$ and $P_2$ are a flat $D_{Y_1}$-module and a flat $D_{Y_2}$-module, respectively, then $P_1 \boxtimes P_2$ is a flat $D_{Y_1\times Y_2}$-module. Hence for $M^{\cdot}, N^{\cdot} \in D^b(D_X)$ we have a canonical isomorphism

$$M^{\cdot} \otimes^L_{\mathcal{O}_X} N^{\cdot} \simeq L\Delta_X^*(M^{\cdot} \boxtimes N^{\cdot})$$

in $D^b(D_X)$.

**Proposition 1.5.18.**

(i) *Let $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ be morphisms of smooth algebraic varieties. Then for $M_1^{\cdot} \in D^b(D_{Y_1})$, $M_2^{\cdot} \in D^b(D_{Y_2})$, we have*

$$L(f_1 \times f_2)^*(M_1^{\cdot} \boxtimes M_2^{\cdot}) \simeq Lf_1^*M_1^{\cdot} \boxtimes Lf_2^*M_2^{\cdot}.$$

(ii) *Let $f : X \to Y$ be a morphism of smooth algebraic varieties. Then for $M^{\cdot}, N^{\cdot} \in D^b(D_Y)$, we have*

$$Lf^*(M^{\cdot} \otimes^L_{\mathcal{O}_Y} N^{\cdot}) \simeq Lf^*M^{\cdot} \otimes^L_{\mathcal{O}_X} Lf^*N^{\cdot}.$$

*Proof.* The statement (i) follows from $(f_1 \times f_2)^*(M_1 \boxtimes M_2) \simeq f_1^*M_1 \boxtimes f_2^*M_2$ for $M_1 \in \mathrm{Mod}(D_{Y_1})$, $M_2 \in \mathrm{Mod}(D_{Y_2})$. The statement (ii) follows from (i) as follows:

$$\begin{aligned} Lf^*(M^{\cdot} \otimes^L_{\mathcal{O}_Y} N^{\cdot}) &\simeq Lf^*L\Delta_Y^*(M^{\cdot} \boxtimes N^{\cdot}) \simeq L\Delta_X^*L(f \times f)^*(M^{\cdot} \boxtimes N^{\cdot}) \\ &\simeq L\Delta_X^*(Lf^*M^{\cdot} \boxtimes Lf^*N^{\cdot}) \simeq Lf^*M^{\cdot} \otimes^L_{\mathcal{O}_X} Lf^*N^{\cdot}. \end{aligned}$$

□

**Proposition 1.5.19.** *Let $M^{\cdot}, N^{\cdot} \in D^b(D_X)$ and $L^{\cdot} \in D^b(D_X^{\mathrm{op}})$. Then we have isomorphisms*

$$(L^{\cdot} \otimes^L_{\mathcal{O}_X} N^{\cdot}) \otimes^L_{D_X} M^{\cdot} \simeq L^{\cdot} \otimes^L_{D_X} (M^{\cdot} \otimes^L_{\mathcal{O}_X} N^{\cdot}) \simeq (L^{\cdot} \otimes^L_{\mathcal{O}_X} M^{\cdot}) \otimes^L_{D_X} N^{\cdot}$$

*of $\mathbb{C}_X$-modules.*

*Proof.* By taking flat resolutions of $M^{\cdot}, N^{\cdot}, L^{\cdot}$ we may assume from the beginning that $M^{\cdot} = M, N^{\cdot} = N \in \mathrm{Mod}(D_X)$ and $L^{\cdot} = L \in \mathrm{Mod}(D_X^{\mathrm{op}})$. Hence the assertion follows from Lemma 1.2.11. □

### Direct images

Let $f : X \to Y$ be a morphism of smooth algebraic varieties. We can define functors

$$\begin{aligned} D^b(D_X) \ni M^{\cdot} &\longmapsto D_{Y \leftarrow X} \otimes^L_{D_X} M^{\cdot} \in D^b(f^{-1}D_Y), \\ D^b(f^{-1}D_Y) \ni N^{\cdot} &\longmapsto Rf_*(N^{\cdot}) \in D^b(D_Y) \end{aligned}$$

by using a flat resolution of $M^{\cdot}$ and an injective resolution of $N^{\cdot}$. Therefore, we obtain a functor

$$\int_f : D^b(D_X) \to D^b(D_Y)$$

given by

$$\int_f M^{\cdot} = Rf_*(D_{Y \leftarrow X} \otimes^L_{D_X} M^{\cdot}) \qquad (M^{\cdot} \in D^b(D_X)).$$

It is true that $\int_f$ sends $D^b_{qc}(D_X)$ to $D^b_{qc}(D_Y)$; however, unlike the case of inverse image functors it does not immediately follow from Proposition 1.5.20 below. We will prove this non-trivial fact later (Proposition 1.5.29).

**Proposition 1.5.20.** *The functor $Rf_* : D^b(\mathcal{O}_X) \to D^b(\mathcal{O}_Y)$ sends $D^b_{qc}(\mathcal{O}_X)$ to $D^b_{qc}(\mathcal{O}_Y)$. If $f$ is proper, it sends $D^b_c(\mathcal{O}_X)$ to $D^b_c(\mathcal{O}_Y)$.*

(See, e.g., [Ha2, III, Corollary 8.6, Theorem 8.8].)
For an integer $k$ we set

$$\int_f^k M^{\cdot} = H^k\Big(\int_f M^{\cdot}\Big).$$

Note that we also have a functor $\int_f : D^b(D_X^{\mathrm{op}}) \to D^b(D_Y^{\mathrm{op}})$ defined by

$$\int_f M^{\cdot} = Rf_*(M^{\cdot} \otimes^L_{D_X} D_{X\to Y}) \qquad (M^{\cdot} \in D^b(D_X^{\mathrm{op}})).$$

By an argument similar to those in Section 1.3 we have a commutative diagram

$$\begin{array}{ccc} D^b(D_X) & \xrightarrow{\int_f} & D^b(D_Y) \\ {\scriptstyle \Omega_X\otimes_{\mathcal{O}_X}(\bullet)}\big\downarrow\wr & & \wr\big\downarrow{\scriptstyle \Omega_Y\otimes_{\mathcal{O}_Y}(\bullet)} \\ D^b(D_X^{\mathrm{op}}) & \xrightarrow[\int_f]{} & D^b(D_Y^{\mathrm{op}}). \end{array}$$

**Proposition 1.5.21.** *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of smooth algebraic varieties. Then we have*

$$\int_{g\circ f} = \int_g \int_f .$$

*Proof.* Similar to the proof of Proposition 1.5.11, we have isomorphisms

$$D_{Z\leftarrow X} \simeq f^{-1}D_{Z\leftarrow Y} \otimes_{f^{-1}D_Y} D_{Y\leftarrow X} \simeq f^{-1}D_{Z\leftarrow Y} \otimes^L_{f^{-1}D_Y} D_{Y\leftarrow X}$$

of (complexes of) $((g\circ f)^{-1}D_Z, D_X)$-bimodules.

For $M^{\cdot} \in D^b(D_X)$ we have

$$\int_g \int_f M^{\cdot} = Rg_*(D_{Z\leftarrow Y} \otimes^L_{D_Y} Rf_*(D_{Y\leftarrow X} \otimes^L_{D_X} M^{\cdot}))$$

by definition. We claim that the canonical morphism

$$D_{Z\leftarrow Y} \otimes^L_{D_Y} Rf_*(D_{Y\leftarrow X} \otimes^L_{D_X} M^{\cdot}) \to Rf_*(f^{-1}D_{Z\leftarrow Y} \otimes^L_{f^{-1}D_Y} (D_{Y\leftarrow X} \otimes^L_{D_X} M^{\cdot}))$$

is an isomorphism. In fact, we show that the canonical morphism

$$F^{\cdot} \otimes^L_{D_Y} Rf_*(G^{\cdot}) \to Rf_*(f^{-1}F^{\cdot} \otimes^L_{f^{-1}D_Y} G^{\cdot})$$

is an isomorphism for any $F^{\cdot} \in D^-_{qc}(D_Y^{\mathrm{op}})$, $G^{\cdot} \in D^b(f^{-1}D_Y)$. Since the question is local, we may assume that $Y$ is affine. Then by Remark 1.4.5 we can replace $F^{\cdot}$ with a complex of free right $D_Y$-modules belonging to $D^-_{qc}(D_Y^{\mathrm{op}})$. Hence we have only to show our claim for $F^{\cdot} = D_Y^{\oplus I}$, where $I$ is a (possibly infinite) index set. Then we

have $F^{\cdot} \otimes^L_{D_Y} Rf_*(G^{\cdot}) \simeq Rf_*(G^{\cdot})^{\oplus I}$ and $Rf_*(f^{-1}F^{\cdot} \otimes^L_{f^{-1}D_Y} G^{\cdot}) \simeq Rf_*((G^{\cdot})^{\oplus I})$. Therefore, the claim follows from Proposition 1.5.4 (for $R = \mathbb{Z}_Y$). Hence we have

$$\begin{aligned}\int_g \int_f M^{\cdot} &\simeq Rg_* Rf_*(f^{-1}D_{Z\leftarrow Y} \otimes^L_{f^{-1}D_Y} (D_{Y\leftarrow X} \otimes^L_{D_X} M^{\cdot})) \\ &\simeq R(g\circ f)_*((f^{-1}D_{Z\leftarrow Y} \otimes^L_{f^{-1}D_Y} D_{Y\leftarrow X}) \otimes^L_{D_X} M^{\cdot}) \\ &\simeq R(g\circ f)_*(D_{Z\leftarrow X} \otimes^L_{D_X} M^{\cdot}) \\ &= \int_{g\circ f} M^{\cdot}.\end{aligned}$$

This completes the proof. □

**Example 1.5.22.** Assume that $U$ is an open subset of a smooth algebraic variety $X$. Let $j : U \hookrightarrow X$ be the embedding. Then we have $D_{X\leftarrow U} = j^{-1}D_X = D_U$ and

$$\int_j = Rj_*$$

(note that $\int_j M = Rj_*M$ may have non-trivial higher cohomology groups $R^i j_*M$, $i > 0$).

**Example 1.5.23.** Assume that $i : X \to Y$ is a closed embedding of smooth algebraic varieties. Take a local coordinate $\{y_k, \partial_{y_k}\}_{1\le k\le n}$ of $Y$ such that $y_{r+1} = \cdots = y_n = 0$ gives a defining equation of $X$. By Example 1.3.5 we have the following local description of $\int_i M$ for $M \in \mathrm{Mod}(D_X)$:

$$\int_i^k M = 0 \ (k \neq 0), \quad \int_i^0 M \simeq \mathbb{C}[\partial_{y_{r+1}}, \dots, \partial_{y_n}] \otimes_{\mathbb{C}} i_*M.$$

The action of $D_Y$ on $\mathbb{C}[\partial_{y_{r+1}}, \dots, \partial_{y_n}] \otimes_{\mathbb{C}} i_*M$ is given by the following. The action of $\partial_{y_k}$ for $k > r$ is given by the multiplication on the first factor $\mathbb{C}[\partial_{y_{r+1}}, \dots, \partial_{y_n}]$. Hence it remains to describe the action of $\varphi \in \mathcal{O}_Y$ and $\partial_{y_k}$ for $k \le r$ on $1 \otimes i_*M \subset \mathbb{C}[\partial_{y_{r+1}}, \dots, \partial_{y_n}] \otimes_{\mathbb{C}} i_*M$. It is given by

$$\begin{aligned}\varphi(1 \otimes m) &= 1 \otimes (\varphi|_X)m \qquad (\varphi \in \mathcal{O}_Y), \\ \partial_{y_k}(1 \otimes m) &= 1 \otimes \partial_{x_k} m \qquad (1 \le k \le r).\end{aligned}$$

The above local consideration gives the following.

**Proposition 1.5.24.** *Let $i : X \to Y$ be a closed embedding of smooth algebraic varieties.*

(i) *For $M \in \mathrm{Mod}(D_X)$ we have $\int_i^k M = 0$ for $k \neq 0$. In particular, $\int_i^0 : \mathrm{Mod}(D_X) \to \mathrm{Mod}(D_Y)$ is an exact functor.*
(ii) *$\int_i^0$ sends $\mathrm{Mod}_{qc}(D_X)$ to $\mathrm{Mod}_{qc}(D_Y)$.*

**Proposition 1.5.25.** *Let $i : X \to Y$ be a closed embedding of smooth algebraic varieties.*

(i) *There exists a functorial isomorphism*

$$R\mathcal{H}om_{D_Y}\left(\int_i M^{\cdot}, N^{\cdot}\right) \simeq i_* R\mathcal{H}om_{D_X}(M^{\cdot}, Ri^{\natural}N^{\cdot})$$
$$(M^{\cdot} \in D^-(D_X),\ N^{\cdot} \in D^+(D_Y)).$$

(ii) *The functor $Ri^{\natural} : D^b(D_Y) \to D^b(D_X)$ is right adjoint to $\int_i : D^b(D_X) \to D^b(D_Y)$.*

*Proof.* The statement (ii) follows from (i) by taking $H^0(R\Gamma(Y, \bullet))$ (note that $H^0(R\,\mathrm{Hom}_{D_Y}(K^{\cdot}, L^{\cdot})) \simeq \mathrm{Hom}_{D^b(D_Y)}(K^{\cdot}, L^{\cdot})$). Let us show (i). Note that for $M \in \mathrm{Mod}(D_X)$, $N \in \mathrm{Mod}(D_Y)$ there exists a canonical isomorphism

$$\mathcal{H}om_{D_X}(M, \mathcal{H}om_{i^{-1}D_Y}(D_{Y\leftarrow X}, i^{-1}N)) \simeq \mathcal{H}om_{i^{-1}D_Y}(D_{Y\leftarrow X} \otimes_{D_X} M, i^{-1}N)$$
$$(\varphi \longleftrightarrow \psi)$$

given by

$$(\varphi(s))(R) = \psi(R \otimes s) \qquad (s \in M,\ R \in D_{Y\leftarrow X}).$$

From this we obtain

$$R\mathcal{H}om_{D_X}(M^{\cdot}, R\mathcal{H}om_{i^{-1}D_Y}(D_{Y\leftarrow X}, i^{-1}N^{\cdot}))$$
$$\simeq R\mathcal{H}om_{i^{-1}D_Y}(D_{Y\leftarrow X} \otimes^L_{D_X} M^{\cdot}, i^{-1}N^{\cdot})$$

for $M^{\cdot} \in D^-(D_X)$, $N^{\cdot} \in D^+(D_Y)$ (see the proof of [KS2, Proposition 2.6.3]). Therefore, we have

$$\begin{aligned}
&R\mathcal{H}om_{D_Y}\left(\int_i M^{\cdot}, N^{\cdot}\right)\\
&\quad\simeq R\mathcal{H}om_{D_Y}(i_*(D_{Y\leftarrow X} \otimes^L_{D_X} M^{\cdot}), N^{\cdot})\\
&\quad\simeq R\mathcal{H}om_{D_Y}(i_*(D_{Y\leftarrow X} \otimes^L_{D_X} M^{\cdot}), R\Gamma_X(N^{\cdot}))\\
&\quad\simeq R\mathcal{H}om_{D_Y}(i_*(D_{Y\leftarrow X} \otimes^L_{D_X} M^{\cdot}), i_*i^{-1}R\Gamma_X(N^{\cdot}))\\
&\quad\simeq i_*R\mathcal{H}om_{D_Y}(i^{-1}i_*(D_{Y\leftarrow X} \otimes^L_{D_X} M^{\cdot}), i^{-1}R\Gamma_X(N^{\cdot}))\\
&\quad\simeq i_*R\mathcal{H}om_{i^{-1}D_Y}(D_{Y\leftarrow X} \otimes^L_{D_X} M^{\cdot}, i^{-1}R\Gamma_X(N^{\cdot}))\\
&\quad\simeq i_*R\mathcal{H}om_{D_X}(M^{\cdot}, R\mathcal{H}om_{i^{-1}D_Y}(D_{Y\leftarrow X}, i^{-1}R\Gamma_X(N^{\cdot})))\\
&\quad\simeq i_*R\mathcal{H}om_{D_X}(M^{\cdot}, Ri^{\natural}N^{\cdot}).
\end{aligned}$$

Here the last equality follows from Proposition 1.5.16 and its proof. □

**Corollary 1.5.26.** *Let $i : X \to Y$ be a closed embedding of smooth algebraic varieties.*

(i) *There exists a functorial isomorphism*

$$\mathcal{H}om_{D_Y}\left(\int_i^0 M, N\right) \simeq i_*\mathcal{H}om_{D_X}(M, i^\natural N)$$
$$(M \in \mathrm{Mod}(D_X), N \in \mathrm{Mod}(D_Y)).$$

(ii) *The functor* $i^\natural : \mathrm{Mod}(D_Y) \to \mathrm{Mod}(D_X)$ *is right adjoint to* $\int_i^0 : \mathrm{Mod}(D_X) \to \mathrm{Mod}(D_Y)$.

In order to analyze the direct images for projections $Y \times Z \to Z$ we need the following.

**Lemma 1.5.27.** *We have the following locally free resolutions of the left* $D_X$*-module* $\mathcal{O}_X$ *and the right* $D_X$*-module* $\Omega_X$*:*

$$0 \to D_X \otimes_{\mathcal{O}_X} \overset{n}{\bigwedge} \Theta_X \to \cdots \to D_X \otimes_{\mathcal{O}_X} \overset{0}{\bigwedge} \Theta_X \to \mathcal{O}_X \to 0, \tag{1.5.6}$$

$$0 \to \Omega_X^0 \otimes_{\mathcal{O}_X} D_X \to \cdots \to \Omega_X^n \otimes_{\mathcal{O}_X} D_X \to \Omega_X \to 0, \tag{1.5.7}$$

*where* $n = \dim X$*, and* $\Omega_X^k = \bigwedge^k \Omega_X^1$ *for* $0 \le k \le n$*. Here*

$$D_X \otimes_{\mathcal{O}_X} \overset{0}{\bigwedge} \Theta_X (= D_X) \to \mathcal{O}_X$$

*and*

$$\Omega_X^n \otimes_{\mathcal{O}_X} D_X (= \Omega_X \otimes_{\mathcal{O}_X} D_X) \to \Omega_X$$

*are given by* $P \mapsto P(1)$ *and* $\omega \otimes P \mapsto \omega P$*, respectively, and*

$$d : D_X \otimes_{\mathcal{O}_X} \overset{k}{\bigwedge} \Theta_X \to D_X \otimes_{\mathcal{O}_X} \overset{k-1}{\bigwedge} \Theta_X$$

*and*

$$d : \quad \Omega_X^k \otimes_{\mathcal{O}_X} D_X \to \Omega_X^{k+1} \otimes_{\mathcal{O}_X} D_X$$

*are given by*

$$\begin{aligned} d(P \otimes \theta_1 \wedge \cdots \wedge \theta_k) &= \sum_i (-1)^{i+1} P\theta_i \otimes \theta_1 \wedge \cdots \wedge \widehat{\theta_i} \cdots \wedge \theta_k \\ &\quad + \sum_{i<j} (-1)^{i+j} P \otimes [\theta_i, \theta_j] \wedge \theta_1 \wedge \cdots \wedge \widehat{\theta_i} \cdots \wedge \widehat{\theta_j} \cdots \wedge \theta_k, \\ d(\omega \otimes P) &= d\omega \otimes P + \sum_i dz_i \wedge \omega \otimes \partial_i P, \end{aligned}$$

*respectively, where* $\{z_i, \partial_i\}$ *is a local coordinate of* $X$ (*we call* (1.5.6) *the* **Spencer resolution** *of* $\mathcal{O}_X$).

*Proof.* The assertion for $\Omega_X$ follows from the one for $\mathcal{O}_X$ using the side-changing operation. In order to show that the complex

$$N^{\cdot} = \left[ D_X \otimes_{\mathcal{O}_X} \overset{n}{\bigwedge} \Theta_X \to \cdots \to D_X \otimes_{\mathcal{O}_X} \overset{0}{\bigwedge} \Theta_X \to \mathcal{O}_X \right]$$

is acyclic we consider its filtration $\{F_p N^{\cdot}\}_p$:

$$F_p N^{\cdot} = \left[ F_{p-n} D_X \otimes_{\mathcal{O}_X} \overset{n}{\bigwedge} \Theta_X \to \cdots \to F_p(D_X) \otimes_{\mathcal{O}_X} \overset{0}{\bigwedge} \Theta_X \to F_p(\mathcal{O}_X) \right],$$

where $F_p(\mathcal{O}_X)$ is $\mathcal{O}_X$ for $p \geq 0$ and is 0 for $p < 0$. Then it is sufficient to show that the associated graded complex $\mathrm{gr}\, N^{\cdot}$ is acyclic. Let $\pi : T^*X \to X$ and $i : X \to T^*X$ be the projection and the embedding by the zero-section, respectively. Then we have $\mathrm{gr}\, N^{\cdot} \simeq \pi_* L^{\cdot}$ with

$$L^{\cdot} = \left[ \mathcal{O}_{T^*X} \otimes_{\pi^{-1}\mathcal{O}_X} \overset{n}{\bigwedge} \pi^{-1}\Theta_X \to \cdots \to \mathcal{O}_{T^*X} \otimes_{\pi^{-1}\mathcal{O}_X} \overset{0}{\bigwedge} \pi^{-1}\Theta_X \to i_*\mathcal{O}_X \right],$$

where $\mathcal{O}_{T^*X} \otimes_{\pi^{-1}\mathcal{O}_X} \bigwedge^0 \pi^{-1}\Theta_X (= \mathcal{O}_{T^*X}) \to i_*\mathcal{O}_X$ is given by $\varphi \mapsto i_*(\varphi \circ i)$ and $d : \mathcal{O}_{T^*X} \otimes_{\pi^{-1}\mathcal{O}_X} \bigwedge^k \pi^{-1}\Theta_X \to \mathcal{O}_{T^*X} \otimes_{\pi^{-1}\mathcal{O}_X} \bigwedge^{k-1} \pi^{-1}\Theta_X$ is given by

$$d(\varphi \otimes \theta_1 \wedge \cdots \wedge \theta_k) = \sum_i (-1)^{i+1} \varphi\sigma_1(\theta_i) \otimes \theta_1 \wedge \cdots \wedge \widehat{\theta_i} \cdots \wedge \theta_k.$$

It is well known that the complex $L^{\cdot}$ is acyclic (the Koszul resolution of the $\mathcal{O}_{T^*X}$-module $i_*\mathcal{O}_X$; see, e.g., [Matm, Theorem 43]). Since $\pi$ is an affine morphism, $\pi_* L^{\cdot}$ is also acyclic. □

Let $Y$ and $Z$ be smooth algebraic varieties and set $X = Y \times Z$. Let $f : X \to Y$ and $g : X \to Z$ be the projections. We consider $\int_f M = Rf_*(D_{Y \leftarrow X} \overset{L}{\otimes}_{D_X} M)$ for $M \in \mathrm{Mod}_{qc}(D_X)$ in the following. To compute $D_{Y \leftarrow X} \overset{L}{\otimes}_{D_X} M$ we use the resolution of the right $D_X$-module $D_{Y \leftarrow X} = D_Y \boxtimes \Omega_Z$ induced by the resolution of the right $D_Z$-module $\Omega_Z$ given in Lemma 1.5.27. Set $n = \dim Z (= \dim X - \dim Y)$ and $\Omega^k_{X/Y} = \mathcal{O}_Y \boxtimes \Omega^k_Z$ for $0 \leq k \leq n$. For $M \in \mathrm{Mod}_{qc}(D_X)$ we define its (*relative*) *de Rham complex* $DR_{X/Y}(M)$ by

$$(DR_{X/Y}(M))^k := \begin{cases} \Omega^{n+k}_{X/Y} \otimes_{\mathcal{O}_X} M & (-n \leq k \leq 0), \\ 0 & (\text{otherwise}), \end{cases}$$

$$d(\omega \otimes s) = d\omega \otimes s + \sum_{i=1}^{n} (dz_i \wedge \omega) \otimes \partial_i s.$$

Here $\{z_i, \partial_i\}_{1 \leq n}$ is a local coordinate of $Z$. Note that each term $(DR_{X/Y}(M))^k = g^{-1}\Omega^{n+k}_Z \otimes_{g^{-1}\mathcal{O}_Z} M$ is an $f^{-1}D_Y$-module by

$$P(\omega \otimes s) = \omega \otimes ((P \otimes 1)s) \qquad (P \in f^{-1}D_Y,\ \omega \in g^{-1}\Omega_Z^{n+k},\ s \in M),$$

where we denote by $P \mapsto P \otimes 1$ the canonical homomorphism $f^{-1}D_Y \to D_X$. Thus $DR_{X/Y}(M)$ is a complex of $f^{-1}D_Y$-modules. By the above lemma we have

$$D_{Y \leftarrow X} \otimes^L_{D_X} M \simeq DR_{X/Y}(M)$$

in the derived category consisting of complexes of $f^{-1}D_Y$-modules.

**Proposition 1.5.28.** *Let $Y$ and $Z$ be smooth algebraic varieties, and let $f : X = Y \times Z \to Y$ be the projection.*

(i) *For $M \in \mathrm{Mod}(D_X)$ we have $\int_f M \simeq Rf_*(DR_{X/Y}(M))$.*

(ii) *For $M \in \mathrm{Mod}(D_X)$ we have $\int_f^j M = 0$ unless $-\dim Z \le j \le \dim Z$.*

(iii) *The functor $\int_f$ sends $D^b_{qc}(D_X)$ to $D^b_{qc}(D_Y)$.*

*Proof.* The assertion (i) follows from the above consideration and the definition of $\int_f$, and (ii) is a consequence of (i) since $f_*$ has cohomological dimension $\dim Z$. In order to show (iii) it is sufficient to show for $M \in \mathrm{Mod}_{qc}(D_X)$ that $R^i f_*(DR_{X/Y}(M)^k)$ is a quasi-coherent $\mathcal{O}_Y$-module for any $i$ and $k$. This follows from Proposition 1.5.20 since $DR_{X/Y}(M)^k$ is a quasi-coherent $\mathcal{O}_X$-module. □

Note that any morphism $f : X \to Y$ of smooth algebraic varieties is a composite of a closed embedding $i : X \to Y \times X$ $(x \mapsto (f(x), x))$ and the projection $Y \times X \to Y$. Hence by Proposition 1.5.21, Proposition 1.5.24 and Proposition 1.5.28 we obtain the following.

**Proposition 1.5.29.** *Let $f : X \to Y$ be a morphism of smooth algebraic varieties. Then $\int_f$ sends $D^b_{qc}(D_X)$ to $D^b_{qc}(D_Y)$.*

**Proposition 1.5.30.** *Let $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ be morphisms of smooth algebraic varieties. Then for $M_1^{\cdot} \in D^b_{qc}(D_{X_1})$, $M_2^{\cdot} \in D^b_{qc}(D_{X_2})$ the canonical morphism*

$$\left(\int_{f_1} M_1^{\cdot}\right) \boxtimes \left(\int_{f_2} M_2^{\cdot}\right) \to \int_{f_1 \times f_2} (M_1^{\cdot} \boxtimes M_2^{\cdot})$$

*is an isomorphism.*

*Proof.* By decomposing $f_1 \times f_2$ into the composite of $X_1 \times X_2 \to Y_1 \times X_2 \to Y_1 \times Y_2$ it is sufficient to show that for a morphism $f : X \to Y$ of smooth algebraic varieties and a smooth algebraic variety $T$ the canonical morphism

$$\left(\int_f M^{\cdot}\right) \boxtimes N^{\cdot} \to \int_{f \times \mathrm{id}_T} (M^{\cdot} \boxtimes N^{\cdot}) \qquad (M \in D^b_{qc}(D_X), N \in D^b_{qc}(T))$$

is an isomorphism. By decomposing $f$ into the composite of $X \to X \times Y$ $(x \mapsto (x, f(x))$ and the projection $X \times Y \to Y$ we may assume that $f$ is either a closed

embedding or a projection. Moreover, we may assume $M^{\cdot} = M \in \mathrm{Mod}_{qc}(D_X)$, $N^{\cdot} = N \in \mathrm{Mod}_{qc}(D_Y)$.

Assume that $i : X \to Y$ is a closed embedding. Since the question is local, we may take a local coordinate $\{y_k, \partial_{y_k}\}_{1 \leq k \leq n}$ of $Y$ such that $y_{r+1}, \ldots, y_n$ give the defining equations of $X$. Then by Example 1.3.5 we have

$$\begin{aligned}\left(\int_i M\right) \boxtimes N &\simeq (\mathbb{C}[\partial_{y_{r+1}}, \ldots, \partial_{y_n}] \otimes_{\mathbb{C}} i_* M) \boxtimes N \\ &\simeq \mathbb{C}[\partial_{y_{r+1}}, \ldots, \partial_{y_n}] \otimes_{\mathbb{C}} (i \times 1)_* (M \boxtimes N) \\ &\simeq \int_{i \times \mathrm{id}_T} (M \boxtimes N).\end{aligned}$$

Assume that $f : X \to Y$ is the projection. Then we have

$$\left(\int_f M\right) \boxtimes N \simeq Rf_*(DR_{X/Y}(M)) \boxtimes N,$$

$$\int_{f \times \mathrm{id}_T} (M \boxtimes N) \simeq R(f \times \mathrm{id}_T)_* (DR_{X \times T / Y \times T}(M \boxtimes N)).$$

Since $DR_{X/Y}(M)^k$ is a quasi-coherent $\mathcal{O}_X$-module, we have

$$\begin{aligned}Rf_*(DR_{X/Y}(M)^k) \boxtimes N &\simeq R(f \times \mathrm{id}_T)_* (DR_{X/Y}(M)^k) \boxtimes N) \\ &\simeq R(f \times \mathrm{id}_T)_* (DR_{X \times T / Y \times T}(M \boxtimes N)^k),\end{aligned}$$

and hence

$$Rf_*(DR_{X/Y}(M)) \boxtimes N \simeq R(f \times \mathrm{id}_T)_* (DR_{X \times T / Y \times T}(M \boxtimes N)).$$

The proof is complete. □

In the proof of Proposition 1.5.30 we have used the following.

**Lemma 1.5.31.** *Let $f : X \to Y$ be a morphism of algebraic varieties and let $T$ be an algebraic variety. For $M^{\cdot} \in D^b_{qc}(\mathcal{O}_X)$ and $N^{\cdot} \in D^b_{qc}(\mathcal{O}_T)$ the canonical morphism*

$$Rf_*(M^{\cdot}) \boxtimes N^{\cdot} \to R(f \times \mathrm{id}_T)_* (M^{\cdot} \boxtimes N^{\cdot})$$

*is an isomorphism.*

*Proof.* Since the question is local, we may assume that $T$ is affine. Then there exists an isomorphism $F^{\cdot} \simeq N^{\cdot}$ in $D^b_{qc}(\mathcal{O}_T)$ such that $F^k$ is a direct summand of a free $\mathcal{O}_T$-module for any $k$ and $F^k = 0$ for $|k| \gg 0$. Hence we may assume from the beginning that $N^{\cdot} = \mathcal{O}_T$. Consider the cartesian square

$$\begin{array}{ccc} X \times T & \xrightarrow{p} & X \\ {\scriptstyle f \times \mathrm{id}_T}\downarrow & & \downarrow{\scriptstyle f} \\ Y \times T & \xrightarrow[q]{} & Y, \end{array}$$

where $p : X \times T \to X$ and $q : Y \times T \to Y$ are the projections. Then we have

$$Rf_*(M^{\cdot}) \boxtimes \mathcal{O}_T \simeq q^* Rf_*(M^{\cdot}) \simeq R(f \times \mathrm{id}_T)_* p^*(M^{\cdot}) \simeq R(f \times \mathrm{id}_T)_*(M^{\cdot} \boxtimes \mathcal{O}_T)$$

by the *base change theorem* (see [Ha2, II, Proposition 5.12]) for $\mathcal{O}$-modules. □

**Remark 1.5.32.** One can investigate some problems in integral geometry such as those for Radon transforms in a purely algebraic (functorial) way using the operations of $D$-modules explained above (see, for example, [Br], [D], [DS2], [Gon], [KT4], [Mar], [MT]).

## 1.6 Kashiwara's equivalence

In Proposition 1.5.24 we saw that for a closed embedding $i : X \hookrightarrow Y$ the direct image functor $\int_i^0 : \mathrm{Mod}_{qc}(D_X) \to \mathrm{Mod}_{qc}(D_Y)$ is an exact functor. In this case, the image of a $D_X$-module by $\int_i^0$ is a $D_Y$-module supported by $X$. Let us denote by $\mathrm{Mod}_{qc}^X(D_Y)$ (resp. $\mathrm{Mod}_c^X(D_Y)$) the full subcategory of $\mathrm{Mod}_{qc}(D_Y)$ (resp. $\mathrm{Mod}_c(D_Y)$) consisting of $D_Y$-modules whose support is contained in $X$. Then we have the following theorem which plays a fundamental role in various studies of $D$-modules.

**Theorem 1.6.1 (Kashiwara's equivalence).** *Let $i : X \hookrightarrow Y$ be a closed embedding.*

(i) *The functor $\int_i^0$ induces equivalences*

$$\mathrm{Mod}_{qc}(D_X) \xrightarrow{\sim} \mathrm{Mod}_{qc}^X(D_Y),$$
$$\mathrm{Mod}_c(D_X) \xrightarrow{\sim} \mathrm{Mod}_c^X(D_Y)$$

*of abelian categories. Their quasi-inverses are given by $i^\natural = H^0 i^\dagger$.*

(ii) *For any $N \in \mathrm{Mod}_{qc}^X(D_Y)$ we have $H^j i^\dagger N = 0$ $(j \neq 0)$.*

*Proof.* In order to show (i) for $\mathrm{Mod}_{qc}$ it is sufficient to show that the canonical homomorphisms

$$M \to i^\natural \int_i^0 M, \quad \int_i^0 i^\natural N \to N \qquad (M \in \mathrm{Mod}_{qc}(D_X),\ N \in \mathrm{Mod}_{qc}^X(D_Y))$$

are isomorphisms (see Corollary 1.5.26). The assertion for $\mathrm{Mod}_c$ follows from that for $\mathrm{Mod}_{qc}$ if we can show that $\int_i^0$ sends $\mathrm{Mod}_c(D_X)$ to $\mathrm{Mod}_c^X(D_Y)$ and that $i^\natural$ sends $\mathrm{Mod}_c^X(D_Y)$ to $\mathrm{Mod}_c(D_X)$. Hence our problem is local. Since (ii) is also a local problem, we may shrink $Y$ if necessary. Moreover, by induction on the codimension of $X$ we can assume that $X$ is a hypersurface. We use the local coordinate $\{y_k, \partial_{y_k}\}_{1 \le k \le n}$ of $Y$ used in Example 1.3.5 ($X$ is defined by $y_n = 0$). We set $y = y_n$, $\partial = \partial_{y_n}$, $\theta = y\partial$. Then we have

$$\int_i^0 M = \mathbb{C}[\partial] \otimes_{\mathbb{C}} i_* M \qquad (M \in \mathrm{Mod}_{qc}(D_X)),$$
$$H^0 i^\dagger N = i^\natural N = \mathrm{Ker}(y : i^{-1}N \to i^{-1}N) \qquad (N \in \mathrm{Mod}^X_{qc}(D_Y)),$$
$$H^1 i^\dagger N = \mathrm{Coker}(y : i^{-1}N \to i^{-1}N) \qquad (N \in \mathrm{Mod}^X_{qc}(D_Y)),$$
$$H^j i^\dagger N = 0 \qquad (j \neq 0, -1,\ N \in \mathrm{Mod}^X_{qc}(D_Y)).$$

Let $N \in \mathrm{Mod}^X_{qc}(D_Y)$. Consider the eigenspaces

$$N^j := \{s \in N \mid \theta s = js\} \quad (j \in \mathbb{Z})$$

of $\theta$ in $N$. By the relation $[\partial,\ y] = 1$ we get $yN^j \subset N^{j+1}$, $\partial N^j \subset N^{j-1}$ and $\theta$ induces an isomorphism $j\times : N^j \xrightarrow{\sim} N^j$ for $^\forall j \neq 0$. Therefore, $\partial y = \theta + 1 : N^j \to N^j$ is an isomorphism for $^\forall j \neq -1$. In particular, if $j < -1$, both morphisms $N^j \xrightarrow{y} N^{j+1} \xrightarrow{\partial} N^j$ are isomorphisms.

Let us show that

$$N = \bigoplus_{i=1}^{\infty} N^{-i}. \tag{1}$$

Since $N$ is a quasi-coherent $\mathcal{O}_Y$-module supported in $X$, any $s \in N$ is annihilated by $y^k$ for a sufficiently large $k$. Hence it suffices to prove the following assertion:

$$\mathrm{Ker}(y^k : N \to N) \subset \bigoplus_{j=1}^{k} N^{-j} \quad (k \geq 1). \tag{2}$$

This is true for $k = 1$ because the condition $ys = 0$ implies $\theta s = (\partial y - 1)s = -s$. Assume that $k > 1$ and that (2) is true for $k - 1$. Then for a section $s \in \mathrm{Ker}(y^k : N \to N)$ we have $y^k s = y^{k-1}(ys) = 0$ and $ys \in \bigoplus_{j=1}^{k-1} N^{-j}$ by the hypothesis of induction. Hence $\partial ys \in \bigoplus_{j=2}^{k} N^{-j}$ and

$$\theta s + s = y\partial s + s = \partial ys \in \bigoplus_{j=2}^{k} N^{-j}. \tag{3}$$

On the other hand, we have $y^{k-1}(\theta s + ks) = y^k \partial s + k y^{k-1} s = \partial y^k s = 0$. Therefore, again by the hypothesis of induction we get

$$\theta s + ks \in \bigoplus_{j=1}^{k-1} N^{-j}. \tag{4}$$

The difference (4)–(3) gives $(k-1)s \in \bigoplus_{j=1}^{k} N^{-j}$. By $k > 1$ we finally obtain $s \in \bigoplus_{i=1}^{k} N^{-i}$ and the proof of (1) is complete.

By (1) we easily see that $H^1 i^\dagger N = 0$, and (ii) is proved. We see also from (1) that

$$N = \mathbb{C}[\partial] \otimes_{\mathbb{C}} N^{-1}, \qquad i^\natural N = i^{-1}N^{-1}.$$

From this we easily obtain $\mathrm{Mod}_{qc}(D_X) \simeq \mathrm{Mod}^X_{qc}(D_Y)$.

It remains to show that $\int_i^0$ sends $\mathrm{Mod}_c(D_X)$ to $\mathrm{Mod}^X_c(D_Y)$ and that $i^\natural$ sends $\mathrm{Mod}^X_c(D_Y)$ to $\mathrm{Mod}_c(D_X)$. We may shrink $Y$ if necessary. If $M \in \mathrm{Mod}_{qc}(D_X)$ is finitely generated over $D_X$, then $\int_i^0 M = \mathbb{C}[\partial] \otimes_{\mathbb{C}} i_* M$ is clearly finitely generated over $D_Y$. Assume that $N \in \mathrm{Mod}^X_{qc}(D_Y)$ is finitely generated over $D_Y$. By $N = \mathbb{C}[\partial] \otimes_{\mathbb{C}} N^{-1}$ the $D_Y$-module $N$ is generated by finitely many sections $s_1, \dots, s_r$ contained in $N^{-1}$. Then $i^\natural N = i^{-1}N^{-1}$ is generated as a $D_X$-module by the sections $s_1, \dots, s_r$. The proof is complete. □

Denote by $D^{b,X}_{qc}(D_Y)$ (resp. $D^{b,X}_c(D_Y)$) the subcategory of $D^b_{qc}(D_Y)$ (resp. $D^b_c(D_Y)$) consisting of complexes $N^\cdot$ whose cohomology sheaves $H^*(N^\cdot)$ are supported by $X$.

**Corollary 1.6.2.** *For $\sharp = qc$ or $c$ the functor*

$$\int_i : D^b_\sharp(D_X) \to D^{b,X}_\sharp(D_Y)$$

*gives an equivalence of triangulated categories. Its quasi-inverse is given by*

$$Ri^\natural = i^\dagger : D^{b,X}_\sharp(D_Y) \to D^b_\sharp(D_X).$$

*Proof.* It is easily seen that $\int_i$ sends $D^b_\sharp(D_X)$ to $D^{b,X}_\sharp(D_Y)$ and $Ri^\natural$ sends $D^{b,X}_\sharp(D_Y)$ to $D^b_\sharp(D_X)$. By Proposition 1.5.25 we have canonical morphisms

$$M^\cdot \to Ri^\natural \int_i M^\cdot, \quad \int_i Ri^\natural N^\cdot \to N^\cdot \qquad (M^\cdot \in D^b_\sharp(D_X), N^\cdot \in D^{b,X}_\sharp(D_Y)).$$

We have only to show that those morphisms are isomorphisms. Let us show that $M^\cdot \to Ri^\natural \int_i M^\cdot$ is an isomorphism for $M^\cdot \in D^b_\sharp(D_X)$. We proceed by induction on the *cohomological length* $l(M^\cdot) := \mathrm{Max}\{\, i \mid H^i(M^\cdot) \neq 0\} - \mathrm{Min}\{\, j \mid H^j(M^\cdot) \neq 0\}$ of $M^\cdot$. Assume that $l(M^\cdot) = 0$. Then we have $M^\cdot = M[k]$ for some $M \in \mathrm{Mod}_\sharp(D_X)$ and $k \in \mathbb{Z}$, and hence we may assume that $M^\cdot = M \in \mathrm{Mod}_\sharp(D_X)$ from the beginning. In this case the assertion is already proved in Theorem 1.6.1. Assume that $l(M^\cdot) > 0$. In this case there exists some $k \in \mathbb{Z}$ such that $l(\tau^{\leqslant k} M^\cdot) < l(M^\cdot)$ and $l(\tau^{>k} M^\cdot) < l(M^\cdot)$, where $\tau^{\leqslant k}$ and $\tau^{>k}$ are the truncation functors (see Appendix B). By applying $Ri^\natural \int_i$ to the distinguished triangle

$$\tau^{\leqslant k} M^\cdot \longrightarrow M^\cdot \longrightarrow \tau^{>k} M^\cdot \xrightarrow{+1}$$

we obtain a distinguished triangle

$$Ri^\natural \int_i \tau^{\leqslant k} M^\cdot \longrightarrow Ri^\natural \int_i M^\cdot \longrightarrow Ri^\natural \int_i \tau^{>k} M^\cdot \xrightarrow{+1}.$$

Moreover, we have a commutative diagram

$$\begin{array}{ccccccc} \tau^{\leqslant k}M^{\cdot} & \longrightarrow & M^{\cdot} & \longrightarrow & \tau^{>k}M^{\cdot} & \xrightarrow{+1} & \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ Ri^{\natural}\textstyle\int_i \tau^{\leqslant k}M^{\cdot} & \longrightarrow & Ri^{\natural}\textstyle\int_i M^{\cdot} & \longrightarrow & Ri^{\natural}\textstyle\int_i \tau^{>k}M^{\cdot} & \xrightarrow{+1} & \end{array}.$$

By our hypothesis on induction $\alpha$ and $\gamma$ are isomorphisms. Hence $\beta$ is also an isomorphism (see Appendix B). We can also show that $\int_i Ri^{\natural}N^{\cdot} \to N^{\cdot}$ is an isomorphism by a similar argument. □

**Remark 1.6.3.** In this book we will frequently use the argument in the proof of Corollary 1.6.2, reducing assertions on complexes to those on objects of abelian categories (regarded as a complex concentrated at degree 0) by induction on the cohomological length.

**Example 1.6.4.** Consider the $D_Y$-module

$$\mathcal{B}_{X|Y} = \int_i^0 \mathcal{O}_X \in \mathrm{Mod}_{qc}^X(D_Y).$$

Take a local coordinate system $\{y_j,\ \partial_j\}_{1\leq j\leq n}$ of $Y$ such that

$$X = \{y_j = 0 \mid j \geq m+1\}.$$

Then we have

$$\mathcal{B}_{X|Y} = D_Y \Big/ \Big(\sum_{l=1}^{m} D_Y\partial_l + \sum_{j=m+1}^{n} D_Y y_j\Big).$$

In particular for $X = \{p\}$ (one point), we get $\mathcal{B}_{\{p\}|Y} = D_Y/D_Y\mathfrak{m}_p = D_Y\delta_p \simeq \mathbb{C}[\partial_1, \partial_2, \ldots, \partial_n]\delta_p$, where $\mathfrak{m}_p = (y_1, y_2, \ldots, y_n)$ is the maximal ideal at $p$ and $\delta_p = 1 \mod \mathfrak{m}_p \in \mathcal{B}_{\{p\}|Y}$. Here, we have used the notation $\delta_p$ since the corresponding system $y_j u = 0$ $(1 \leq j \leq n)$ of differential equations is the one satisfied by the Dirac delta function supported by $\{p\}$. By Kashiwara's equivalence, we have the correspondence

$$\mathrm{Mod}_{qc}^{\{p\}}(D_Y) \simeq \{\text{the category of } \mathbb{C}\text{-vector spaces}\} \quad (\mathcal{B}_{\{p\}|Y} \longleftrightarrow \mathbb{C}).$$

Hence objects of $\mathrm{Mod}_{qc}^{\{p\}}(D_Y)$ are direct sums of $\mathcal{B}_{\{p\}|Y}$.

We will give an application of Kashiwara's equivalence.

**Theorem 1.6.5.** *A product of a projective space and a smooth affine variety is D-affine.*

*Proof.* Let $Y$ be a smooth affine algebraic variety. We set $X = \mathbb{P}^n(\mathbb{C}) \times Y$, $V = \mathbb{C}^{n+1}$, $V^\bullet = V \setminus \{0\}$. Let $\pi : \widetilde{X} = V^\bullet \times Y \to \mathbb{P}^n(\mathbb{C}) \times Y$ be the projection. Then for $M \in \mathrm{Mod}_{qc}(D_X)$ we obtain a natural action of the complex multiplicative group $\mathbb{C}^\times$ on the space $\Gamma(\widetilde{X},\, \pi^*M)$ of the global sections of its inverse image $\pi^*M = \mathcal{O}_{\widetilde{X}} \otimes_{\pi^{-1}\mathcal{O}_X} \pi^{-1}M \in \mathrm{Mod}_{qc}(D_{\widetilde{X}})$, because $\Gamma(\widetilde{X},\, \mathcal{O}_{\widetilde{X}})$ has a natural action of $\mathbb{C}^\times$. Considering $\Gamma(\widetilde{X},\, \pi^*M)$ as a $\mathbb{C}^\times$-module, we get its weight space decomposition

$$\Gamma(\widetilde{X},\, \pi^*M) = \bigoplus_{l \in \mathbb{Z}} \gamma(M)^{(l)},$$

where $z \in \mathbb{C}^\times$ acts on $\gamma(M)^{(l)}$ by $z^l$. In particular, $\Gamma(X,\, M) = \gamma(M)^{(0)}$. Now let us consider the Euler vector field $\theta = \sum_{i=0}^n x_i \partial_i$ (here $\{x_i\}$ is a linear coordinate system of $V$ and $\partial_i = \partial/\partial x_i$) on $V$. If we define the action of $\theta$ on $\pi^*M$ by $\theta \otimes \mathrm{Id}$, we have

$$\gamma(M)^{(l)} = \{u \in \Gamma(\widetilde{X},\, \pi^*M) \mid \theta u = lu\,\}.$$

Moreover, we can easily check

$$x_i(\gamma(M)^{(l)}) \subset \gamma(M)^{(l+1)}, \qquad \partial_i(\gamma(M)^{(l)}) \subset \gamma(M)^{(l-1)}.$$

Set $Z = \{0\} \times Y \subset V \times Y$, and let $j : \widetilde{X} \hookrightarrow V \times Y$ and $k : Z \hookrightarrow V \times Y$ be the embeddings.

Let us show that $\Gamma(X, \bullet)$ is exact on $\mathrm{Mod}_{qc}(D_X)$. Let

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$$

be an exact sequence in $\mathrm{Mod}_{qc}(D_X)$. Since $\pi$ is smooth, the sequence

$$0 \longrightarrow \pi^*M_1 \longrightarrow \pi^*M_2 \longrightarrow \pi^*M_3 \longrightarrow 0$$

is also exact. Hence we obtain the long exact sequence

$$0 \to j_*\pi^*M_1 \to j_*\pi^*M_2 \to j_*\pi^*M_3 \to R^1 j_*\pi^*M_1 \to \cdots$$

in $\mathrm{Mod}_{qc}(D_{V\times Y})$. On the other hand, the supports of the first cohomology sheaves $\int_j^1 \pi^*M_i = R^1 j_*\pi^*M_i \quad (i = 1, 2, 3)$ are contained in $Z$. Therefore, by Kashiwara's equivalence (Theorem 1.6.1), there exists a unique $D_Z$-module $N \in \mathrm{Mod}_{qc}(D_Z)$ such that

$$R^1 j_*\pi^*M_1 \simeq \int_k^0 N \simeq \mathbb{C}[\partial_0, \partial_1, \partial_2, \ldots, \partial_n] \otimes_{\mathbb{C}} N$$

As we have seen in Example 1.3.5 the action of $D_V$ on $\mathbb{C}[\partial_0, \partial_1, \partial_2, \ldots, \partial_n] \otimes_{\mathbb{C}} N$ is given by $x_i(\partial_j^k \otimes u) = -k\delta_{ij}(\partial_j^{k-1} \otimes u)$ $(u \in N)$ for $k \geq 0$ ($\delta_{ij}$ is Kronecker's delta). So the Euler operator $\theta$ acts on it by

$$\theta(\partial^\alpha \otimes u) = -(|\alpha| + (n+1))(\partial^\alpha \otimes u) \quad (u \in N).$$

Hence the eigenvalues of the action of $\theta$ on $\Gamma(V \times Y, \int_k^0 N) = \Gamma(V \times Y, R^1 j_*\pi^*M_1)$ are negative integers. Since $V \times Y$ is affine, $\Gamma(V \times Y, \bullet)$ is an exact functor and we get a long exact sequence

$$0 \longrightarrow \Gamma(V \times Y, j_*\pi^* M_1) \longrightarrow \cdots \longrightarrow \Gamma(V \times Y, R^1 j_*\pi^* M_1) \longrightarrow \cdots .$$

Note that $\Gamma(V \times Y, j_*\pi^* M_i) = \Gamma(\widetilde{X}, \pi^* M_i)$, and the eigenvalues of $\theta$ on $\Gamma(V \times Y, R^1 j_*\pi^* M_1)$ are negative integers. Hence taking the 0-eigenspaces of $\theta$ in the above long exact sequence, we finally obtain the exact sequence

$$0 \longrightarrow \gamma(M_1)^{(0)} \longrightarrow \gamma(M_2)^{(0)} \longrightarrow \gamma(M_3)^{(0)} \longrightarrow 0$$

i.e.,

$$0 \longrightarrow \Gamma(X, M_1) \longrightarrow \Gamma(X, M_2) \longrightarrow \Gamma(X, M_3) \longrightarrow 0.$$

The exactness of $\Gamma(X, \bullet)$ on $\mathrm{Mod}_{qc}(D_X)$ is verified.

Let us show that $\Gamma(X, M) = 0$ for $M \in \mathrm{Mod}_{qc}(D_X)$ implies $M = 0$. Assume $M \neq 0$. As before, let us consider the eigenspace decomposition

$$\Gamma(\widetilde{X}, \pi^* M) = \bigoplus_{l \in \mathbb{Z}} \gamma(M)^{(l)}$$

with respect to the $\theta$-action. Since $M \neq 0$ and $\widetilde{X} \to X$ is a smooth surjective (hence, faithfully flat) morphism, we get $\pi^* M \neq 0$. As a consequence, there exists an integer $l_0 \in \mathbb{Z}$ satisfying $\gamma(M)^{(l_0)} \neq 0$. Assume $l_0 > 0$, and take a section $u \neq 0 \in \gamma(M)^{(l_0)}$. If $\partial_i u = 0$ for any $i$, then we get $\theta u = 0$ and it contradicts our assumption $l_0 > 0$. So for some $i$ we should have $0 \neq \partial_i u \in \gamma(M)^{(l_0-1)}$. By repeating this procedure, we can show $\Gamma(X, M) \simeq \gamma(M)^{(0)} \neq 0$. This is a contradiction. Next assume $l_0 < 0$, $\gamma(M)^{(l_0)} \neq 0$, and take a non-zero section $0 \neq u \in \gamma(M)^{(l_0)}$. If $x_i u = 0$ $(0 \leq i \leq n)$, then supp $u \subset Z$ and $u$ should be zero globally on $\widetilde{X}$. This implies $x_i u \in \gamma(M)^{(l_0+1)} \neq 0$ for some $x_i$. We can repeat this argument until we get $\Gamma(X, M) \simeq \gamma(M)^{(0)} \neq 0$. This is also a contradiction. Hence we have $M = 0$. □

## 1.7 A base change theorem for direct images

Let $X$ be a topological space, $Z$ a closed subset, and $U = X \setminus Z$ the complementary open subset of $X$. We denote by $i : Z \to X$ and $j : U \to X$ the embeddings

$$Z \overset{i}{\hookrightarrow} X \overset{j}{\hookleftarrow} U.$$

Then for an injective sheaf $F$ on $X$ we get an exact sequence

$$0 \longrightarrow \Gamma_Z(F) \longrightarrow F \longrightarrow j_* j^{-1} F \longrightarrow 0,$$

where $\Gamma_Z(F)$ is the sheaf of sections of $F$ supported by $Z$. Hence for any $F^\cdot \in D^b(\mathbb{C}_X)$, there exists a distinguished triangle

$$R\Gamma_Z(F^\cdot) \longrightarrow F^\cdot \longrightarrow Rj_* j^{-1} F^\cdot \overset{+1}{\longrightarrow} .$$

Considering this distinguished triangle in the case where $X$ is a smooth algebraic variety and $F^\cdot \in D^b(D_X)$, we obtain the following.

**Proposition 1.7.1.** *Let $X$ be a smooth algebraic variety and let $Z$ be its closed subset. Set $U = X \setminus Z$. Denote by $i : Z \to X$ and $j : U \to X$ the embeddings.*

(i) *For $M^{\cdot} \in D^b_{qc}(D_X)$ we have a canonical distinguished triangle*

$$R\Gamma_Z(M^{\cdot}) \longrightarrow M^{\cdot} \longrightarrow \int_j j^{\dagger} M^{\cdot} \xrightarrow{+1} .$$

(ii) *Assume that $Z$ is smooth. Then for $M^{\cdot} \in D^b_{qc}(D_U)$ we have*

$$i^{\dagger} \int_j M^{\cdot} = 0.$$

(iii) *Assume that $Z$ is smooth. Then for $M^{\cdot} \in D^b_{qc}(D_X)$ we have*

$$R\Gamma_Z(M^{\cdot}) \simeq \int_i i^{\dagger} M^{\cdot}.$$

*Proof.* We have

$$\begin{cases} Lj^* M^{\cdot} = j^{\dagger} M^{\cdot} = j^{-1} M^{\cdot} & (M^{\cdot} \in D^b(D_X)), \\ \int_j N^{\cdot} = Rj_* N^{\cdot} & (N^{\cdot} \in D^b(D_U)). \end{cases}$$

Hence for $M^{\cdot} \in D^b(D_X)$ we have $Rj_* j^{-1} M^{\cdot} \simeq \int_j j^{\dagger} M^{\cdot}$. The assertion (i) is proved.

The statement (ii) follows from Lemma 1.5.17.

Let us show (iii). Since $\int_j j^{\dagger} M^{\cdot}$ belongs to $D^b_{qc}(D_X)$, we have $R\Gamma_Z(M^{\cdot}) \in D^{b,Z}_{qc}(D_X)$. Hence we obtain $R\Gamma_Z(M^{\cdot}) \simeq \int_i (i^{\dagger} R\Gamma_Z(M^{\cdot}))$ by Corollary 1.6.2. Therefore, it is sufficient to show $i^{\dagger} R\Gamma_Z(M^{\cdot}) \simeq i^{\dagger} M^{\cdot}$. It is seen by applying $i^{\dagger}$ to the distinguished triangle in (i) that this assertion is equivalent to $i^{\dagger} Rj_*(j^{-1} M^{\cdot}) = 0$. This follows from (ii). □

**Remark 1.7.2.** In some literature, $\int_j j^{\dagger} M^{\cdot}$ is denoted by $R\Gamma_{X|Z}(M^{\cdot})$.

Now let us state our main theorem in this section.

**Theorem 1.7.3 (Base change theorem).** *For two morphisms $f : Y \to X$, $g : Z \to X$ of algebraic varieties consider the fiber product (cartesian square)*

$$\begin{array}{ccc} Y_Z & \xrightarrow{\tilde{g}} & Y \\ {\scriptstyle \tilde{f}}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{g} & X \end{array}$$

*($Y_Z := Y \times_X Z$). Assume that the four varieties $X$, $Y$, $Z$ and $Y_Z$ are smooth. Then there exists an isomorphism*

$$g^{\dagger} \int_f \simeq \int_{\tilde{f}} \tilde{g}^{\dagger} : D^b_{qc}(D_Y) \longrightarrow D^b_{qc}(D_Z) \tag{1.7.1}$$

*of functors.*

*Proof.* We can decompose the morphism $g : Z \to X$ as $g : Z \hookrightarrow Z \times X \to X$, where $Z \hookrightarrow Z \times X$ is the *graph embedding* of $g$ and $Z \times X \to X$ is the projection. Hence by Proposition 1.5.11 it is enough to prove the theorem in the case where $g$ is a projection or a closed embedding (such that $Y_Z$ is smooth).

(i) Let $g = \mathrm{pr}_X : Z = T \times X \to X$ be a projection ($T$ is smooth). In this case we have a cartesian square

$$\begin{array}{ccc} T \times Y & \xrightarrow{\tilde{g}} & Y \\ {\scriptstyle \tilde{f}}\downarrow & & \downarrow{\scriptstyle f} \\ T \times X & \xrightarrow{g} & X, \end{array}$$

where $\tilde{g} = \mathrm{pr}_Y$, $\tilde{f} = \mathrm{id}_T \times f$. Then for $M^{\cdot} \in D^b_{qc}(D_Y)$ we have

$$\int_{\tilde{f}} \tilde{g}^{\dagger} M^{\cdot}[-\dim T] \simeq \int_{\mathrm{id}_T \times f} (\mathcal{O}_T \boxtimes M^{\cdot}) \simeq \mathcal{O}_T \boxtimes \int_f M^{\cdot} \simeq g^{\dagger} \int_f M^{\cdot}[-\dim T].$$

(ii) Let $i = g : Z \hookrightarrow X$ be a closed embedding such that $Y_Z$ is smooth. Then we have the two cartesian squares

$$\begin{array}{ccccc} Y_Z = f^{-1}(Z) & \xrightarrow{\tilde{i}} & Y & \xleftarrow{\tilde{j}} & V = f^{-1}(U) \\ {\scriptstyle \tilde{f}}\downarrow & & {\scriptstyle f}\downarrow & & {\scriptstyle h}\downarrow \\ Z & \xrightarrow{i} & X & \xleftarrow{j} & U = X \setminus Z. \end{array}$$

By Kashiwara's equivalence we have $i^{\dagger} \int_i \simeq \mathrm{Id}$ and hence

$$\int_{\tilde{f}} \tilde{i}^{\dagger} \simeq i^{\dagger} \int_i \int_{\tilde{f}} \tilde{i}^{\dagger} \simeq i^{\dagger} \int_f \int_{\tilde{i}} \tilde{i}^{\dagger}.$$

Hence the canonical morphism $\int_{\tilde{i}} \tilde{i}^{\dagger} \to \mathrm{Id}$ (see Proposition 1.7.1) yields the morphism $\int_{\tilde{f}} \tilde{i}^{\dagger} \to i^{\dagger} \int_f$. We need to show

$$i^{\dagger} \int_f \int_{\tilde{i}} \tilde{i}^{\dagger} M^{\cdot} \simeq i^{\dagger} \int_f M^{\cdot}$$

for $M^{\cdot} \in D_{qc}(D_Y)$. Applying $i^{\dagger} \int_f$ to the distinguished triangle

$$\int_{\tilde{i}} \tilde{i}^{\dagger} M^{\cdot} \longrightarrow M^{\cdot} \longrightarrow \int_{\tilde{j}} \tilde{j}^{\dagger} M^{\cdot} \xrightarrow{+1}$$

we see that our assertion is equivalent to $i^{\dagger} \int_f \int_{\tilde{j}} \tilde{j}^{\dagger} M^{\cdot} = 0$. By $i^{\dagger} \int_f \int_{\tilde{j}} \tilde{j}^{\dagger} M^{\cdot} = i^{\dagger} \int_j \int_h \tilde{j}^{\dagger} M^{\cdot}$ this follows from Proposition 1.7.1 (ii). □

**Corollary 1.7.4.** *We keep the notation of Theorem* 1.7.3. *If $g(Z) \cap f(Y) = \emptyset$, then we have $\int_{\tilde{f}} \tilde{g}^{\dagger} = 0$.*

**Corollary 1.7.5 (Projection formula).** *Let $f : X \to Y$ be a morphism of smooth algebraic varieties. Then for $M \in D^b_{qc}(D_X)$, $N \in D^b_{qc}(D_Y)$ we have*

$$\int_f (M \otimes^L_{\mathcal{O}_X} Lf^* N) \simeq \left(\int_f M\right) \otimes^L_{\mathcal{O}_Y} N. \tag{1.7.2}$$

*Proof.* Applying Theorem 1.7.3 to the cartesian square

$$\begin{array}{ccc} X & \xrightarrow{(\mathrm{id}_X \times f)\circ \Delta_X} & X \times Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f\times \mathrm{id}_Y} \\ Y & \xrightarrow{\Delta_Y} & Y \times Y \end{array}$$

we have

$$\begin{aligned} \int_f (M \otimes^L_{\mathcal{O}_X} Lf^* N) &\simeq \int_f L((\mathrm{id}_X \times f) \circ \Delta_X)^* (M \boxtimes N) \simeq L\Delta_Y^* \int_{f\times \mathrm{id}_Y} (M \boxtimes N) \\ &\simeq \left(\int_f M\right) \otimes^L_{\mathcal{O}_Y} N. \end{aligned}$$

The proof is complete. □

**Remark 1.7.6.** It is not obvious whether the isomorphism (1.7.1) constructed in the proof of Theorem 1.7.3 is actually canonical. Our proof of Theorem 1.7.3 only implies that the canonicality is ensured when we restrict ourselves to the case $g$ is a closed embedding. In particular, the isomorphism (1.7.2) giving the projection formula is canonical.

# 2

# Coherent $D$-Modules

As described in the introduction, any system of linear partial differential equations can be considered as a coherent $D$-module. In this chapter we focus our attention on coherent $D$-modules and study their basic properties. Among other things, for a coherent $D_X$-module $M$ we define its characteristic variety as a subvariety of the cotangent bundle $T^*X$ of $X$. This plays an important role for the geometric (or microlocal) study of $M$.

## 2.1 Good filtrations

Recall that the ring $D_X$ has the order filtration $\{F_i D_X\}_{i\in\mathbb{Z}}$ such that the associated graded ring $\mathrm{gr}^F D_X = \bigoplus_{i=0}^{\infty} F_i D_X / F_{i-1} D_X$ is naturally isomorphic to the sheaf $\pi_* \mathcal{O}_{T^*X}$ of commutative rings consisting of symbols of differential operators, where $\pi : T^*X \to X$ denotes the cotangent bundle (see Section 1.1). By the aid of the commutative approximation $\mathrm{gr}^F D$ of the non-commutative ring $D$, we will deduce various results on $D$ using techniques from commutative algebra (algebraic geometry).

We note that some of the results in this chapter can be formulated for more general filtered rings, in which cases they are presented and proved in Appendix D. Hence readers should occasionally consult Appendix D according to references to it in this chapter.

Our first task is to give a commutative approximation of modules over $D$. Let $M$ be a $D_X$-module quasi-coherent over $\mathcal{O}_X$. We consider a filtration of $M$ by quasi-coherent $\mathcal{O}_X$-submodules $F_i M$ $(i \in \mathbb{Z})$ satisfying the conditions:

$$\begin{cases} F_i M \subset F_{i+1} M, \\ F_i M = 0 \quad (i \ll 0), \\ M = \bigcup_{i\in\mathbb{Z}} F_i M, \\ (F_j D_X)(F_i M) \subset F_{i+j} M. \end{cases}$$

In this case, we call $(M,\ F)$ a *filtered $\boldsymbol{D_X}$-module*; the module

$$\mathrm{gr}^F M := \bigoplus_{i\in\mathbb{Z}} F_i M / F_{i-1} M$$

obtained by $F$ is a graded module over $\mathrm{gr}^F D_X = \pi_* \mathcal{O}_{T^*X}$. This module is clearly quasi-coherent over $\mathcal{O}_X$.

**Proposition 2.1.1.** *Let $(M, F)$ be a filtered $D_X$-module. Then the following conditions are equivalent to each other:*

(i) $\mathrm{gr}^F M$ *is coherent over* $\pi_* \mathcal{O}_{T^*X}$.
(ii) $F_i M$ *is coherent over* $\mathcal{O}_X$ *for each* $i$, *and there exists* $i_0 \gg 0$ *satisfying*

$$(F_j D_X)(F_i M) = F_{j+i} M \quad (j \geq 0,\ i \geq i_0).$$

(iii) *There exist locally a surjective $D_X$-linear morphism* $\Phi : D_X^{\oplus m} \to M$ *and integers* $n_j$ $(j = 1, 2, \ldots, m)$ *such that*

$$\Phi(F_{i-n_1} D_X \oplus F_{i-n_2} D_X \oplus \cdots \oplus F_{i-n_m} D_X) = F_i M \quad (i \in \mathbb{Z}).$$

*Proof.* By Proposition D.1.1 the conditions (i) and (iii) are equivalent. It is easily checked that (iii) holds if and only if $F_i M$ is coherent over $\mathcal{O}_X$ for each $i$ and one can find $i_0$ as in (ii) locally on $X$. Then the global existence of $i_0$ follows from this since $X$ is quasi-compact. □

**Definition 2.1.2.** Let $(M, F)$ be a filtered $D_X$-module. We say that $F$ is a *good filtration* of $M$ if the equivalent conditions in Proposition 2.1.1 are satisfied.

**Theorem 2.1.3.**

(i) *Any coherent $D_X$-module admits a (globally defined) good filtration. Conversely, a $D_X$-module endowed with a good filtration is coherent.*
(ii) *Let $F$, $F'$ be two filtrations of a $D_X$-module $M$ and assume that $F$ is good. Then there exists $i_0 \gg 0$ such that*

$$F_i M \subset F'_{i+i_0} M \quad (i \in \mathbb{Z}).$$

*If, moreover, $F'$ is also a good filtration, there exists $i_0 \gg 0$ such that*

$$F'_{i-i_0} M \subset F_i M \subset F'_{i+i_0} M \quad (i \in \mathbb{Z}).$$

*Proof.* (i) By Corollary D.1.2 an object of $\mathrm{Mod}_{qc}(D_X)$ is coherent if and only if it admits a good filtration locally on $X$. Hence it is sufficient to show that any coherent $D_X$-module $M$ admits a global good filtration. By Corollary 1.4.17 (i), $M$ is generated by a globally defined coherent $\mathcal{O}_X$-submodule $M_0$. If we set $F_i M = (F_i D_X) M_0$ $(i \in \mathbb{N})$, then this is a global good filtration of $M$. The statement (ii) follows from Proposition D.1.3. □

## 2.2 Characteristic varieties (singular supports)

Let $M$ be a coherent $D_X$-module and choose a good filtration $F$ on it (Theorem 2.1.3). Let $\pi : T^*X \to X$ be the cotangent bundle of $X$. Since we have $\mathrm{gr}^F D_X \simeq \pi_* \mathcal{O}_{T^*X}$, the graded module $\mathrm{gr}^F M$ of $M$ obtained by $F$ is a coherent module over $\pi_* \mathcal{O}_{T^*X}$ by Proposition 2.1.1. We call the support of the coherent $\mathcal{O}_{T^*X}$-module

$$\widetilde{\mathrm{gr}^F M} := \mathcal{O}_{T^*X} \otimes_{\pi^{-1}\pi_* \mathcal{O}_{T^*X}} \pi^{-1}(\mathrm{gr}^F M)$$

the *characteristic variety* of $M$ and denote it by $\mathrm{Ch}(M)$ (it is sometimes called the *singular support* of $M$ and denoted by $\mathrm{SS}(M)$). As we see below $\mathrm{Ch}(M)$ does not depend on the choice of a good filtration $F$ on $M$. Since $\widetilde{\mathrm{gr}^F M}$ is a graded module over the graded ring $\mathcal{O}_{T^*X}$, $\mathrm{Ch}(M)$ is a closed *conic* (i.e., stable by the scalar multiplication of complex numbers on the fibers) algebraic subset in $T^*X$.

Let $U$ be an affine open subset of $X$. Then $T^*U$ is an affine open subset of $T^*X$ with coordinate algebra $\mathrm{gr}^F D_U(U)$, and $\mathrm{Ch}(M) \cap T^*U$ coincides with the support of the coherent $\mathcal{O}_{T^*U}$-module associated to the finitely generated $\mathrm{gr}^F D_U(U)$-module $\mathrm{gr}^F M(U)$. Hence in the notation of Section D.3 we have

$$\mathrm{Ch}(M) \cap T^*U = \{p \in T^*U \mid f(p) = 0 \ (\forall f \in J_{M(U)})\},$$

and its decomposition into irreducible components is given by

$$\mathrm{Ch}(M) \cap T^*U = \bigcup_{\mathfrak{p} \in \mathrm{SS}_0(M(U))} \{p \in T^*U \mid f(p) = 0 \ (\forall f \in \mathfrak{p})\}.$$

By Lemmas D.3.1 and D.3.3 we have the following.

**Theorem 2.2.1.**

(i) *Let $M$ be a coherent $D_X$-module. Then the set* $\mathrm{Ch}(M)$ *does not depend on the choice of a good filtration $F$.*

(ii) *For a short exact sequence*

$$0 \longrightarrow M \longrightarrow N \longrightarrow L \longrightarrow 0$$

*of coherent $D_X$-modules, we have*

$$\mathrm{Ch}(N) = \mathrm{Ch}(M) \cup \mathrm{Ch}(L).$$

By the above theorem, the characteristic variety is a geometric invariant of a coherent $D$-module.

From now on we introduce the notion of the characteristic cycle, which is a finer invariant of a coherent $D$-module (obtained by taking the multiplicities into account).

Let $V$ be a smooth algebraic variety and assume that we are given a coherent $\mathcal{O}_V$-module $G$. Then we can define an algebraic cycle $\mathrm{Cyc}\, G$ associated to $G$ as follows. Denote by $I(\mathrm{supp}\, G)$ the set of the irreducible components of the support of $G$. Let $C \in I(\mathrm{supp}\, G)$. Take an affine open subset $U$ of $V$ such that $\overline{C \cap U} = C$,

and denote the defining ideal of $C \cap U$ by $\mathfrak{p}_C \subset \mathcal{O}_U(U)$. Then we obtain a local ring $\mathcal{O}_U(U)_{\mathfrak{p}_C}$ with maximal ideal $\mathfrak{p}_C \mathcal{O}_U(U)_{\mathfrak{p}_C}$ and an $\mathcal{O}_U(U)_{\mathfrak{p}_C}$-module $G(U)_{\mathfrak{p}_C}$. Note that $\mathcal{O}_U(U)_{\mathfrak{p}_C}$ and $G(U)_{\mathfrak{p}_C}$ do not depend on the choice of $U$ (in scheme-theoretical language they are the stalks of $\mathcal{O}_V$ and $G$ at the generic point of $C$). By a standard fact in commutative algebra $G(U)_{\mathfrak{p}_C}$ is an artinian $\mathcal{O}_U(U)_{\mathfrak{p}_C}$-module, and its length $m_C(G)$ is defined. We call it the *multiplicity* of $G$ along $C$. For an irreducible subvariety $C$ of $V$ with $C \not\subset \operatorname{supp} G$ we set $m_C(G) = 0$. We call the formal sum

$$\operatorname{Cyc} G := \sum_{C \in I(\operatorname{supp} G)} m_C(G) C$$

the *associated cycle* of $G$.

Let $M$ be a coherent $D_X$-module. By choosing a good filtration $F$ of $M$ we can consider a coherent $\mathcal{O}_{T^*X}$-module $\widetilde{\operatorname{gr}^F M}$. By Lemma D.3.1 the cycle $\operatorname{Cyc}(\widetilde{\operatorname{gr}^F M})$ does not depend on the choice of a good filtration $F$.

**Definition 2.2.2.** For a coherent $D_X$-module $M$ we define the *characteristic cycle* of $M$ by

$$\mathbf{CC}(M) := \operatorname{Cyc}(\widetilde{\operatorname{gr}^F M}) = \sum_{C \in I(\operatorname{Ch}(M))} m_C(\widetilde{\operatorname{gr}^F M}) C,$$

where $F$ is a good filtration of $M$. For $d \in \mathbb{N}$ we denote its degree $d$ part by

$$\mathbf{CC}_d(M) := \sum_{\substack{C \in I(\operatorname{Ch}(M)) \\ \dim C = d}} m_C(\widetilde{\operatorname{gr}^F M}) C.$$

By Lemma D.3.3 we have the following.

**Theorem 2.2.3.** *Let*

$$0 \longrightarrow M \longrightarrow N \longrightarrow L \longrightarrow 0$$

*be an exact sequence of coherent $D_X$-modules. Then for any irreducible subvariety $C$ of $T^*X$ such that $C \in I(\operatorname{Ch}(N))$ we have*

$$m_C(\widetilde{\operatorname{gr}^F N}) = m_C(\widetilde{\operatorname{gr}^F M}) + m_C(\widetilde{\operatorname{gr}^F L}).$$

*In particular, for $d = \dim \operatorname{Ch}(N)$ we have*

$$\mathbf{CC}_d(N) = \mathbf{CC}_d(M) + \mathbf{CC}_d(L).$$

**Example 2.2.4.** Let $M$ be an integrable connection of rank $r > 0$. Set $F_i M = 0$ $(i < 0)$, $F_i M = M$ $(i \geq 0)$. Then $F$ defines a good filtration on $M$ and $\operatorname{gr}^F M \simeq M \simeq \mathcal{O}_X^r$ holds locally. Moreover, since $\Theta_X \subset \operatorname{Ann}_{\pi_* \mathcal{O}_{T^*X}}(\operatorname{gr}^F M)$, we get $\operatorname{Ch}(M) = T^*_X X = s(X) \simeq X$ ($s : x \mapsto (x, 0)$, the *zero-section* of $T^*X$) and $\mathbf{CC}(M) = r\, T^*_X X$.

Conversely, integrable connections are characterized by their characteristic varieties as follows.

**Proposition 2.2.5.** *For a non-zero coherent $D_X$-module $M$ the following three conditions are equivalent:*

(i) *$M$ is an integrable connection.*
(ii) *$M$ is coherent over $\mathcal{O}_X$.*
(iii) $\mathrm{Ch}(M) = T^*_X X \simeq X$ *(the zero-section of $T^*X$).*

*Proof.* Since the equivalence (i) $\Leftrightarrow$ (ii) is already proved in Theorem 1.4.10 and (i) $\Rightarrow$ (iii) is explained in Example 2.2.4, it remains to prove the part (iii) $\Rightarrow$ (ii). Since the problem is local, we may assume that $X$ is an affine algebraic variety with a local coordinate system $\{x_i, \partial_i\}_{1\le i\le n}$. Then we have $T^*X = X \times \mathbb{C}^n$. Assume that $\mathrm{Ch}(M) = T^*_X X$. This means that for a good filtration $F$ of $M$ we have

$$\sqrt{\mathrm{Ann}_{\mathcal{O}_X[\xi_1,\dots,\xi_n]}(\mathrm{gr}^F M)} = \sum_{i=1}^{n} \mathcal{O}_X[\xi]\,\xi_i.$$

Here we denote by $\xi_i$ the principal symbol of $\partial_i$, and we identify $\pi_*\mathcal{O}_{T^*X}$ with $\mathcal{O}_X[\xi_1, \dots, \xi_n]$. Now let us set $I = \sum_{i=1}^n \mathcal{O}_X[\xi]\,\xi_i$. Since the ideal $I$ is noetherian, we have

$$I^{m_0} \subset \mathrm{Ann}_{\mathcal{O}_X[\xi_1,\dots,\xi_n]}(\mathrm{gr}^F M)$$

for $m_0 \gg 0$. Since the set $\{\xi^\alpha \mid |\alpha| = m_0\}$ generates the ideal $I^{m_0}$, we have

$$\partial^\alpha F_j M \subset F_{j+m_0-1} M \quad (|\alpha| = m_0).$$

On the other hand, since $F$ is a good filtration, we have $F_i D_X F_j M = F_{i+j} M$ $(j \gg 0)$. It follows that

$$\begin{aligned} F_{m_0+j} M &= (F_{m_0} D_X)(F_j M) \\ &= \sum_{|\alpha|\le m_0} \mathcal{O}_X \partial^\alpha F_j M \\ &\subset F_{j+m_0-1} M \quad (j \gg 0). \end{aligned}$$

This means $F_{j+1}M = F_j M = M$ $(j \gg 0)$. Since each $F_j M$ is coherent over $\mathcal{O}_X$, $M$ is also $\mathcal{O}_X$-coherent. □

**Exercise 2.2.6.** For a coherent $D_X$-module $M = D_X u \simeq D_X/I$ $(I = \mathrm{Ann}_{D_X} u)$ consider the good filtration $F_i M = (F_i D_X)u$. If we define a filtration on $I$ by $F_i I := F_i D_X \cap I$, we have $\mathrm{gr}^F M \simeq \mathrm{gr}^F D_X/\,\mathrm{gr}^F I$. In this case, the graded ideal $\mathrm{gr}^F I := \sum_{i\ge 0} F_i I/F_{i-1}I$ is generated by the principal symbols $\sigma(P)$ of $P \in I$. Therefore, for an arbitrary chosen set $\{\sigma(P_i) \mid 1 \le i \le m\}$ of generators of $\mathrm{gr}^F I$, we have $I = \sum_{i=1}^m D_X P_i$ and

$$\mathrm{Ch}(M) = \{\,(x,\xi) \in T^*X \mid \sigma(P_i)(x,\xi) = 0,\ 1 \le i \le m\,\}.$$

However, for a set $\{Q_i\}$ of generators of $I$, the equality

$$\mathrm{Ch}(M) = \{\,(x,\xi) \mid \sigma(Q_i)(x,\xi) = 0,\ 1 \le i \le m\}$$

does not always hold. In general, we have only the inclusion

$$\mathrm{Ch}(M) \subset \{\,(x,\xi) \mid \sigma(Q_i)(x,\xi) = 0,\ \ 1 \le i \le m\}.$$

Find an example so that this inclusion is strict.

**Remark 2.2.7.** In general, it is not easy to compute the characteristic variety of a given coherent $D$-module as seen from Exercise 2.2.6. However, thanks to recent advances in the theory of *computational algebraic analysis* we now have an effective algorithm to compute their characteristic varieties. Moreover, we can now compute most of the operations of $D$-modules by computer programs. For example, we refer to [Oa1], [Oa2], [SST], [Ta]. It is also an interesting problem to determine various invariants of special holonomic $D$-modules introduced in the theory of hypergeometric functions of several variables (see [AK], [GKZ]).

## 2.3 Dimensions of characteristic varieties

One of the most fundamental results in the theory of $D$-modules is the following result about the characteristic varieties of coherent $D$-modules.

**Theorem 2.3.1.** *The characteristic variety of any coherent $D_X$-module is* **involutive** *with respect to the symplectic structure of the cotangent bundle $T^*X$.*

This result was first established by Sato–Kawai–Kashiwara [SKK] by an analytic method. Different proofs were also given by Malgrange [Ma5], Gabber [Ga], and Kashiwara–Schapira [KS2]. Here, we only note that in view of Lemma E.2.3 it is a consequence of Gabber's theorem (Theorem D.3.4), which is a deep result on a certain class of filtered rings (the proof of Theorem D.3.4 is not given in this book).

An important consequence of Theorem 2.3.1 is the following result.

**Corollary 2.3.2.** *Let $M$ be a coherent $D_X$-module. Then for any irreducible component $\Lambda$ of $\mathrm{Ch}(M)$ we have $\dim \Lambda \ge \dim X$. In particular, we have $\dim \mathrm{Ch}(M) \ge \dim X$ if $M \ne 0$.*

**Remark 2.3.3.** Note that Corollary 2.3.2 is weaker than Theorem 2.3.1; however, the weaker statement Corollary 2.3.2 is almost sufficient for arguments in this book. In fact, we will need the stronger statement Theorem 2.3.1 (or rather its analytic counterpart Theorem 4.1.3 below) only in the proof of Kashiwara's constructibility theorem for solutions of analytic holonomic $D$-modules (Theorem 4.6.3 below). Since we will also present a proof of the corresponding fact for algebraic holonomic $D$-modules due to Beilinson–Bernstein without using Theorem 2.3.1 (see Theorem 4.7.7 below), the readers who are only interested in algebraic $D$-modules can skip Section 4.6.

In the rest of this section we will give a direct proof of Corollary 2.3.2 following Kashiwara [Kas16].

We first establish the following result.

**Theorem 2.3.4.** *For any coherent $D_X$-module $M$ there exists a canonical filtration*

$$0 = C^{2\dim X+1}M \subset C^{2\dim X}M \subset \cdots \subset C^1M \subset C^0M = M$$

*of $M$ by coherent $D_X$-modules such that any irreducible component of*

$$\mathrm{Ch}(C^sM/C^{s+1}M)$$

*is $s$-codimensional in $T^*X$.*

*Proof.* Let $U$ be an affine open subset of $X$. We apply the result in Section D.5 to $A = D_X(U)$. Then by Lemma D.5.1 and Theorem D.5.3, together with Proposition 1.4.13, we obtain a filtration

$$0 = C^{2\dim X+1}(M|_U) \subset C^{2\dim X}(M|_U) \subset \cdots \subset C^0(M|_U) = M|_U$$

of $M|_U$ by coherent $D_U$-modules such that any irreducible component of $\mathrm{Ch}(C^s(M|_U)/C^{s+1}(M|_U))$ is $s$-codimensional in $T^*U$. We see by the cohomological description of the filtration given in Proposition D.5.2 that it is canonical and globally defined on $X$. □

**Lemma 2.3.5.** *Let $S$ be a smooth closed subvariety of $X$ and let $M$ be a coherent $D_S$-module. Set $N = \int_i^0 M$, where $i : S \hookrightarrow X$ denotes the embedding. Let $\rho_i : S \times_X T^*X \to T^*S$ and let $\varpi_i : S \times_X T^*X \hookrightarrow T^*X$ be natural morphisms induced by $i$. Then we have*

$$\mathrm{Ch}(N) = \varpi_i\rho_i^{-1}(\mathrm{Ch}(M)).$$

*Proof.* Note that the problem is local on $S$. By induction on the codimension of $S$ one can reduce the problem to the case where $S$ is a hypersurface of $X$ (see Proposition 1.5.21 and Lemma 2.4.1 below). Assume that $S$ is a hypersurface of $X$ defined by $x = 0$. Take a local coordinate $\{x_i, \partial_i\}_{1\le i\le n}$ of $X$ such that $x = x_1$ and set $\partial = \partial_1$. Then we obtain a local identification $N \simeq \mathbb{C}[\partial] \otimes_{\mathbb{C}} i_*M$ (see Section 1.5). Take a good filtration $G$ of $M$ such that $G_{-1}M = 0$, and define a filtration $F$ of $N$ by

$$F_jN = \sum_{l=0}^{j}\sum_{k\le l} \mathbb{C}\partial^k \otimes i_*G_{j-l}(M).$$

Then $F$ is a good filtration of $N$ satisfying

$$F_jN/F_{j-1}N = \bigoplus_{l=0}^{j} \mathbb{C}\partial^l \otimes i_*(G_{j-l}M/G_{j-l-1}M).$$

Hence we have

$$\mathrm{gr}^F N \simeq \mathbb{C}[\xi] \otimes_{\mathbb{C}} \mathrm{gr}^G M \simeq (\mathbb{C}[x,\xi]/\mathbb{C}[x,\xi]x) \otimes_{\mathbb{C}} \mathrm{gr}^G M,$$

where $\xi$ is the principal symbol of $\partial$. From this we easily see that

$$\mathrm{Ch}(N) = \mathrm{supp}\,\widetilde{\mathrm{gr}^F N} = \varpi_i\rho_i^{-1}(\mathrm{supp}\,\widetilde{\mathrm{gr}^G M} = \varpi_i\rho_i^{-1}(\mathrm{Ch}(M)).$$

The proof is complete. □

*Proof of Corollary* 2.3.2. By Theorem 2.3.4 we have only to show that

$$\dim \mathrm{Ch}(M) \geq \dim X$$

for any non-zero coherent $D_X$-module $M$. We prove it by induction on $\dim X$. It is trivial in the case $\dim X = 0$. Assume that $\dim X > 0$. If $\mathrm{supp}\, M = X$, then we have $\mathrm{Ch}(M) \supset T^*_X X$, and hence $\dim \mathrm{Ch}(M) \geq \dim T^*_X X = \dim X$. Therefore, we may assume from the beginning that $\mathrm{supp}\, M$ is a proper closed subset of $X$. By replacing $X$ with a suitable open subset (if necessary) we may further assume that $\mathrm{supp}\, M$ is contained in a smooth hypersurface $S$ in $X$. Let $i : S \to X$ be the embedding. By Kashiwara's equivalence there exists a non-zero coherent $D_S$-module $L$ satisfying $M = \int_i^0 L$. Then by Lemma 2.3.5 we have $\mathrm{Ch}(M) = \varpi_i \rho_i^{-1}(\mathrm{Ch}(L))$ and hence $\dim \mathrm{Ch}(M) = \dim \mathrm{Ch}(L) + 1$. On the other hand, the hypothesis of induction implies $\dim \mathrm{Ch}(L) \geq \dim S = \dim X - 1$. It follows that $\dim \mathrm{Ch}(M) \geq \dim S + 1 = \dim X$. □

**Definition 2.3.6.** A coherent $D_X$-module $M$ is called a *holonomic* $D_X$-module (or a *holonomic system*, or a *maximally overdetermined system*) if it satisfies $\dim \mathrm{Ch}(M) \leqq \dim X$.

By Theorem 2.3.1 characteristic varieties of holonomic $D$-modules are $\mathbb{C}^\times$-invariant Lagrangian subset of $T^*X$.

Holonomic $D_X$-modules are the coherent $D_X$-modules whose characteristic variety has minimal possible dimension $\dim X$. Assume that the dimension of the characteristic variety $\mathrm{Ch}(M)$ is "small." This means that the ideal defining the corresponding system of differential equations is "large," and hence the space of the solutions should be "small." In fact, we will see later that the holonomicity is related to the finite dimensionality of the solution space.

**Example 2.3.7.** Integrable connections are holonomic by Proposition 2.2.5.

**Example 2.3.8.** The $D_X$-module $\mathcal{B}_{Y|X}$ for a closed smooth subvariety $Y$ of $X$ is holonomic (see Example 1.6.4). In this case the characteristic variety $\mathrm{Ch}(\mathcal{B}_{Y|X})$ is the conormal bundle $T^*_Y X$ of $Y$ in $X$.

## 2.4 Inverse images in the non-characteristic case

We have shown in Proposition 1.5.13 that the inverse image of a coherent $D$-module with respect to a smooth morphism is again coherent; however, the inverse images with respect to non-smooth morphisms do not necessarily preserve coherency as we saw in Example 1.5.10. In this section we will give a sufficient condition on a coherent $D$-module $M$ so that its inverse image is again coherent.

For a morphism $f : X \to Y$ of smooth algebraic varieties there are associated natural morphisms

$$T^*X \xleftarrow{\rho_f} X \times_Y T^*Y \xrightarrow{\varpi_f} T^*Y.$$

Note that if $f$ is a closed embedding (resp. smooth), then $\rho_f$ is smooth (resp. a closed embedding) and $\varpi_f$ is a closed embedding (resp. smooth). We set

$$T^*_X Y := \rho_f^{-1}(T^*_X X) \subset X \times_Y T^*Y.$$

When $f$ is a closed embedding, $T^*_X Y$ is the conormal bundle of $X$ in $Y$.

The following is easily checked.

**Lemma 2.4.1.** *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of smooth algebraic varieties. Then we have the natural commutative diagram*

$$\begin{array}{ccccc}
T^*X & \xleftarrow{\rho_f} & X \times_Y T^*Y & \xleftarrow{\varphi} & X \times_Z T^*Z \\
 & & \downarrow{\scriptstyle \varpi_f} & & \downarrow{\scriptstyle \psi} \\
 & & T^*Y & \xleftarrow[\rho_g]{} & Y \times_Z T^*Z \\
 & & & & \downarrow{\scriptstyle \varpi_g} \\
 & & & & T^*Z
\end{array}$$

*such that $\rho_f \circ \varphi = \rho_{g\circ f}$, $\varpi_g \circ \psi = \varpi_{g\circ f}$, and the square in the right upper corner is cartesian.*

**Definition 2.4.2.** Let $f : X \to Y$ be a morphism of smooth algebraic varieties and let $M$ be a coherent $D_Y$-module. We say that $f$ is *non-characteristic* with respect to $M$ if the condition

$$\varpi_f^{-1}(\mathrm{Ch}(M)) \cap T^*_X Y \subset X \times_Y T^*_Y Y$$

is satisfied.

**Remark 2.4.3.** We can easily show that if a closed embedding $f : X \hookrightarrow Y$ is non-characteristic with respect to a coherent $D_Y$-module $M$, then $\rho_f|_{\varpi_f^{-1}(\mathrm{Ch}(M))} : \varpi_f^{-1}(\mathrm{Ch}(M)) \longrightarrow T^*X$ is a finite morphism.

This definition is motivated by the theory of linear partial differential equations, as we see below.

**Example 2.4.4.** Consider the case where $f : X \to Y$ is the embedding of a hypersurface. Then the conormal bundle $T^*_X Y$ is a line bundle on $X$. Let $P \in D_Y$ be a differential operator of order $m \geq 0$ and set $M = D_Y/D_Y P$. In this case $\mathrm{Ch}(M)$ is exactly the zero set of the principal symbol $\sigma_m(P)$, and hence $f$ is non-characteristic with respect to the coherent $D_Y$-module $M$ if and only if

$$(\sigma_m(P))(\xi) \neq 0 \quad (\forall \xi \in T^*_X Y \setminus (\text{the zero-section of } T^*_X Y)).$$

Take a local coordinate $\{z_i, \partial_i\}_{1\leq i\leq n}$ of $Y$ such that $z_1$ is the defining equation of $X$, and let $(z_1, \ldots, z_n; \zeta_1, \ldots, \zeta_n)$ be the corresponding coordinate of $T^*Y$. Then the condition can be written as

$$\sigma_m(P)\big(0, z_2, \dots, z_n; 1, 0, \dots, 0\big) \neq 0 \qquad (\forall (z_2, \dots, z_n)),$$

or equivalently as

$$\frac{\partial^m}{\partial \zeta_1^m}\sigma_m(P)\big(0, z_2, \dots, z_n; 0, \dots, 0\big) \neq 0 \qquad (\forall (z_2, \dots, z_n)).$$

In the classical analysis, if this is the case, we say that $Y$ is a non-characteristic hypersurface of $X$ with respect to the differential operator $P$.

Let us show that $H^0(Lf^*M)$ is a locally free $D_X$-module of rank $m$. By definition we have

$$\begin{aligned} &H^0(Lf^*M) \\ &\quad = (D_Y/z_1 D_Y) \otimes_{D_Y} (D_Y/D_Y P) \\ &\quad \simeq D_Y/(z_1 D_Y + D_Y P). \end{aligned}$$

Set $D' = \sum_{(j_2,\dots,j_n)} \mathcal{O}_Y \partial_2^{j_2} \dots \partial_n^{j_n} \subset D_Y$. By the above consideration we may assume that $P$ is of the form

$$P = \partial_1^m + \sum_{i=0}^{m-1} P_i \partial_1^i \qquad (P_i \in D').$$

We will show that

$$\begin{array}{ccc} D_X^{\oplus m} & \longrightarrow & D_Y/(z_1 D_Y + D_Y P) \\ \cup & & \cup \\ (Q_0, Q_1, \dots, Q_{m-1}) & \longmapsto & \sum_{j=0}^{m-1} Q_j \partial_1^j \end{array}$$

is an isomorphism of $D_X$-modules. For this we have only to show that for any $R \in D_Y$ there exist uniquely $Q \in D_Y$ and $R_0, \dots, R_{m-1} \in D'$ satisfying

$$R = QP + \sum_{j=0}^{m-1} R_j \partial_1^j.$$

Note that $D_Y = \bigoplus_{j=0}^{\infty} D' \partial_1^j$. Hence we can write uniquely that

$$R = \sum_{j=0}^{p} S_j \partial_1^j \ (S_j \in D').$$

If $p \geq m$, then $R - S_p \partial_1^{p-m} P \in \sum_{j=0}^{p-1} D' \partial_1^j$. Hence we obtain the existence of $Q$ and $R_0, \dots, R_{m-1}$ as above by induction on $p$. In order to show uniqueness it is sufficient to show that $D_Y P \cap (\sum_{j=0}^{m-1} D' \partial_1^j) = 0$. Assume that for $Q \in D_Y$ we have $QP \in \bigoplus_{j=0}^{m-1} D' \partial_1^j$. If $Q \neq 0$, we can write $Q = \sum_{j=0}^{p} T_j \partial_i^j \ (T_j \in D')$ with $T_p \neq 0$. Then we have $QP \in T_p \partial_1^{m+p} + \sum_{j=0}^{m+p-1} D' \partial_1^j$. This is a contradiction. Hence we have $Q = 0$.

**Example 2.4.5.** A smooth morphism $f : X \to Y$ is non-characteristic with respect to any coherent $D_Y$-module.

The aim of this section is to prove the following.

**Theorem 2.4.6.** *Let $f : X \to Y$ be a morphism of smooth algebraic varieties and let $M$ be a coherent $D_Y$-module. Assume that $f$ is non-characteristic with respect to $M$.*

(i) $H^j(Lf^*M) = 0$ *for* $^\forall j \neq 0$.
(ii) $H^0(Lf^*M)$ *is a coherent $D_X$-module.*
(iii) $\mathrm{Ch}(H^0(Lf^*M)) \subset \rho_f \varpi_f^{-1}(\mathrm{Ch}\, M)$.

For the proof we need the following.

**Lemma 2.4.7.** *Let $f : X \to Y$ be an embedding of a hypersurface and let $M$ be a coherent $D_Y$-module. Assume that $f$ is non-characteristic with respect to $M$. Then for any $u \in M$ there exists locally a differential operator $P \in D_Y$ such that $Pu = 0$ and $f$ is non-characteristic with respect to $D_Y/D_Y P$. In particular, there exists locally an exact sequence*

$$\bigoplus_{i=1}^{r} D_Y/D_Y P_i \to M \to 0,$$

*where $f$ is non-characteristic with respect to $D_Y/D_Y P_i$ for any $i$.*

*Proof.* It follows from $\mathrm{Ch}(D_Y u) \subset \mathrm{Ch}(M)$ that $f$ is also non-characteristic with respect to the $D_Y$-submodule $D_Y u$ of $M$. Note that $\mathrm{Ch}(D_Y u)$ is the zero-set of $\mathrm{gr}^F I$ for $I = \{Q \in D_Y \mid Qu = 0\}$. Since $T^*_X Y$ is a line bundle on $X$, there exists locally $P \in I$ such that $f$ is non-characteristic with respect to $D_Y/D_Y P$. □

*Proof of Theorem* 2.4.6.
**(Step 1)** We first consider the case when $X$ is a hypersurface $\{z_1 = 0\}$ of $Y$.

Let us show (i). Since $Lf^*M \in D^b(D_X)$ is represented by the complex

$$f^{-1}M \xrightarrow{z_1} f^{-1}M$$

concentrated in degrees $-1$ and 0, it suffices to show that $f^{-1}M \xrightarrow{z_1} f^{-1}M$ is injective. Assume that $u \in M$ satisfies $z_1(f^{-1}u) = 0$. By Lemma 2.4.7 there exists $P \in D_Y$ such that $Pu = 0$ and $f$ is non-characteristic with respect to $D_Y/D_Y P$. Then $P \in D_Y$ is a differential operator of the form in Example 2.4.4. Let $m \geq 0$ be the order of $P$ and set $\mathrm{ad}_{z_1}(P) = [z_1, P] = z_1 P - P z_1 \in D_Y$. Then $\mathrm{ad}^m_{z_1}(P) \in D_Y$ is a multiplication by an invertible function. Hence from $\mathrm{ad}^m_{z_1}(P)u = 0$ we obtain $u = 0$. The assertion (i) is proved.

Let us show (ii) and (iii). Take a good filtration $F$ of $M$. Then $\mathrm{gr}^F M$ is a coherent $\mathrm{gr}^F D_Y$-module such that the support of the associated coherent $\mathcal{O}_{T^*Y}$-module

$$\widetilde{\mathrm{gr}^F M} := \mathcal{O}_{T^*Y} \otimes_{\pi_Y^{-1}\mathrm{gr}^F D_Y} \pi_Y^{-1}(\mathrm{gr}^F M)$$

is $\mathrm{Ch}(M)$, where $\pi_Y : T^*Y \to Y$ denotes the projection. We set $N = f^*M$ $(= H^0(Lf^*M))$ and define a filtration $F$ of $N$ by

$$F_iN = \mathrm{Im}(f^*F_iM \to f^*M).$$

It is sufficient to show that $\mathrm{gr}^F N$ is a coherent $\mathrm{gr}^F D_X$-module such that the support of the associated coherent $\mathcal{O}_{T^*X}$-module

$$\widetilde{\mathrm{gr}^F N} := \mathcal{O}_{T^*X} \otimes_{\pi_X^{-1}\mathrm{gr}^F D_X} \pi_X^{-1}(\mathrm{gr}^F N)$$

is contained in $\rho_f \varpi_f^{-1}(\mathrm{Ch}(M))$. Note that we have a canonical epimorphism $f^*\mathrm{gr}^F M \to \mathrm{gr}^F N$. Set

$$\widetilde{f^*\mathrm{gr}^F M} := \mathcal{O}_{T^*X} \otimes_{\pi_X^{-1}\mathrm{gr}^F D_X} \pi_X^{-1}(f^*\mathrm{gr}^F M).$$

Since $f$ is non-characteristic with respect to $M$, we have that the restriction $\varpi_f^{-1}(\mathrm{supp}\,\widetilde{\mathrm{gr}^F M}) \longrightarrow T^*X$ of $\rho_f$ to $\varpi_f^{-1}(\mathrm{supp}\,\widetilde{\mathrm{gr}^F M})$ is a finite morphism. Hence it follows from a standard fact in algebraic geometry that

$$\widetilde{f^*\mathrm{gr}^F M} = (\rho_f)_*\varpi_f^*\widetilde{\mathrm{gr}^F M}.$$

In particular, $\widetilde{f^*\mathrm{gr}^F M}$ is a coherent $\mathcal{O}_{T^*X}$-module whose support is contained in $\rho_f\varpi_f^{-1}(\mathrm{Ch}(M))$. The coherence of $\widetilde{f^*\mathrm{gr}^F M}$ over $\mathcal{O}_{T^*X}$ implies the coherence of $f^*\mathrm{gr}^F M$ over $\mathrm{gr}^F D_X$. It remains to show that $\mathrm{gr}^F N$ is a coherent $\mathrm{gr}^F D_X$-module. Since $f^*\mathrm{gr}^F M$ is a coherent $\mathrm{gr}^F D_X$-module, it is sufficient to show that $F_iN$ is coherent over $\mathcal{O}_X$ for each $i$ (see Proposition 2.1.1). This follows from the definition of $F_iN$ since $f^*F_iM$ is coherent and $f^*M$ is quasi-coherent over $\mathcal{O}_X$.

**(Step 2)** We treat the case when $f : X \longrightarrow Y$ is a general closed embedding. We can prove the assertion by induction on the codimension of $X$ using Lemma 2.4.1 as follows (details are left to the readers). The case when $\mathrm{codim}_Y X = 1$ was treated in **Step 1**. In the general case, we can locally factorize $f : X \hookrightarrow Y$ as a composite of $X \overset{g}{\hookrightarrow} Z \overset{h}{\hookrightarrow} Y$ where $g$ and $h$ are closed embeddings of smooth varieties with $\mathrm{codim}_Z X, \mathrm{codim}_Y Z < \mathrm{codim}_Y X$. Lemma 2.4.1 and our assumption on $M$ implies that there exists an open neighborhood $U$ of $X$ in $Z$ satisfying $\varpi_h^{-1}(\mathrm{Ch}(M)) \cap T^*_U Y \subset U \times_Y T^*_Y Y$. Hence we may assume that $Z$ is non-characteristic with respect to $M$ from the beginning. Then by our hypothesis of induction we have $H^i(Lh^*M) = 0$ for $i \neq 0$ and $L = H^0(Lh^*M)$ is a coherent $D_Z$-module with $\mathrm{Ch}(L) \subset \rho_h\varpi_h^{-1}(\mathrm{Ch}(M))$. We easily see by Lemma 2.4.1 that $g$ is non-characteristic with respect to $L$. Hence by our hypothesis of induction we have $H^i(Lf^*M) = H^i(Lg^*L) = 0$ for $i \neq 0$ and $H^0(Lf^*M) = H^0(Lg^*L)$ is a coherent $D_X$-module satisfying

$$\begin{aligned}\mathrm{Ch}(H^0(Lf^*M)) &= \mathrm{Ch}(H^0(Lg^*L)) \subset \rho_g\varpi_g^{-1}(\mathrm{Ch}(L)) \\ &\subset \rho_g\varpi_g^{-1}\rho_h\varpi_h^{-1}(\mathrm{Ch}(M)) = \rho_f\varpi_f^{-1}(\mathrm{Ch}(M)).\end{aligned}$$

**(Step 3)** If $f : X = Y \times Z \longrightarrow Y$ is the first projection, then the assertions follows easily from the isomorphism $Lf^*M \simeq M \boxtimes \mathcal{O}_Z$.

**(Step 4)** To handle the case of a general morphism $f : Y \longrightarrow X$, we may factorize $f$ as $Y \xrightarrow{g} Y \times X \xrightarrow{p} X$, where $g$ is the graph embedding defined by $y \mapsto (y, f(y))$ and $p$ is the second projection. Then the result follows from **Step 2** and **Step 3** by using Lemma 2.4.1 and the arguments similar to those in **Step 2**. □

**Remark 2.4.8.** Under the assumption of Theorem 2.4.6 it is known that we have actually

$$\mathrm{Ch}(H^0(Lf^*M)) = \rho_f \varpi_f^{-1}(\mathrm{Ch}(M))$$

(see [Kas8] and [Kas18]).

## 2.5 Proper direct images

In this section we show the following.

**Theorem 2.5.1.** *Let $f : X \to Y$ be a proper morphism. Then for an object $M^\cdot$ in $D_c^b(D_X)$ the direct image $\int_f M^\cdot$ belongs to $D_c^b(D_Y)$.*

*Proof.* Since we assumed that $X$ and $Y$ are quasi-projective, $f$ is a projective morphism. Namely, $f$ is factorized as

$$X \xhookrightarrow{i} Y \times \mathbb{P}^n \xrightarrow{p} Y$$

by a closed embedding $i$ ($i(x) = (f(x), j(x))$, $j : X \hookrightarrow \mathbb{P}^n$) and a projection $p = \mathrm{pr}_Y$ to $Y$. Hence it is enough to prove our theorem for each case.

(i) The case of closed embeddings $i : X \hookrightarrow Y$: The problem being local on $Y$, we may take a free resolution $F^\cdot \simeq M^\cdot$ of $M^\cdot \in D_c^b(D_X)$ such that each term $F^j$ is isomorphic to $D_X^{n_j}$. Using the exactness of the functor $\int_i$ we have only to prove the coherence of $\int_i D_X$ over $D_Y$. We see by

$$\begin{aligned}\int_i D_X &= i_*(D_{Y \leftarrow X} \otimes_{D_X} D_X) = i_*(i^{-1}(D_Y \otimes_{\mathcal{O}_Y} \Omega_Y^{\otimes -1}) \otimes_{i^{-1}\mathcal{O}_Y} \Omega_X) \\ &= D_Y \otimes_{\mathcal{O}_Y} (\Omega_Y^{\otimes -1} \otimes_{\mathcal{O}_Y} i_*\Omega_X)\end{aligned}$$

that $\int_i D_X$ is locally isomorphic to $D_Y/D_Y I_X$ where $I_X \subset \mathcal{O}_Y$ is the defining ideal of $X$. In particular, it is coherent.

(ii) The case of projections $p : X = Y \times \mathbb{P}^n \to Y$: Since the problem is local on $Y$, we may assume that $Y$ is an affine variety. By Theorem 1.6.5 and Proposition 1.4.13 there exists a resolution $F^\cdot \simeq M^\cdot$ of $M^\cdot$ in $D_c^b(D_X)$, where $F^\cdot$ is a bounded complex of $D_X$-modules such that each term $F^j$ of $F^\cdot$ is a direct summand of a free $D_X$-module of finite rank. Then it is sufficient to show $\int_p F^j \in D_c^b(D_Y)$ for any $j$. Assume that $F^j$ is a direct summand of $D_X^n$. Then $\int_p F^j \in D_{qc}^b(D_Y)$ is a direct summand of $\int_p D_X^n$. Hence it is enough to show $\int_p D_X \in D_c^b(D_Y)$. By

$$D_{Y\leftarrow X} = \Omega_{Y\times\mathbb{P}^n} \otimes_{\mathcal{O}_{Y\times\mathbb{P}^n}} p^*\big(D_Y \otimes_{\mathcal{O}_Y} \Omega_Y^{\otimes -1}\big) \simeq D_Y \boxtimes \Omega_{\mathbb{P}^n},$$

we have

$$\int_p D_X = Rp_*(D_{Y\leftarrow X} \otimes^L_{D_X} D_X) \simeq Rp_*(D_Y \boxtimes \Omega_{\mathbb{P}^n}) \simeq D_Y \otimes_{\mathbb{C}} R\Gamma(\mathbb{P}^n, \Omega_{\mathbb{P}^n}).$$

Now we recall that the only non-vanishing cohomology group of $R\Gamma(\mathbb{P}^n, \Omega_{\mathbb{P}^n})$ is $H^n(\mathbb{P}^n, \Omega_{\mathbb{P}^n}) \simeq \mathbb{C}$. Therefore, we get that $R\Gamma(\mathbb{P}^n, \Omega_{\mathbb{P}^n}) \simeq \mathbb{C}[-n]$ and $\int_p D_X \simeq D_Y[-n]$. □

**Remark 2.5.2.** Under the assumption of Theorem 2.5.1 it is known that we have

$$\mathrm{Ch}\Big(\int_f M^{\cdot}\Big) \subset \varpi_f \rho_f^{-1}(\mathrm{Ch}(M^{\cdot})).$$

As we saw in Lemma 2.3.5 the equality holds in the case where $f$ is a closed embedding. The proof for the general case is more involved.

## 2.6 Duality functors

We first try to find heuristically a candidate for the "*dual*" of a left $D$-module. Let $M$ be a left $D_X$-module. Then $\mathcal{H}om_{D_X}(M, D_X)$ is a *right* $D_X$-module by right multiplication of $D_X$ on $D_X$. By the side-changing functor $\otimes_{\mathcal{O}_X} \Omega_X^{\otimes -1}$ we obtain a left $D_X$-module $\mathcal{H}om_{D_X}(M, D_X) \otimes_{\mathcal{O}_X} \Omega_X^{\otimes -1}$. Since the functor $\mathcal{H}om_{D_X}(\bullet, D_X)$ is not exact, it is more natural to consider the complex $R\mathcal{H}om_{D_X}(M, D_X) \otimes_{\mathcal{O}_X} \Omega_X^{\otimes -1}$ of left $D_X$-modules. In order to judge which cohomology group of this complex deserves to be called "dual," let us consider the following example. Let $X = \mathbb{C}$ (or an open subset of $\mathbb{C}$) and $M = D_X/D_X P$ ($P \neq 0$). By applying the functor $\mathcal{H}om_{D_X}(\bullet, D_X)$ to the exact sequence

$$0 \longrightarrow D_X \xrightarrow{\times P} D_X \longrightarrow M \longrightarrow 0$$

of left $D_X$-modules we get an exact sequence

$$0 \longrightarrow \mathcal{H}om_{D_X}(M, D_X) \longrightarrow D_X \xrightarrow{P\times} D_X$$

(note $\mathcal{H}om_{D_X}(D_X, D_X) \simeq D_X$). Hence in this case, we have

$$\mathcal{E}xt^0_{D_X}(M, D_X) = \mathcal{H}om_{D_X}(M, D_X) = \mathrm{Ker}(P : D_X \to D_X) = 0$$

and the only non-vanishing cohomology group is the first one

$$\mathcal{E}xt^1_{D_X}(M, D_X) \simeq D_X/P D_X.$$

The left $D_X$-module obtained by the side changing $\otimes\, \Omega_X^{\otimes -1}$ is isomorphic to

$$\mathcal{E}xt^1_{D_X}(M,\ D_X)\otimes_{\mathcal{O}_X}\Omega_X^{\otimes -1}\simeq D_X/D_XP^*,$$

where $P^*$ is the formal adjoint of $P$. From this calculation, we see that $\mathcal{E}xt^1$ is more suited than $\mathcal{E}xt^0$ to be called "dual" of $M$. More generally, if $n=\dim X$ and $M$ is a holonomic $D_X$-module, then we can (and will) prove that only the term $\mathcal{E}xt^n_{D_X}(M,D_X)$ survives and the resulting left $D_X$-module $\mathcal{E}xt^n_{D_X}(M,D_X)\otimes_{\mathcal{O}_X}\Omega_X^{\otimes -1}$ is also holonomic. Hence the correct definition of the dual $\mathbb{D}M$ of a holonomic $D_X$-module $M$ is given by $\mathbb{D}M=\mathcal{E}xt^n_{D_X}(M,D_X)\otimes_{\mathcal{O}_X}\Omega_X^{\otimes -1}$. For a non-holonomic $D_X$-module one may have other non-vanishing cohomology groups, and hence the duality functor should be defined as follows for the derived categories.

**Definition 2.6.1.** We define the *duality functor* $\mathbb{D}=\mathbb{D}_X: D^-(D_X)\to D^+(D_X)^{\mathrm{op}}$ by

$$\begin{aligned}\mathbb{D}M^{\cdot}&:=\mathrm{R}\mathcal{H}om_{D_X}(M^{\cdot},\ D_X)\otimes_{\mathcal{O}_X}\Omega_X^{\otimes -1}[\dim X]\\&=\mathrm{R}\mathcal{H}om_{D_X}(M^{\cdot},\ D_X\otimes_{\mathcal{O}_X}\Omega_X^{\otimes -1}[\dim X])\qquad (M^{\cdot}\in D^-(D_X)).\end{aligned}$$

We use the following notation since shifts of complexes by dimensions of varieties will often appear in the subsequent parts.

**Notation 2.6.2.** For an algebraic variety $X$ we denote its dimension $\dim X$ by $d_X$.

**Example 2.6.3.** We have

$$H^k(\mathbb{D}D_X)=\begin{cases}D_X\otimes_{\mathcal{O}_X}\Omega_X^{\otimes -1} & (k=-d_X),\\ 0 & (k\neq -d_X).\end{cases}$$

**Lemma 2.6.4.** *Let $M$ be a coherent $D_X$-module. Then for any affine open subset $U$ of $X$ we have*

$$(\mathcal{E}xt^i_{D_X}(M,D_X))(U)=\mathrm{Ext}^i_{D_X(U)}(M(U),D_X(U)).$$

*Proof.* Take a resolution $P^{\cdot}\to M|_U$ of $M|_U$ by free $D_U$-modules of finite rank. Since $U$ is affine, $P^{\cdot}(U)\to M(U)$ gives a resolution of $M(U)$ by free $D_X(U)$-modules of finite rank. By definition we have $(\mathcal{E}xt^i_{D_X}(M,D_X))(U)=(H^i(\mathcal{H}om_{D_U}(P^{\cdot},D_U)))(U)$. Set $L^{\cdot}=\mathcal{H}om_{D_U}(P^{\cdot},D_U)$. Since $U$ is affine and $L^{\cdot}$ is a complex of coherent right $D_U$-modules, we have $H^i(L^{\cdot})(U)=H^i(L^{\cdot}(U))$ (see Remark 1.4.5 (ii)). Moreover, we have

$$L^{\cdot}(U)=\mathrm{Hom}_{D_U}(P^{\cdot},D_U)=\mathrm{Hom}_{D_X(U)}(P^{\cdot}(U),D_X(U)).$$

Here, the first equality is obvious, and the second equality follows easily from the fact that $P^{\cdot}$ is a complex of free $D_U$-modules (or one can use the $D$-affinity of $U$). Therefore, we obtain

$$\begin{aligned}(\mathcal{E}xt^i_{D_X}(M,D_X))(U)&=H^i(\mathrm{Hom}_{D_X(U)}(P^{\cdot}(U),D_X(U)))\\&=\mathrm{Ext}^i_{D_X(U)}(M(U),D_X(U)).\end{aligned}$$

The proof is complete. □

**Proposition 2.6.5.**
(i) *The functor* $\mathbb{D}$ *sends* $D^b_c(D_X)$ *to* $D^b_c(D_X)^{\mathrm{op}}$.
(ii) $\mathbb{D}^2 \simeq \mathrm{Id}$ *on* $D^b_c(D_X)$.

*Proof.* (i) We may assume that $M^{\cdot} = M \in \mathrm{Mod}_c(D_X)$. Then we see from (the proof of) Lemma 2.6.4 that $H^i(\mathbb{D}M) \in \mathrm{Mod}_c(D_X)$ for any $i$. The boundedness of $\mathbb{D}M$ also follows from Lemma 2.6.4 and Proposition 1.4.6 (ii).

(ii) We first construct a canonical morphism $M^{\cdot} \to \mathbb{D}^2 M^{\cdot}$ for $M^{\cdot} \in D^b(D_X)$. First note that

$$\mathbb{D}^2 M^{\cdot} \simeq R\mathcal{H}om_{D_X^{\mathrm{op}}}(R\mathcal{H}om_{D_X}(M^{\cdot}, D_X), D_X),$$

where $R\mathcal{H}om_{D_X}(M^{\cdot}, D_X)$ and $D_X$ are regarded as objects of $D^b(D_X^{\mathrm{op}})$ (complexes of right $D_X$-modules) by the right multiplication of $D_X$ on $D_X$, and the left $D_X$-action on the right-hand side is induced from the left multiplication of $D_X$ on $D_X$. Set $H^{\cdot} = R\mathcal{H}om_{D_X}(M^{\cdot}, D_X)$. By applying $H^0(R\Gamma(X, \bullet))$ to

$$R\mathcal{H}om_{D_X \otimes_{\mathbb{C}} D_X^{\mathrm{op}}}(M^{\cdot} \otimes_{\mathbb{C}} H^{\cdot}, D_X) \simeq R\mathcal{H}om_{D_X}(M^{\cdot}, R\mathcal{H}om_{D_X^{\mathrm{op}}}(H^{\cdot}, D_X)),$$

we obtain

$$\mathrm{Hom}_{D_X \otimes_{\mathbb{C}} D_X^{\mathrm{op}}}(M^{\cdot} \otimes_{\mathbb{C}} H^{\cdot}, D_X) \simeq \mathrm{Hom}_{D_X}(M^{\cdot}, R\mathcal{H}om_{D_X^{\mathrm{op}}}(H^{\cdot}, D_X)).$$

Hence the canonical morphism $M^{\cdot} \otimes_{\mathbb{C}} H^{\cdot} (= M^{\cdot} \otimes_{\mathbb{C}} R\mathcal{H}om_{D_X}(M^{\cdot}, D_X)) \to D_X$ in $D^b(D_X \otimes_{\mathbb{C}} D_X^{\mathrm{op}})$ gives rise to a canonical morphism

$$M^{\cdot} \to R\mathcal{H}om_{D_X^{\mathrm{op}}}(H^{\cdot}, D_X) (= \mathbb{D}^2 M)$$

in $D^b(D_X)$. It remains to show that $M^{\cdot} \to \mathbb{D}^2 M^{\cdot}$ is an isomorphism for $M^{\cdot} \in D^b_c(D_X)$. Since the question is local, we may assume that $X$ is affine. Then we can replace $M^{\cdot}$ with $D_X$ by Proposition 1.4.13 (see the proof of Theorem 2.5.1). In this case the assertion is clear. □

**Corollary 2.6.6.** $\mathbb{D}$ *is fully faithful on* $D^b_c(D_X)$.

The following theorem gives an estimate for the dimensions of the characteristic varieties $\mathrm{Ch}(H^i(\mathbb{D}M))$ for $M \in \mathrm{Mod}_c(D_X)$.

**Theorem 2.6.7.** *Let $X$ be a smooth algebraic variety and $M$ a coherent $D_X$-module.*
(i) $\mathrm{codim}_{T^*X} \mathrm{Ch}(\mathcal{E}xt^i_{D_X}(M,\ D_X) \otimes_{\mathcal{O}_X} \Omega_X^{\otimes -1}) \geq i$.
(ii) $\mathcal{E}xt^i_{D_X}(M,\ D_X) = 0 \quad (i < \mathrm{codim}_{T^*X} \mathrm{Ch}(M))$.

This theorem is a consequence of Theorem D.4.3 and Lemma 2.6.4.

**Corollary 2.6.8.** *Let $M$ be a coherent $D_X$-module.*
(i) $H^i(\mathbb{D}M) = 0$ *unless* $-(d_X - \mathrm{codim}_{T^*X} \mathrm{Ch}(M)) \leq i \leq 0$.

(ii) $\operatorname{codim}_{T^*X} \operatorname{Ch}(H^i(\mathbb{D}M)) \geq d_X + i$.
(iii) *$M$ is holonomic if and only if $H^i(\mathbb{D}M) = 0$ $(i \neq 0)$.*
(iv) *If $M$ is holonomic, then $\mathbb{D}M \simeq H^0(\mathbb{D}M)$ is also holonomic.*

*Proof.* The statements (i) and (ii) are just restatements of Theorem 2.6.7. The statement (iv) and the "only if" part of (iii) follows from (i), (ii) and Corollary 2.3.2. Let us show the "if" part of (iii). Assume that $H^i(\mathbb{D}M) = 0$ $(i \neq 0)$, i.e., $\mathbb{D}M \simeq H^0(\mathbb{D}M)$. Set $M^* = H^0(\mathbb{D}M)$. Then we have $\mathbb{D}M^* = \mathbb{D}^2M \simeq M$ and $H^0(\mathbb{D}M^*) \simeq M$ by the preceding result $\mathbb{D}^2 = \mathrm{Id}$. On the other hand by (ii) we have $\operatorname{codim} \operatorname{Ch}(H^0(\mathbb{D}M^*)) \geq d_X$, and hence $\mathbb{D}M^* \simeq M$ is a holonomic $D_X$-module. □

**Example 2.6.9.** Let $X = \mathbb{C}$ and $Y = \{0\}$. Then we have $\mathcal{B}_{Y|X} \simeq D_X/D_Xx$, where $x$ is the coordinate of $X$. Hence by the first part of this section we have

$$\mathbb{D}\mathcal{B}_{Y|X} \simeq D_X/D_Xx \simeq \mathcal{B}_{Y|X}.$$

More generally, we have $\mathbb{D}\mathcal{B}_{Y|X} \simeq \mathcal{B}_{Y|X}$ for any smooth closed subvariety $Y$ of a smooth variety $X$. This follows from Example 2.6.10, Theorem 2.7.2 below and $\mathcal{B}_{Y|X} = \int_i \mathcal{O}_Y$, where $i : Y \to X$ is the embedding.

**Example 2.6.10.** Let $M$ be an integrable connection. Then by Proposition 1.2.9, $\mathcal{H}om_{\mathcal{O}_X}(M, \mathcal{O}_X)$ is a left $D_X$-module (an integrable connection). Let us show that

$$\mathbb{D}M \simeq \mathcal{H}om_{\mathcal{O}_X}(M, \mathcal{O}_X).$$

First consider the locally free resolution

$$0 \to D_X \otimes_{\mathcal{O}_X} \bigwedge^{d_X} \Theta_X \to \cdots \to D_X \otimes_{\mathcal{O}_X} \Theta_X \to D_X \to \mathcal{O}_X \to 0$$

of $\mathcal{O}_X$ given in Lemma 1.5.27. Since $M$ is locally free over $\mathcal{O}_X$, $D_X \otimes_{\mathcal{O}_X} \bigwedge^{\cdot} \Theta_X \otimes_{\mathcal{O}_X} M$ is a locally free resolution of $M$. Using this resolution we can calculate $\mathcal{E}xt^{d_X}_{D_X}(M, D_X)$ by the complex

$$\begin{array}{ccc}
\cdots \to \mathcal{H}om_D\Big(D \otimes_{\mathcal{O}} \bigwedge^{d_X-1} \Theta \otimes_{\mathcal{O}} M,\ D\Big) & \to & \mathcal{H}om_D\Big(D \otimes_{\mathcal{O}} \bigwedge^{d_X} \Theta \otimes_{\mathcal{O}} M,\ D\Big) \to 0 \\
\downarrow\wr & & \downarrow\wr \\
\mathcal{H}om_{\mathcal{O}}\Big(\bigwedge^{d_X-1} \Theta \otimes_{\mathcal{O}} M,\ D\Big) & \longrightarrow & \mathcal{H}om_{\mathcal{O}}\Big(\bigwedge^{d_X} \Theta \otimes_{\mathcal{O}} M,\ D\Big) \\
\downarrow\wr & & \downarrow\wr \\
\mathcal{H}om_{\mathcal{O}}\big(M,\ \Omega^{d_X-1} \otimes_{\mathcal{O}} D\big) & \xrightarrow{\delta} & \mathcal{H}om_{\mathcal{O}}\big(M,\ \Omega^{d_X} \otimes_{\mathcal{O}} D\big).
\end{array}$$

On the other hand since $M$ is locally free over $\mathcal{O}_X$, we have an exact sequence

$$\mathcal{H}om_{\mathcal{O}}(M,\ \Omega^{d_X-1} \otimes_{\mathcal{O}} D) \xrightarrow{\delta} \mathcal{H}om_{\mathcal{O}}(M,\ \Omega^{d_X} \otimes_{\mathcal{O}} D) \to \mathcal{H}om_{\mathcal{O}}(M,\ \Omega^{d_X}) \to 0$$

of right $D_X$-modules. Hence as a right $D_X$-module we have

$$\mathcal{E}xt_D^{d_X}(M,\ D) \simeq \mathcal{H}om_{\mathcal{O}}(M,\ \Omega^{d_X}).$$

Passing to a left $D_X$-module by the side-changing functor, we finally obtain

$$\mathbb{D}M \simeq \mathcal{H}om_{\mathcal{O}}(M,\ \Omega^{d_X}) \otimes_{\mathcal{O}} (\Omega^{d_X})^{\otimes -1} \simeq \mathcal{H}om_{\mathcal{O}}(M,\ \mathcal{O}).$$

**Theorem 2.6.11.**

(i) *The rings $D_X(U)$ and $D_{X,x}$, where $U$ is an affine open subset of $U$ and $x$ is a point of $X$, have left and right global dimensions $d_X$.*

(ii) *Any $M \in \mathrm{Mod}_{qc}(D_X)$ admits a resolution*

$$0 \longrightarrow P_{d_X} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

*of length $d_X$ by locally projective $D_X$-modules. If $M \in \mathrm{Mod}_c(D_X)$, we can take all $P_i$'s to be of finite rank.*

*Proof.* (i) Since the category of right $D$-modules is equivalent to that of left $D$-modules we only need to show the statement for left global dimensions. Since $D_X(U)$ is a left noetherian ring with finite left global dimension, its left global dimension coincides with the largest integer $m$ such that there exists a finitely generated $D_X(U)$-module $M$ satisfying $\mathrm{Ext}^m_{D_X(U)}(M, D_X(U)) \neq 0$. By Theorem 2.6.7 we have $\mathrm{Ext}^i_{D_X(U)}(M, D_X(U)) = 0$ for any finitely generated $D_X(U)$-module $M$ and $i > d_X$. Moreover, by Example 2.6.10 $\mathrm{Ext}^{d_X}_{D_X(U)}(\mathcal{O}_X(U), D_X(U)) = \Omega_X(U) \neq 0$. Hence the left global dimension of $D_X(U)$ is exactly $d_X$. The statement for $D_{X,x}$ follows from this.

(ii) follows from (i) and the proof of Corollary 1.4.20 (ii).

We note the following basic result, which is a consequence of Proposition D.4.2.

**Proposition 2.6.12.** *Let $X$ be a smooth algebraic variety and $M$ a coherent $D_X$-module. Then we have*

$$\mathrm{Ch}(M) = \bigcup_{0 \le i \le d_X} \mathrm{Ch}(\mathcal{E}xt^i_{D_X}(M,\ D_X) \otimes_{\mathcal{O}_X} \Omega_X^{\otimes -1}).$$

*In particular, if $M$ is holonomic, then the characteristic varieties of $M$ and its dual $\mathbb{D}M$ are the same.*

In the rest of this section we give a description of $R\mathcal{H}om_{D_X}(M^{\cdot}, N^{\cdot})$ for $M^{\cdot} \in D^b_c(D_X)$, $N^{\cdot} \in D^b(D_X)$ in terms of the duality functor.

**Lemma 2.6.13.** *For $M^{\cdot} \in D^b_c(D_X)$ and $N^{\cdot} \in D^b(D_X)$, we have*

$$R\mathcal{H}om_{D_X}(M^{\cdot}, N^{\cdot}) \simeq R\mathcal{H}om_{D_X}(M^{\cdot}, D_X) \otimes^L_{D_X} N^{\cdot}.$$

*Proof.* Note that there exists a canonical morphism

$$R\mathcal{H}om_{D_X}(M^{\cdot}, D_X) \otimes^L_{D_X} N^{\cdot} \to R\mathcal{H}om_{D_X}(M^{\cdot}, N^{\cdot}).$$

Hence we may assume that $M^{\cdot} = D_X$. In this case the assertion is obvious since both sides are isomorphic to $N^{\cdot}$. □

**Proposition 2.6.14.** *For $M^{\cdot} \in D^b_c(D_X)$, $N^{\cdot} \in D^b(D_X)$ we have isomorphisms*

$$\begin{aligned} R\mathcal{H}om_{D_X}(M^{\cdot},\ N^{\cdot}) &\simeq (\Omega_X \otimes^L_{\mathcal{O}_X} \mathbb{D}_X M^{\cdot}) \otimes^L_{D_X} N^{\cdot}[-d_X] \\ &\simeq \Omega_X \otimes^L_{D_X} (\mathbb{D}_X M^{\cdot} \otimes^L_{\mathcal{O}_X} N^{\cdot})[-d_X] \\ &\simeq R\mathcal{H}om_{D_X}(\mathcal{O}_X, \mathbb{D}_X M^{\cdot} \otimes^L_{\mathcal{O}_X} N^{\cdot}) \end{aligned} \tag{2.6.1}$$

*in $D^b(\mathbb{C}_X)$. In particular, we have*

$$R\mathcal{H}om_{D_X}(\mathcal{O}_X, N^{\cdot}) \simeq \Omega_X \otimes^L_{D_X} N^{\cdot}[-d_X] \tag{2.6.2}$$

*for $N^{\cdot} \in D^b(D_X)$.*

*Proof.* We first show (2.6.2). By Lemma 2.6.13 we may assume that $N^{\cdot} = D_X$. In this case we have

$$\begin{aligned} &R\mathcal{H}om_{D_X}(\mathcal{O}_X, D_X) \\ &\simeq \left[ \mathcal{H}om_{D_X}\left(D_X \otimes_{\mathcal{O}_X} \bigwedge^0 \Theta_X, D_X\right) \to \cdots \to \mathcal{H}om_{D_X}\left(D_X \otimes_{\mathcal{O}_X} \bigwedge^{d_X} \Theta_X, D_X\right)\right] \\ &\simeq \left[ \mathcal{H}om_{\mathcal{O}_X}\left(\bigwedge^0 \Theta_X, D_X\right) \to \cdots \to \mathcal{H}om_{\mathcal{O}_X}\left(\bigwedge^{d_X} \Theta_X, D_X\right)\right] \\ &\simeq \left[ \bigwedge^0 \Omega^1_X \otimes_{\mathcal{O}_X} D_X \to \cdots \to \bigwedge^{d_X} \Omega^1_X \otimes_{\mathcal{O}_X} D_X \right] \\ &\simeq \Omega_X[-d_X] \end{aligned}$$

by Lemma 1.5.27. The isomorphism (2.6.2) is proved. Let us show (2.6.1). We have

$$\begin{aligned} R\mathcal{H}om_{D_X}(M^{\cdot},\ N^{\cdot}) &\simeq R\mathcal{H}om_{D_X}(M^{\cdot},\ D_X) \otimes^L_{D_X} N^{\cdot} \\ &\simeq (\Omega_X \otimes^L_{\mathcal{O}_X} \mathbb{D}_X M^{\cdot}) \otimes^L_{D_X} N^{\cdot}[-d_X] \end{aligned}$$

by Lemma 2.6.13. The second and the third isomorphisms follow from Proposition 1.5.19 and (2.6.2), respectively. □

Applying $R\Gamma(X, \bullet)$ to (2.6.1), we obtain the following.

**Corollary 2.6.15.** *Let $p : X \to \mathrm{pt}$ be the projection to a point. Then for $M^{\cdot} \in D^b_c(D_X)$ and $N^{\cdot} \in D^b(D_X)$ we have isomorphisms*

$$\begin{aligned} R\,\mathrm{Hom}_{D_X}(M^{\cdot},\ N^{\cdot}) &\simeq \int_p \left(\mathbb{D}_X M^{\cdot} \otimes^L_{\mathcal{O}_X} N^{\cdot}\right)[-d_X] \\ &\simeq R\,\mathrm{Hom}_{D_X}(\mathcal{O}_X, \mathbb{D}_X M^{\cdot} \otimes^L_{\mathcal{O}_X} N^{\cdot}). \end{aligned}$$

## 2.7 Relations among functors

### 2.7.1 Duality functors and inverse images

The main result in this subsection is the following.

**Theorem 2.7.1.** *Let $f : X \to Y$ be a morphism of smooth algebraic varieties, and let $M$ be a coherent $D_Y$-module.*

(i) *Assume $Lf^*M \in D^b_c(D_X)$. Then there exists a canonical morphism*

$$\mathbb{D}_X(Lf^*M) \to Lf^*(\mathbb{D}_Y M).$$

(ii) *Assume that $f$ is non-characteristic with respect to $M$ (hence $Lf^*M = f^*M$ and $f^*M$ is coherent by Theorem* 2.4.6*). Then we have*

$$\mathbb{D}_X(Lf^*M) \simeq Lf^*(\mathbb{D}_Y M).$$

*Proof.* (i) By Proposition 2.6.14 and Proposition 1.5.18 (ii) we have a sequence

$$\begin{aligned}
&\mathrm{Hom}_{D^b(D_Y)}(M, M)\\
&\quad\simeq \mathrm{Hom}_{D^b(D_Y)}(\mathcal{O}_Y, \mathbb{D}_Y M \otimes^L_{\mathcal{O}_Y} M)\\
&\quad\to \mathrm{Hom}_{D^b(D_X)}(Lf^*\mathcal{O}_Y, Lf^*(\mathbb{D}_Y M) \otimes^L_{\mathcal{O}_X} Lf^*M)\\
&\quad\simeq \mathrm{Hom}_{D^b(D_X)}(\mathcal{O}_X, Lf^*M \otimes^L_{\mathcal{O}_X} Lf^*(\mathbb{D}_Y M))\\
&\quad\simeq \mathrm{Hom}_{D^b(D_X)}(\mathbb{D}_X(Lf^*M), Lf^*(\mathbb{D}_Y M))
\end{aligned}$$

of morphisms, and hence we obtain a canonical morphism

$$\mathbb{D}_X(Lf^*M) \to Lf^*(\mathbb{D}_Y M)$$

as the image of $\mathrm{id}_M$.

(ii) By using the decomposition of $f$ into a composite of the graph embedding $X \to X \times Y$ and the projection $X \times Y \to Y$ we may assume that $f$ is either a closed embedding or a projection.

Assume that $f : X = T \times Y \to Y$ is the projection. Since the question is local on $Y$, we may assume that $Y$ is affine. In this case we may further assume that $M = D_Y$. Then we have

$$\mathbb{D}_X(Lf^*D_Y) \simeq \mathbb{D}_X(\mathcal{O}_T \boxtimes D_Y) \simeq \mathcal{O}_T \boxtimes (D_Y \otimes_{\mathcal{O}_Y} \Omega_Y^{\otimes -1})[d_Y] \simeq Lf^*(\mathbb{D}_Y D_X).$$

Assume that $f : X \to Y$ is a closed embedding. In this case we may assume that $f$ is an embedding of a hypersurface (see the proof of Theorem 2.4.6). By Lemma 2.4.7 we may further assume that $M = D_Y/D_Y P$. Choose a local coordinate $\{z_i, \partial_i\}_{1 \le i \le n}$ as in Example 2.4.4. Then we have $\mathbb{D}_Y M \simeq D_Y/D_Y P^*[d_Y - 1] = D_Y/D_Y P^*[d_X]$, where $P^*$ is the formal adjoint of $P$ with respect to the chosen coordinate (see Section 2.6). Denote by $m$ the order of the differential operator $P$. By Example 2.4.4 we have

$$\mathbb{D}_X(Lf^*M) \simeq \mathbb{D}_X(D_X^{\oplus m}) \simeq D_X^{\oplus m}[d_X], \qquad Lf^*(\mathbb{D}_Y M) \simeq D_X^{\oplus m}[d_X].$$

The proof that the canonical morphism $\mathbb{D}_X(Lf^*D_X) \to Lf^*(\mathbb{D}_Y M)$ is actually an isomorphism is left to the readers. □

### 2.7.2 Duality functors and direct images

In this section we will prove the commutativity of duality functors with proper direct images.

Let $f : X \to Y$ be a proper morphism of smooth algebraic varieties. We first construct a morphism

$$\mathrm{Tr}_f : \int_f \mathcal{O}_X[d_X] \longrightarrow \mathcal{O}_Y[d_Y]$$

in $D^b_c(D_Y)$ ($d_X = \dim X$, $d_Y = \dim Y$), which is called the *trace map* of $f$. In the case of analytic $D$-modules, this morphism can be constructed using resolutions by currents (Schwartz distributions) (Morihiko Saito, Kashiwara, Schneiders [Sch] or see [Bj2, p. 120]). In our situation dealing with algebraic $D$-modules we decompose $f$ into a composite of a closed embedding and a projection and construct the trace map in each case.

First, assume that $i : X \hookrightarrow Y$ is a closed embedding. By applying the canonical morphism $\int_i i^\dagger \to \mathrm{Id}$ to $\mathcal{O}_Y$ we get a morphism $\int_i i^\dagger \mathcal{O}_Y \to \mathcal{O}_Y$ in $D^b_c(D_Y)$. By $i^\dagger \mathcal{O}_Y = i^* \mathcal{O}_Y[d_X - d_Y] = \mathcal{O}_X[d_X - d_Y]$ it gives $\int_i \mathcal{O}_X[d_X - d_Y] \to \mathcal{O}_Y$. We obtain the required morphism $\mathrm{Tr}_i$ after taking the shift $[d_Y]$.

Next consider the case of a projection $X = \mathbb{P}^n \times Y \to Y$. By $\mathcal{O}_X = \mathcal{O}_{\mathbb{P}^n} \boxtimes \mathcal{O}_Y$ the problem is reduced to the case where $Y$ consists of a single point. So let us only consider the case $p : \mathbb{P}^n \to \mathrm{pt}$, where pt denotes the algebraic variety consisting of a single point. In this case $\int_p \mathcal{O}_{\mathbb{P}^n}$ is given by

$$R\Gamma\left(\mathbb{P}^n, \left[\mathcal{O}_{\mathbb{P}^n} \to \Omega^1_{\mathbb{P}^n} \to \cdots \to \Omega^n_{\mathbb{P}^n}\right]\right).$$

Hence there exist isomorphisms

$$H^0\Big(\int_p \mathcal{O}_{\mathbb{P}^n}[n]\Big) \simeq \tau^{\geqslant 0}\Big(\int_p \mathcal{O}_{\mathbb{P}^n}[n]\Big) \simeq H^n\big(\mathbb{P}^n, \Omega_{\mathbb{P}^n}\big)$$

(use the Hodge spectral sequence). Using the canonical isomorphism

$$H^n\big(\mathbb{P}^n, \Omega_{\mathbb{P}^n}\big) \simeq \mathbb{C}$$

given by the standard trace morphism in algebraic geometry, we obtain the desired morphism

$$\int_p \mathcal{O}_{\mathbb{P}^n}[n] \longrightarrow \tau^{\geqslant 0}\Big(\int_p \mathcal{O}_{\mathbb{P}^n}[n]\Big) \simeq \mathbb{C} = \mathcal{O}_{\mathrm{pt}}.$$

Let $f : X \to Y$ be a general proper morphism of smooth algebraic varieties. We can decompose $f$ into a composite of a closed embedding $i : X \to \mathbb{P}^n \times Y$ and the projection $p : \mathbb{P}^n \times Y \to Y$. Then the trace morphism

$$\mathrm{Tr}_f : \int_f \mathcal{O}_X[d_X] \longrightarrow \mathcal{O}_Y[d_Y]$$

is defined as the composite of

$$\int_f \mathcal{O}_X[d_X] = \int_p \int_i \mathcal{O}_X[d_X] \longrightarrow \int_p \mathcal{O}_{\mathbb{P}^n \times Y}[d_Y + n] \longrightarrow \mathcal{O}_Y[d_Y].$$

One can show that the trace morphism $\mathrm{Tr}_f$ does not depend on the choice of the decomposition $f = p \circ i$ and that it is functorial in the sense that for two proper morphisms $f : X \to Y$ and $g : Y \to Z$ we have $\mathrm{Tr}_{g \circ f} = \mathrm{Tr}_g \circ \int_g \mathrm{Tr}_f$. We omit the details.

The main result in this section is the following.

**Theorem 2.7.2.** *Let $f : X \to Y$ be a proper morphism. Then we have a canonical isomorphism*

$$\int_f \mathbb{D}_X \xrightarrow{\sim} \mathbb{D}_Y \int_f : D^b_c(D_X) \longrightarrow D^b_c(D_Y)$$

*of functors.*

*Proof.* We first construct a canonical morphism $\int_f \mathbb{D}_X \to \mathbb{D}_Y \int_f$ of functors. Let $M^{\cdot} \in D^b_c(D_X)$. By

$$\begin{aligned}
\int_f \mathbb{D}_X M &= Rf_*(R\mathcal{H}om_{D_X}(M^{\cdot}, D_X) \otimes^L_{D_X} D_{X \to Y}) \otimes^L_{\mathcal{O}_Y} \Omega_Y^{\otimes -1}[d_X] \\
&= Rf_*(R\mathcal{H}om_{D_X}(M^{\cdot}, D_{X \to Y})) \otimes^L_{\mathcal{O}_Y} \Omega_Y^{\otimes -1}[d_X], \\
\mathbb{D}_Y \int_f M &= R\mathcal{H}om_{D_Y}\Big(\int_f M^{\cdot}, D_Y\Big) \otimes^L_{\mathcal{O}_Y} \Omega_Y^{\otimes -1}[d_Y],
\end{aligned}$$

it is sufficient to construct a canonical morphism

$$\Phi(M^{\cdot}) : Rf_*(R\mathcal{H}om_{D_X}(M^{\cdot}, D_{X \to Y}[d_X])) \to R\mathcal{H}om_{D_Y}\Big(\int_f M^{\cdot}, D_Y[d_Y]\Big)$$

in $D^b_c(D_Y^{\mathrm{op}})$. By the projection formula (Corollary 1.7.5) we have

$$\int_f D_{X \to Y}[d_X] = \int_f Lf^* D_Y[d_X] \simeq \int_f \mathcal{O}_X[d_X] \otimes^L_{\mathcal{O}_Y} D_Y$$

and hence the trace morphism $\mathrm{Tr}_f$ induces a canonical morphism

$$\int_f D_{X \to Y}[d_X] \to D_Y[d_Y].$$

Using this $\Phi(M^{\cdot})$ is defined as the composite of

$$\begin{aligned}
&Rf_*(R\mathcal{H}om_{D_X}(M^{\cdot}, D_{X \to Y}[d_X])) \\
&\quad \to Rf_* R\mathcal{H}om_{f^{-1}D_Y}(D_{Y \leftarrow X} \otimes^L_{D_X} M^{\cdot}, D_{Y \leftarrow X} \otimes^L_{D_X} D_{X \to Y}[d_X])) \\
&\quad \to R\mathcal{H}om_{D_Y}(Rf_*(D_{Y \leftarrow X} \otimes^L_{D_X} M^{\cdot}), Rf_*(D_{Y \leftarrow X} \otimes^L_{D_X} D_{X \to Y})[d_X])
\end{aligned}$$

$$
\begin{aligned}
&= R\mathcal{H}om_{D_Y}\Big(\int_f M^{\cdot}, \int_f D_{X\to Y}[d_X]\Big)\\
&\to R\mathcal{H}om_{D_Y}\Big(\int_f M^{\cdot}, D_Y[d_Y]\Big).
\end{aligned}
$$

It remains to prove that $\Phi(M^{\cdot})$ is an isomorphism for any $M^{\cdot}$. By decomposing $f$ into a composite of a closed embedding and a projection we may assume from the beginning that $f$ is either a closed embedding $i : X \hookrightarrow Y$ or a projection $p : X = \mathbb{P}^n \times Y \to Y$. In each case there exists locally on $Y$ a resolution $F^{\cdot} \simeq M^{\cdot}$ of $M^{\cdot}$ in $D^b_c(D_X)$, where $F^{\cdot}$ is a bounded complex of $D_X$-modules such that each term $F^j$ of $F^{\cdot}$ is a direct summand of a free $D_X$-module of finite rank. This is obvious in the case of a closed embedding. In the case of a projection this is a consequence of Theorem 1.6.5 and Proposition 1.4.13. Therefore, we may assume from the beginning that $M^{\cdot} = D_X$ (see the proof of Theorem 2.5.1).

Let $i : X \hookrightarrow Y$ be a closed embedding. In this case $\Phi(D_X)$ is given by the composite of

$$
\begin{aligned}
&i_*(R\mathcal{H}om_{D_X}(D_X, i^*D_Y)[d_X]\\
&\quad\simeq R\mathcal{H}om_{D_Y}\Big(\int_i D_X, \int_i i^*D_Y\Big)[d_X]\\
&\quad= R\mathcal{H}om_{D_Y}\Big(\int_i D_X, \int_i i^{\dagger}D_Y\Big)[d_Y]\\
&\quad\to R\mathcal{H}om_{D_Y}\Big(\int_i D_X, D_Y\Big)[d_Y],
\end{aligned}
$$

where the first isomorphism is a consequence of Kashiwara's equivalence. Hence it is sufficient to show that $R\mathcal{H}om_{D_Y}(\int_i D_X, \int_i i^{\dagger}D_Y) \to R\mathcal{H}om_{D_Y}(\int_i D_X, D_Y)$ is an isomorphism. Set $U = Y \setminus X$ and let $j : U \to X$ be the embedding. By the distinguished triangle

$$
\int_i i^{\dagger}D_Y \longrightarrow D_Y \longrightarrow \int_j j^{\dagger}D_Y \xrightarrow{+1}
$$

we have only to show that $R\mathcal{H}om_{D_Y}(\int_i D_X, \int_j j^*D_Y) = 0$. By Propositions 1.5.25 and 1.7.1 (ii), we obtain

$$
\begin{aligned}
R\mathcal{H}om_{D_Y}\Big(\int_i D_X, \int_j j^*D_Y\Big) &\simeq i_*R\mathcal{H}om_{D_X}\Big(D_X, i^!\int_j j^*D_Y\Big)\\
&= i_*i^!\int_j j^*D_Y = 0.
\end{aligned}
$$

Let $p : X = \mathbb{P}^n \times Y \to Y$ be the projection. By $D_X = D_{\mathbb{P}^n} \boxtimes D_Y$ the problem is easily reduced to the case when $Y$ consists of a single point and we can only consider the case $p : X = \mathbb{P}^n \to \mathrm{pt}$, where pt is the algebraic variety consisting of a single point. In this case we have $D_{\mathbb{P}^n \to \mathrm{pt}} = \mathcal{O}_{\mathbb{P}^n}$, $D_{\mathrm{pt} \leftarrow \mathbb{P}^n} = \Omega_{\mathbb{P}^n}$ and hence

$$
\begin{aligned}
&Rp_*(R\mathcal{H}om_{D_X}(D_X, D_{X\to Y}[d_X])) \\
&\quad = R\operatorname{Hom}_{D_{\mathbb{P}^n}}(D_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n})[n] = R\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n})[n] \simeq \mathbb{C}[n], \\
&R\mathcal{H}om_{D_Y}\Big(\int_p D_X, D_Y[d_Y]\Big) \\
&\quad = R\operatorname{Hom}_{\mathbb{C}}(R\Gamma(\mathbb{P}^n, \Omega^n_{\mathbb{P}}), \mathbb{C}) \simeq \operatorname{Hom}_{\mathbb{C}}(\mathbb{C}[-n], \mathbb{C}) = \mathbb{C}[n].
\end{aligned}
$$

Therefore, it is sufficient to show that $\Phi(D_{\mathbb{P}^n})$ is non-trivial. Note that $\Phi(D_{\mathbb{P}^n})[-n]$ is given by the composite of

$$
\begin{aligned}
&R\operatorname{Hom}_{D_{\mathbb{P}^n}}(D_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n}) \\
&\quad \to R\operatorname{Hom}_{\mathbb{C}}(\Omega_{\mathbb{P}^n}, \Omega_{\mathbb{P}^n} \otimes^L_{D_{\mathbb{P}^n}} \mathcal{O}_{\mathbb{P}^n}) \\
&\quad \to R\operatorname{Hom}_{\mathbb{C}}(R\Gamma(\mathbb{P}^n, \Omega_{\mathbb{P}^n}), R\Gamma(\mathbb{P}^n, \Omega_{\mathbb{P}^n} \otimes^L_{D_{\mathbb{P}^n}} \mathcal{O}_{\mathbb{P}^n})) \\
&\quad \to R\operatorname{Hom}_{\mathbb{C}}(R\Gamma(\mathbb{P}^n, \Omega_{\mathbb{P}^n}), \tau^{\geqslant n} R\Gamma(\mathbb{P}^n, \Omega_{\mathbb{P}^n} \otimes^L_{D_{\mathbb{P}^n}} \mathcal{O}_{\mathbb{P}^n})) \\
&\quad \simeq R\operatorname{Hom}_{\mathbb{C}}(R\Gamma(\mathbb{P}^n, \Omega_{\mathbb{P}^n}), \mathbb{C}[-n]) \\
&\quad \simeq R\operatorname{Hom}_{\mathbb{C}}(R\Gamma(\mathbb{P}^n, \Omega_{\mathbb{P}^n}), R\Gamma(\mathbb{P}^n, \Omega_{\mathbb{P}^n})).
\end{aligned}
$$

We easily see that the morphism

$$
R\operatorname{Hom}_{D_{\mathbb{P}^n}}(D_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n}) \to R\operatorname{Hom}_{\mathbb{C}}(R\Gamma(\mathbb{P}^n, \Omega_{\mathbb{P}^n}), R\Gamma(\mathbb{P}^n, \Omega_{\mathbb{P}^n}))
$$

is induced by the canonical morphism $\mathcal{O}_{\mathbb{P}^n} \to \mathcal{H}om_{\mathcal{O}_{\mathbb{P}^n}}(\Omega_{\mathbb{P}^n}, \Omega_{\mathbb{P}^n})$ and it is non-trivial. □

**Corollary 2.7.3 (Adjunction formula).** *Let $f : X \to Y$ be a proper morphism. Then we have an isomorphism*

$$
R\mathcal{H}om_{D_Y}\Big(\int_f M^{\cdot},\ N^{\cdot}\Big) \simeq Rf_* R\mathcal{H}om_{D_X}(M^{\cdot},\ f^{\dagger}N^{\cdot})
$$

*for $M^{\cdot} \in D^b_c(D_X)$ and $N^{\cdot} \in D^b(D_Y)$.*

*Proof.* We have

$$
\begin{aligned}
&Rf_* R\mathcal{H}om_{D_X}(M^{\cdot},\ f^{\dagger}N^{\cdot}) \\
&\quad \simeq Rf_*((\Omega_X \otimes^L_{\mathcal{O}_X} \mathbb{D}_X M^{\cdot}) \otimes^L_{D_X} Lf^* N^{\cdot})[-d_Y] \\
&\quad \simeq Rf_*((\Omega_X \otimes^L_{\mathcal{O}_X} \mathbb{D}_X M^{\cdot}) \otimes^L_{D_X} D_{X\to Y} \otimes^L_{f^{-1}D_Y} f^{-1}N^{\cdot})[-d_Y] \\
&\quad \simeq Rf_*((\Omega_X \otimes^L_{\mathcal{O}_X} \mathbb{D}_X M^{\cdot}) \otimes^L_{D_X} D_{X\to Y}) \otimes^L_{D_Y} N^{\cdot}[-d_Y] \\
&\quad \simeq \int_f \mathbb{D}_X M^{\cdot} \otimes^L_{D_Y} N^{\cdot}[-d_Y] \\
&\quad \simeq \mathbb{D}_Y \int_f M^{\cdot} \otimes^L_{D_Y} N^{\cdot}[-d_Y] \\
&\quad \simeq R\mathcal{H}om_{D_Y}\Big(\int_f M^{\cdot},\ N^{\cdot}\Big)
\end{aligned}
$$

by Proposition 2.6.14 and Theorem 2.7.2. □

# 3

# Holonomic $D$-Modules

In this chapter we study functorial behaviors of holonomic systems and show that any simple object in the abelian category of holonomic $D_X$-modules is a minimal extension of an integrable connection on a locally closed smooth subvariety $Y$ of $X$.

## 3.1 Basic results

Recall that the dimension of the characteristic variety $\mathrm{Ch}(M)$ of a coherent $D_X$-module $M$ ($\neq 0$) satisfies the inequality $\dim \mathrm{Ch}(M) \geq \dim X$ and that a coherent $D_X$-module $M$ is called holonomic if $\dim \mathrm{Ch}(M) = \dim X$ or $M = 0$.

**Notation 3.1.1.** We denote by $\mathrm{Mod}_h(D_X)$ the full subcategory of $\mathrm{Mod}_c(D_X)$ consisting of holonomic $D_X$-modules.

The next proposition implies that $\mathrm{Mod}_h(D_X)$ is a thick abelian subcategory of $\mathrm{Mod}_c(D_X)$.

**Proposition 3.1.2.**

(i) *For an exact sequence*

$$0 \to M \to N \to L \to 0$$

*in* $\mathrm{Mod}_c(D_X)$ *we have*

$$N \in \mathrm{Mod}_h(D_X) \iff M,\ L \in \mathrm{Mod}_h(D_X).$$

(ii) *Any holonomic $D_X$-module has finite length. In other words, the category* $\mathrm{Mod}_h(D_X)$ *is* **artinian**.

*Proof.* The statement (i) is a consequence of $\mathrm{Ch}(N) = \mathrm{Ch}(M) \cup \mathrm{Ch}(L)$.

The statement (ii) is proved using the characteristic cycle as follows. For a holonomic $D_X$-module $M$ consider its characteristic cycle

$$\mathbf{CC}(M) = \sum_{C \in I(\mathrm{Ch}(M))} m_C(M)\, C.$$

Note that $\dim C = d_X$ for any $C \in I(\mathrm{Ch}(M))$. Define the *total multiplicity* of $M$ by

$$m(M) := \sum_{C \in I(\mathrm{Ch}(M))} m_C(M).$$

By Theorem 2.2.3 the total multiplicity is additive in the sense that we have $m(M) = m(L) + m(N)$ for any short exact sequence

$$0 \to L \to M \to N \to 0$$

in $\mathrm{Mod}_h(D_X)$. Moreover, we have $m(M) = 0 \iff \mathrm{Ch}(M) = \emptyset \iff M = 0$ by the definition of characteristic varieties. Hence the assertion follows by induction on $m(M)$. □

**Notation 3.1.3.** We denote by $D^b_h(D_X)$ the full subcategory of $D^b_c(D_X)$ consisting of objects $M^{\cdot} \in D^b_c(D_X)$ whose cohomology groups are holonomic, that is, $H^i(M^{\cdot}) \in \mathrm{Mod}_h(D_X)$ for $^{\forall}i \in \mathbb{Z}$.

We easily see the following from Propositions 3.1.2 and B.4.7.

**Corollary 3.1.4.** *$D^b_h(D_X)$ is a full triangulated subcategory of $D^b_c(D_X)$.*

**Remark 3.1.5.** It is known that

$$D^b(\mathrm{Mod}_h(D_X)) \xrightarrow{\sim} D^b_h(D_X)$$

(see Beilinson [Bei]).

The following result is the first important step in the study of holonomic $D$-modules. Namely, we can say "A holonomic $D$-module is generically an integrable connection."

**Proposition 3.1.6.** *Let $M$ be a holonomic $D_X$-module. Then there exists an open dense subset $U \subset X$ such that $M|_U$ is coherent over $\mathcal{O}_U$. In other words, $M|_U$ is an integrable connection on $U$.*

*Proof.* Let $T^*_X X \subset T^*X$ be the zero section of $T^*X$ and set $S := \mathrm{Ch}(M) \setminus T^*_X X$. If $S = \emptyset$, then $M$ itself is coherent over $\mathcal{O}_X$ by Proposition 2.2.5. Assume that $S \neq \emptyset$. Since $S$ is conic, the dimension of each fiber of $\pi|_S : S \to \pi(S)$ ($\pi : T^*X \to X$) is $\geq 1$ and hence $\dim \pi(S) < \dim S \leq \dim X$. Therefore, there exists an open subset $U \subset X$ such that $X \setminus \pi(S) \supset U \neq \emptyset$. In this case we have $\mathrm{Ch}(M|_U) \setminus T^*_U U = \emptyset$ and hence $M|_U$ is coherent over $\mathcal{O}_U$ by Proposition 2.2.5. □

The following result, which can be proved by duality, is also important.

**Proposition 3.1.7.** *Let $M \in \mathrm{Mod}_{qc}(D_X)$. For an open subset $U \subset X$ suppose that we are given a holonomic submodule $N$ of $M|_U$. Then there exists a holonomic submodule $\tilde{N}$ of $M$ such that $\tilde{N}|_U = N$.*

*Proof.* By Corollary 1.4.17 we may assume that $M$ is coherent and $M|_U = N$. Set $L = H^0(\mathbb{D}_X M)$. By Corollary 2.6.8 (ii) we have $\operatorname{codim} \operatorname{Ch}(L) \geq d_X$ and hence $L$ is a holonomic $D_X$-module. Moreover, its dual $\tilde{N} = \mathbb{D}_X L$ is also holonomic by Corollary 2.6.8 (vi). By $L = H^0(\mathbb{D}_X M) \simeq \tau^{\geqslant 0}(\mathbb{D}_X M)$ we have a distinguished triangle

$$K^{\cdot} \longrightarrow \mathbb{D}_X M \longrightarrow L \overset{+1}{\longrightarrow},$$

where $K^{\cdot} = \tau^{\leqslant -1}(\mathbb{D}_X M)$. By applying $\mathbb{D}_X$ we obtain

$$\tilde{N} \longrightarrow M \longrightarrow \mathbb{D}_X K^{\cdot} \overset{+1}{\longrightarrow}.$$

Since the duality functors commute with restrictions to open subsets, we have

$$\tilde{N}|_U = \mathbb{D}_U(L|_U) = \mathbb{D}_U^2(M|_U) = M|_U = N.$$

It remains to show that the canonical morphism $\tilde{N} \to M$ is injective. For this we have only to show $H^{-1}(\mathbb{D}_X K^{\cdot}) = 0$. In fact, we will show that

$$H^i(\mathbb{D}_X(\tau^{\geqslant -k} K^{\cdot})) = 0 \qquad (i < 0,\ k > 0) \tag{3.1.1}$$

(note that $\tau^{\geqslant -k} K^{\cdot} \simeq K^{\cdot}$ for $k \gg 0$). Let us first show

$$H^i(\mathbb{D}_X(H^{-k}(K^{\cdot})[k])) = 0 \qquad (i < 0,\ k > 0). \tag{3.1.2}$$

For $k > 0$ we have $H^{-k}(K^{\cdot}) \simeq H^{-k}(\mathbb{D}_X M)$ and hence $\operatorname{codim} \operatorname{Ch}(H^{-k}(K^{\cdot})) \geq d_X - k$ by Corollary 2.6.8 (ii). Hence the assertion is a consequence of Corollary 2.6.8 (i). Now we prove (3.1.1) by induction on $k$. If $k = 1$, then we have $\tau^{\geqslant -k} K^{\cdot} = H^{-k}(K^{\cdot})[k]$, and hence the assertion follows from (3.1.2). Assume $k \geqq 2$. By applying $\mathbb{D}_X$ to the distinguished triangle

$$H^{-k}(K^{\cdot})[k] \longrightarrow \tau^{\geqslant -k} K^{\cdot} \longrightarrow \tau^{\geqslant -(k-1)} K^{\cdot} \overset{+1}{\longrightarrow}$$

we obtain a distinguished triangle

$$\mathbb{D}_X(\tau^{\geqslant -(k-1)} K^{\cdot}) \longrightarrow \mathbb{D}_X(\tau^{\geqslant -k} K^{\cdot}) \longrightarrow \mathbb{D}_X(H^{-k}(K^{\cdot})[k]) \overset{+1}{\longrightarrow}.$$

Hence the assertion follows from (3.1.2) and the hypothesis of induction. □

## 3.2 Functors for holonomic $D$-modules

### 3.2.1 Stability of holonomicity

We first note the following, which is an obvious consequence of Corollary 2.6.8.

**Proposition 3.2.1.** *The duality functor $\mathbb{D}_X$ induces isomorphisms*

$$\mathbb{D}_X : \operatorname{Mod}_h(D_X) \xrightarrow{\sim} \operatorname{Mod}_h(D_X)^{\mathrm{op}},$$

$$\mathbb{D}_X : \quad D^b_h(D_X) \xrightarrow{\sim} D^b_h(D_X)^{\mathrm{op}}.$$

The following is also obvious by $\mathrm{Ch}(M \boxtimes N) = \mathrm{Ch}(M) \times \mathrm{Ch}(N)$.

**Proposition 3.2.2.** *The external tensor product $\boxtimes$ induces the functors*

$$(\bullet) \boxtimes (\bullet) : \mathrm{Mod}_h(D_X) \times \mathrm{Mod}_h(D_Y) \to \mathrm{Mod}_h(D_{X\times Y}),$$
$$(\bullet) \boxtimes (\bullet) : \quad D^b_h(D_X) \times D^b_h(D_Y) \to D^b_h(D_{X\times Y}).$$

Recall that for a morphism $f : X \to Y$ of smooth algebraic varieties we have functors

$$\int_f : D^b_{qc}(D_X) \longrightarrow D^b_{qc}(D_Y),$$
$$f^\dagger : D^b_{qc}(D_Y) \longrightarrow D^b_{qc}(D_X).$$

Moreover, if $f$ is proper (resp. smooth), $\int_f$ (resp. $f^\dagger$) preserves the coherency and we have the functors

$$\int_f : D^b_c(D_X) \to D^b_c(D_Y) \qquad (\text{resp. } f^\dagger : D^b_c(D_Y) \to D^b_c(D_X)).$$

However, neither $\int_f$ nor $f^\dagger$ preserves the coherency for general morphisms $f$. A surprising fact, which we will show in this section, is that the holonomicity is nevertheless preserved by these functors for any morphism $f : X \to Y$. Namely, we have the following.

**Theorem 3.2.3.** *Let $f : X \to Y$ be a morphism of smooth algebraic varieties.*

(i) *$\int_f$ sends $D^b_h(D_X)$ to $D^b_h(D_Y)$.*
(ii) *$f^\dagger$ sends $D^b_h(D_Y)$ to $D^b_h(D_X)$.*

**Corollary 3.2.4.** *The internal tensor product $\otimes^L_{\mathcal{O}_X}$ induces the functor*

$$(\bullet) \otimes^L_{\mathcal{O}_X} (\bullet) : D^b_h(D_X) \times D^b_h(D_X) \to D^b_h(D_X).$$

*Proof.* This follows from Proposition 3.2.2 and Theorem 3.2.3 (ii) noting that $(\bullet)\otimes^L_{\mathcal{O}_X} (\bullet) = L\Delta^*_X \circ ((\bullet) \boxtimes (\bullet))$, where $\Delta_X : X \to X \times X$ is the diagonal embedding. □

The proof of Theorem 3.2.3 will be completed in the next subsection. In the rest of this subsection we reduce it to that of Theorem 3.2.3 (i) in the case when $f$ is the projection $\mathbb{C}^n \to \mathbb{C}^{n-1}$.

**Lemma 3.2.5.** *Let $i : X \to Y$ be a closed embedding. Then for $M^\cdot \in D^b_c(D_X)$ we have*

$$M^\cdot \in D^b_h(D_X) \Longleftrightarrow \int_i M^\cdot \in D^b_h(D_Y).$$

*Proof.* Since $\int_i$ is exact, we may assume that $M = M^{\cdot} \in \mathrm{Mod}_c(D_X)$. Let

$$T^*Y \overset{\varpi}{\hookleftarrow} X \times_Y T^*Y \overset{\rho}{\twoheadrightarrow} T^*X$$

be the canonical morphisms. Then we have

$$\mathrm{Ch}\Big(\int_i M\Big) = \varpi\rho^{-1}(\mathrm{Ch}(M)),$$

by Lemma 2.3.5. Since $\varpi$ is a closed embedding and $\rho$ is a smooth surjective morphism with one-dimensional fibers, we have

$$\dim \mathrm{Ch}\Big(\int_i M\Big) = \dim \mathrm{Ch}(M) + 1,$$

form which we obtain the desired result. □

Next we reduce the proof of Theorem 3.2.3 (i) to the case when $f$ is the projection $\mathbb{C}^n \to \mathbb{C}^{n-1}$. In order to prove Theorem 3.2.3 (i) it is sufficient to show $\int_f M \in D^b_h(D_Y)$ for $M \in \mathrm{Mod}_h(D_X)$. By considering the decomposition of $f$ into a composite of a closed embedding and a projection we may assume that $f$ is either a closed embedding or a projection. The case of a closed embedding has already been dealt with in Lemma 3.2.5, and hence we can only consider the case when $f$ is the projection $X = Z \times Y \to Y$. Since the problem is local on $Y$, we may assume that $Y$ is affine. Take a finite affine open covering $Z = \bigcup_{i=0}^r Z_i$ of $Z$ such that $Z \setminus Z_i$ is a divisor on $Z$ for each $i$, and set $X_i = Z_i \times Y$. Then $X = \bigcup_{i=0}^r X_i$ is an affine open covering of $X$. For $0 \le i_0 < \cdots < i_k \le r$ let $j_{i_0,\dots,i_k} : X_{i_0,\dots,i_k} = \bigcap_{p=0}^k X_{i_p} \to X$ be the embedding (note that $X_{i_0,\dots,i_k}$ is affine by the choice of $Z_i$'s). Then $M$ is quasi-isomorphic to the *Čech complex*

$$\cdots \longrightarrow 0 \longrightarrow C^0(M) \longrightarrow C^1(M) \longrightarrow \cdots \longrightarrow C^r(M) \longrightarrow 0 \longrightarrow \cdots$$

with

$$C^k(M) = \bigoplus_{i_0<\cdots<i_k} j_{i_0,\dots,i_k *}(M|_{X_{i_0,\dots,i_k}})$$

(note $j_{i_0,\dots,i_k *}(M|_{X_{i_0,\dots,i_k}}) \simeq \int_{j_{i_0,\dots,i_k}} j^*_{i_0,\dots,i_k} M$). Hence it is sufficient to show $\int_{f\circ j_{i_0,\dots,i_k}} j^*_{i_0,\dots,i_k} M (= \int_f \int_{j_{i_0,\dots,i_k}} j^*_{i_0,\dots,i_k} M) \in D^b_h(D_Y)$ for any $(i_0, \dots, i_k)$. Therefore, we may assume from the beginning that $X$ and $Y$ are affine. Fix closed embeddings $\alpha : X \hookrightarrow \mathbb{C}^n$, $\beta : Y \hookrightarrow \mathbb{C}^m$, and consider the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle g} & & \\ X \times Y & & \downarrow{\scriptstyle \beta} \\ \downarrow{\scriptstyle \alpha\times\beta} & & \\ \mathbb{C}^{n+m} & \xrightarrow[p]{} & \mathbb{C}^m, \end{array}$$

where $g$ is the graph embedding associated to $f$ and $p$ is the projection. By Lemma 3.2.5 $\int_f M \in D^b_h(D_Y)$ if and only if $\int_\beta \int_f M \in D^b_h(D_{\mathbb{C}^m})$. Note that

$$\int_\beta \int_f M = \int_{\beta\circ f} M = \int_p \int_{(\alpha\times\beta)\circ g} M.$$

Since $(\alpha \times \beta) \circ g$ is a closed embedding, we have

$$\int_{(\alpha\times\beta)\circ g} M \in \mathrm{Mod}_h(D_{\mathbb{C}^{n+m}})$$

by Lemma 3.2.5, and hence the problem is reduced to the case when $f$ is the projection $\mathbb{C}^{n+m} \to \mathbb{C}^m$. Since $\mathbb{C}^{n+m} \to \mathbb{C}^m$ is a composite of morphisms $\mathbb{C}^k \to \mathbb{C}^{k-1}$, the problem is finally reduced to the case when $f$ is the projection $\mathbb{C}^n \to \mathbb{C}^{n-1}$.

Let us show that Theorem 3.2.3 (i) implies Theorem 3.2.3 (ii). So we assume that Theorem 3.2.3 (i) holds and show $f^\dagger M \in D^b_h(D_X)$ for any $M \in \mathrm{Mod}_h(D_Y)$. By decomposing $f$ into a composite of a closed embedding and a projection we may further assume that $f$ is either a closed embedding or a projection. Consider first the case where $f$ is the projection $X = Z \times Y \to Y$. Then $f^*$ is an exact functor and the complex $f^\dagger M = f^* M[\dim Z]$ is concentrated in the degree $-\dim Z$. Moreover, we have $f^* M \simeq \mathcal{O}_Z \boxtimes M$ and it is holonomic by

$$\mathrm{Ch}(\mathcal{O}_Z \boxtimes M) = \mathrm{Ch}(\mathcal{O}_Z) \times \mathrm{Ch}(M) = T^*_Z Z \times \mathrm{Ch}(M),$$

and hence $f^\dagger M \in D^b_h(D_X)$. Let us consider the case of a closed embedding $i : X \hookrightarrow Y$. Let $j : U := Y \setminus X \hookrightarrow Y$ be the corresponding open embedding. Then by the results in Section 1.7 there exists a distinguished triangle

$$\int_i i^\dagger M \longrightarrow M \longrightarrow \int_j j^\dagger M \xrightarrow{+1} .$$

We have $j^\dagger M = M|_U \in \mathrm{Mod}_h(D_U)$, and hence (i) implies $\int_j j^\dagger M^\cdot \in D^b_h(D_Y)$. Therefore, we see by the above distinguished triangle that $\int_i i^\dagger M^\cdot \in D^b_h(D_Y)$. This implies $i^\dagger M^\cdot \in D^b_h(D_X)$ by Lemma 3.2.5. Theorem 3.2.3 (ii) is verified assuming Theorem 3.2.3 (i).

### 3.2.2 Holonomicity of modules over Weyl algebras

In the last subsection the proof Theorem 3.2.3 (i), (ii) was reduced to that of (i) in the case when $f$ is the projection $p : \mathbb{C}^n = \mathbb{C} \times \mathbb{C}^{n-1} \to \mathbb{C}^{n-1}$. The aim of this subsection is to prove it using the theory of $D$-modules on $\mathbb{C}^n$.

Set

$$D_n := \Gamma(\mathbb{C}^n,\ D_{\mathbb{C}^n}) = \bigoplus_{\alpha,\beta} \mathbb{C} x^\alpha \partial^\beta,$$

where $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ for $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\partial^\beta = \partial_1^{\beta_1} \partial_2^{\beta_2} \cdots \partial_n^{\beta_n}$ for $\beta = (\beta_1, \ldots, \beta_n)$. The algebra $D_n$ is called the *Weyl algebra*. Since $\mathbb{C}^n$ is affine, we have equivalences of categories

$$\mathrm{Mod}_{qc}(D_{\mathbb{C}^n}) \xrightarrow{\sim} \mathrm{Mod}(D_n),$$

$$\mathrm{Mod}_{c}(D_{\mathbb{C}^n}) \xrightarrow{\sim} \mathrm{Mod}_f(D_n)$$

given by $M \mapsto \Gamma(\mathbb{C}^n, M)$. For $N \in \mathrm{Mod}(D_n)$ we denote the corresponding $D_{\mathbb{C}^n}$-module by $\widetilde{N}$. A $D_n$-module $N$ is called holonomic if $\widetilde{N}$ is a holonomic $D_{\mathbb{C}^n}$-module.

Let $N$ be a $D_n$-module. We define its *Fourier transform* $\widehat{N}$ as follows. As an additive group $\widehat{N}$ is the same as $N$, and the action of the generators $x_i$, $\partial_i$ of $D_n$ on $\widehat{N}$ is given by

$$x_i \circ s := -\partial_i s, \quad \partial_i \circ s := x_i s.$$

It is easily checked that $\widehat{N}$ is a left $D_n$-module with respect to this action $\circ$. This definition of the Fourier transform $\widehat{N}$ is motivated by the classical Fourier transform. The Fourier transform induces equivalences of categories

$$\widehat{(\bullet)} : \mathrm{Mod}(D_n) \xrightarrow{\sim} \mathrm{Mod}(D_n),$$

$$\widehat{(\bullet)} : \mathrm{Mod}_f(D_n) \xrightarrow{\sim} \mathrm{Mod}_f(D_n).$$

The corresponding equivalences for the categories of $D_{\mathbb{C}^n}$-modules are also denoted by

$$\widehat{(\bullet)} : \mathrm{Mod}_{qc}(D_{\mathbb{C}^n}) \xrightarrow{\sim} \mathrm{Mod}_{qc}(D_{\mathbb{C}^n}),$$

$$\widehat{(\bullet)} : \mathrm{Mod}_{c}(D_{\mathbb{C}^n}) \xrightarrow{\sim} \mathrm{Mod}_{c}(D_{\mathbb{C}^n}).$$

**Proposition 3.2.6.** *Let $p : \mathbb{C}^n (= \mathbb{C} \times \mathbb{C}^{n-1}) \to \mathbb{C}^{n-1}$ be the projection and let $i : \mathbb{C}^{n-1} (= \{0\} \times \mathbb{C}^{n-1}) \hookrightarrow \mathbb{C}^n (= \mathbb{C} \times \mathbb{C}^{n-1})$ be the embedding. For $M \in \mathrm{Mod}_{qc}(D_{\mathbb{C}^n})$ we have*

$$\widehat{H^k\Big(\int_p M\Big)} \simeq H^k(Li^* \widehat{M})$$

*for any $k$.*

*Proof.* Set $N = \Gamma(\mathbb{C}^n, M)$. Since $p$ is an affine morphism, we have

$$\int_p M \simeq Rp_*(DR_{\mathbb{C}^n/\mathbb{C}^{n-1}}(M)) \simeq \big[p_*M \xrightarrow{\partial_1} p_*M\big],$$

and hence

$$\Gamma\left(\mathbb{C}^{n-1}, H^k\Big(\int_p M\Big)\right) \simeq \begin{cases} \mathrm{Ker}\big[N \xrightarrow{\partial_1} N\big] & (k = -1), \\ \mathrm{Coker}\big[N \xrightarrow{\partial_1} N\big] & (k = 0), \\ 0 & (k \neq 0, -1). \end{cases}$$

Therefore, we have

$$\Gamma\left(\mathbb{C}^{n-1}, \widehat{H^k\Big(\int_p M\Big)}\right) \simeq \begin{cases} \mathrm{Ker}\big[\widehat{N} \xrightarrow{x_1} \widehat{N}\big] & (k=-1), \\ \mathrm{Coker}\big[\widehat{N} \xrightarrow{x_1} \widehat{N}\big] & (k=0), \\ 0 & (k \neq 0, -1), \end{cases}$$
$$\simeq \Gamma(\mathbb{C}^{n-1}, H^k(Li^*\widehat{M}))$$

from which we obtain the desired result. □

In proving Theorem 3.2.3 we also need the following results.

**Proposition 3.2.7.** *A coherent $D_{\mathbb{C}^n}$-module $M$ is holonomic if and only if $\widehat{M}$ is as well.*

**Proposition 3.2.8.** *Let $j : (\mathbb{C} \setminus \{0\}) \times \mathbb{C}^{n-1} \to \mathbb{C}^n$ be the embedding. If $M$ is a holonomic $D_{\mathbb{C}^n}$-module, then so is $H^0(\int_j j^\dagger M)$ (since $(\mathbb{C} \setminus \{0\}) \times \mathbb{C}^{n-1}$ is an affine open subset of $\mathbb{C}^n$ we have $H^k(\int_j j^\dagger M) = 0$ for $k \neq 0$).*

Let us complete the proof of Theorem 3.2.3 assuming Propositions 3.2.7 and 3.2.8. By Propositions 3.2.6 and 3.2.7 and the arguments in the last subsection it is sufficient to show $i^\dagger M \in D^b_h(D_{\mathbb{C}^{n-1}})$ for $M \in \mathrm{Mod}_h(D_{\mathbb{C}^n})$, where $i : \mathbb{C}^{n-1} \hookrightarrow \mathbb{C}^n$ is as in Proposition 3.2.6. Let $j : (\mathbb{C} \setminus \{0\}) \times \mathbb{C}^{n-1} \hookrightarrow \mathbb{C}^n$ be as in Proposition 3.2.8. By the distinguished triangle

$$\int_i i^\dagger M \longrightarrow M \longrightarrow \int_j j^\dagger M \xrightarrow{+1}$$

we obtain an exact sequence

$$0 \longrightarrow H^0\Big(\int_i i^\dagger M\Big) \longrightarrow M \longrightarrow H^0\Big(\int_j j^\dagger M\Big) \longrightarrow H^1\Big(\int_i i^\dagger M\Big) \longrightarrow 0.$$

Since $H^0(\int_j j^\dagger M)$ is holonomic by Proposition 3.2.8, we obtain $\int_i i^\dagger M \in D^b_h(D_{\mathbb{C}^n})$ (note $H^k(\int_i i^\dagger M) = 0$ for $k \neq 0, 1$). Hence we have $i^\dagger M \in D^b_h(D_{\mathbb{C}^{n-1}})$ by Lemma 3.2.5.

The rest of this subsection is devoted to proving Proposition 3.2.7 and Proposition 3.2.8.

In addition to the usual order filtration $F$, the Weyl algebra $D_n$ has another filtration $B$ defined by

$$B_i D_n := \sum_{|\alpha|+|\beta| \leq i} \mathbb{C} x^\alpha \partial^\beta \subset D_n.$$

We call it the *Bernstein filtration* of the Weyl algebra $D_n$. The graded algebra $\mathrm{gr}^B D_n$ associated to the Bernstein filtration $B$ is commutative and isomorphic to the polynomial ring $\mathbb{C}[x, \xi]$ ($x = (x_1, x_2, \ldots, x_n)$, $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$), as in the case of the usual order filtration. For a $D_n$-module $M$ we can also define good filtrations $F$ on it with respect to the Bernstein filtration $B$. Any finitely generated $D_n$-module has a good filtration. The Bernstein filtration has the advantage that for any good filtration $F$ of a finitely generated $D_n$-module $M$ each $F_i M$ is *finite dimensional* over $\mathbb{C}$. Therefore, we can apply results on Hilbert polynomials to the associated graded $\mathrm{gr}^B D_n$-module.

**Proposition 3.2.9.**

(i) *Let $F$ be a good filtration on a non-zero module $M \in \mathrm{Mod}_f(D_n)$ with respect to the Bernstein filtration. Then there exists a unique polynomial $\chi(M,\ F;\ T) \in \mathbb{Q}[T]$ such that*

$$\chi(M,\ F;\ i) = \dim_{\mathbb{C}} F_i M \quad (i \gg 0).$$

(ii) *If the degree of $\chi(M,\ F;\ T)$ is $d$, then the coefficient of the degree $d$ (the highest degree) part of $\chi(M,\ F;\ T)$ is $m/d!$ for some integer $m > 0$. These two integers $d$ and $m$ do not depend on the choice of the good filtration $F$. They depend only on $M$ itself.*

*Proof.* By $\dim_{\mathbb{C}} F_i M = \sum_{k \le i} \dim_{\mathbb{C}} \mathrm{gr}_k^F M$ most of the statements are well known in algebraic geometry [Ha2, Chapter 1]. Let us show that $d$ and $m$ are independent of the choice of a good filtration. Let $F$ and $F'$ be good filtrations of a finitely generated $D_n$-module $M$. By Proposition D.1.3 there exists $i_0 > 0$ satisfying

$$F'_{i-i_0} M \subset F_i M \subset F'_{i+i_0} M,$$

and hence

$$\chi(M, F'; i - i_0) \le \chi(M, F; i) \le \chi(M, F'; i + i_0)$$

for $i \gg 0$. The desired result easily follows from this. □

We call $d = d_B(M)$ the *dimension* of $M$, and $m = m_B(M)$ the *multiplicity* of $M$.

**Proposition 3.2.10.** *Let*

$$0 \longrightarrow L \to M \longrightarrow N \longrightarrow 0$$

*be an exact sequence of finitely generated $D_n$-modules.*

(i) *We have $d_B(M) = \mathrm{Max}\{d_B(L), d_B(N)\}$.*
(ii) *We have*

$$m_B(M) = \begin{cases} m_B(L) + m_B(N) & (d_B(L) = d_B(N)), \\ m_B(L) & (d_B(L) > d_B(N)), \\ m_B(N) & (d_B(L) < d_B(N)). \end{cases}$$

*Proof.* Take a good filtration $F$ on $M$. With respect to the induced filtrations on $L$ and $N$ we have an exact sequence

$$0 \longrightarrow \mathrm{gr}^F L \to \mathrm{gr}^F M \longrightarrow \mathrm{gr}^F N \longrightarrow 0$$

of graded $\mathrm{gr}^B D_n$-modules. The desired result follows from this. □

**Proposition 3.2.11.** *For a non-zero finitely generated $D_n$-module $M$ we have*

$$\dim \mathrm{Ch}(\widetilde{M}) = d_B(M).$$

*Proof.* Set $j(M) := \mathrm{Min}\{\, i \mid Ext^i_{D_n}(M,\ D_n) \neq 0 \,\}$. By applying Theorem D.4.3 to the two filtrations $F$ and $B$ of $D_n$ we have

$$\dim \mathrm{Ch}(\widetilde{M}) = 2n - j(M) = \dim \mathrm{supp}(\widetilde{\mathrm{gr}^F M}),$$

where $F$ is a good filtration on $M$ with respect to the Bernstein filtration and $\widetilde{\mathrm{gr}^F M}$ denotes the corresponding coherent $\mathcal{O}_{\mathbb{C}^{2n}}$-module. It is well known in algebraic geometry that we have $\dim \mathrm{supp}(\widetilde{\mathrm{gr}^F M}) = d_B(M)$ [Ha2, Chapter 1]. □

By Proposition 3.2.10 a coherent $D_{\mathbb{C}^n}$-module $\widetilde{M}$ associated to $M$ is holonomic if and only if $d_B(M) = n$. We can use the following estimate as a useful criterion for the holonomicity of $M$.

**Proposition 3.2.12.** *Let $M$ be a (not necessarily finitely generated) non-zero $D_n$-module. We assume that $M$ has a filtration $F$ bounded from below (with respect to the Bernstein filtration $B$ of $D_n$) such that there exist constants $c$, $c'$ satisfying the condition*

$$\dim_{\mathbb{C}} F_i M \leq \frac{c}{n!} i^n + c' i^{n-1}$$

*for any $i$. Then $M$ is holonomic and $m_B(M) \leq c$.*

*Proof.* We first show that any finitely generated non-zero $D_n$-submodule $N$ of $M$ is holonomic and satisfies $m_B(N) \leq c$. Take a good filtration $G$ on $N$. By Proposition D.1.3 we have

$$G_i N \subset N \cap F_{i+i_0} M \subset F_{i+i_0} M \qquad (\forall i)$$

for some $i_0$, and hence

$$\chi(N, G; i) \leq \frac{c}{n!}(i + i_0)^n + c'(i + i_0)^{n-1}.$$

It follows that $d_B(N) \leq n$. By $N \neq 0$ and $d_B(N) = \dim \mathrm{Ch}(N)$ we obtain $d_B(N) = n$ and $m_B(N) \leq c$.

It remains to show that $M$ is finitely generated. It is sufficient to show that any increasing sequence

$$0 \neq N_1 \subset N_2 \subset \cdots \subset M$$

of finitely generated submodules of $M$ is stationary. We have shown that $N_i$ is holonomic and satisfies $m_B(N_i) \leq c$. Moreover, we have

$$m_B(N_1) \leq m_B(N_2) \leq m_B(N_3) \leq \cdots \leq c$$

by Proposition 3.2.10, and hence the sequence $\{m_B(N_i)\}$ is stationary. This implies the desired result by Proposition 3.2.10. □

Now we are ready to give proofs of Proposition 3.2.7 and Proposition 3.2.8.

*Proof of Proposition* 3.2.7. Set $N = \Gamma(\mathbb{C}^n, M)$. By the definition of $B$ and the Fourier transform we have $d_B(N) = d_B(\widehat{N})$. Hence by Proposition 3.2.11 we have $\dim \mathrm{Ch}(M) = \dim \mathrm{Ch}(\widehat{M})$. This implies the desired result. □

*Proof of Proposition* 3.2.8. Set $N = \Gamma(\mathbb{C}^n, M)$. Note that $\Gamma(\mathbb{C}^n, H^0(\int_j j^\dagger M))$ is isomorphic to the localization $N_{x_1} = \mathbb{C}[x, x_1^{-1}] \otimes_{\mathbb{C}[x]} N$. Hence it is sufficient to show that $N_{x_1}$ is holonomic. Take a good filtration $F$ of $N$ and define $F_i N_{x_1}$ to be the image of $F_{2i}N \ni s \mapsto x_1^{-i} s \in N_{x_1}$. It is easily checked that this defines a filtration of $N_{x_1}$ with respect to the Bernstein filtration. Moreover, we have

$$\begin{aligned}\dim_{\mathbb{C}} F_i N_{x_1} &\le \dim_{\mathbb{C}} F_{2i} N \\ &= \frac{m_B(M)}{n!}(2i)^n + O(i^{n-1}) \\ &= \frac{m_B(M)2^n}{n!} i^n + O(i^{n-1}),\end{aligned}$$

and hence $N_{x_1}$ is holonomic by Proposition 3.2.12. □

### 3.2.3 Adjunction formulas

Let $f : X \to Y$ be a morphism of smooth algebraic varieties.

**Definition 3.2.13.** We define new functors by

$$\begin{aligned}\int_{f!} &:= \mathbb{D}_Y \int_f \mathbb{D}_X : D_h^b(D_X) \longrightarrow D_h^b(D_Y), \\ f^\bigstar &:= \mathbb{D}_X f^\dagger \mathbb{D}_Y : D_h^b(D_Y) \longrightarrow D_h^b(D_X).\end{aligned}$$

**Theorem 3.2.14.** *For $M^\cdot \in D_h^b(D_X)$ and $N^\cdot \in D_h^b(D_Y)$ we have natural isomorphisms*

$$\begin{aligned}R\mathcal{H}om_{D_Y}\left(\int_{f!} M^\cdot,\ N^\cdot\right) &\xrightarrow{\sim} Rf_* R\mathcal{H}om_{D_X}(M^\cdot,\ f^\dagger N^\cdot), \\ Rf_* R\mathcal{H}om_{D_X}(f^\bigstar N^\cdot,\ M^\cdot) &\xrightarrow{\sim} R\mathcal{H}om_{D_Y}\left(N^\cdot,\ \int_f M^\cdot\right).\end{aligned}$$

*Proof.* We have

$$\begin{aligned}&Rf_* R\mathcal{H}om_{D_X}(M^\cdot,\ f^\dagger N^\cdot) \\ &\quad\simeq Rf_*\left(\left(\Omega_X \otimes^L_{\mathcal{O}_X} \mathbb{D}_X M^\cdot\right) \otimes^L_{D_X} f^\dagger N^\cdot\right)[-d_X] \\ &\quad\simeq Rf_*\left(\left(\Omega_X \otimes^L_{\mathcal{O}_X} \mathbb{D}_X M^\cdot\right) \otimes^L_{D_X} D_{X\to Y} \otimes^L_{f^{-1}D_Y} f^{-1} N^\cdot\right)[-d_Y] \\ &\quad\simeq Rf_*\left(\left(\Omega_X \otimes^L_{\mathcal{O}_X} \mathbb{D}_X M^\cdot\right) \otimes^L_{D_X} D_{X\to Y}\right) \otimes^L_{D_Y} N^\cdot[-d_Y]\end{aligned}$$

$$\simeq \left(\Omega_Y \otimes^L_{\mathcal{O}_Y} \int_f \mathbb{D}_X M^{\cdot}\right) \otimes^L_{D_Y} N^{\cdot}[-d_Y]$$
$$\simeq \left(\Omega_Y \otimes^L_{\mathcal{O}_Y} \mathbb{D}_Y \int_{f!} M^{\cdot}\right) \otimes^L_{D_Y} N^{\cdot}[-d_Y]$$
$$\simeq R\mathcal{H}om_{D_Y}\left(\int_{f!} M^{\cdot},\ N^{\cdot}\right).$$

The first isomorphism is established. The second isomorphism follows from the first by duality. □

By applying $H^0(R\Gamma(Y, \bullet))$ to the isomorphisms in Theorem 3.2.14, we obtain the following.

**Corollary 3.2.15.** *For $M^{\cdot} \in D^b_h(D_X)$ and $N^{\cdot} \in D^b_h(D_Y)$ we have natural isomorphisms*

$$\mathrm{Hom}_{D^b_h(D_Y)}\left(\int_{f!} M^{\cdot},\ N^{\cdot}\right) \xrightarrow{\sim} \mathrm{Hom}_{D^b_h(D_X)}(M^{\cdot},\ f^{\dagger}N^{\cdot}),$$
$$\mathrm{Hom}_{D^b_h(D_X)}(f^{\bigstar}N^{\cdot},\ M^{\cdot}) \xrightarrow{\sim} \mathrm{Hom}_{D^b_h(D_Y)}\left(N^{\cdot},\ \int_f M^{\cdot}\right).$$

*Namely, $\int_{f!}$ (resp. $f^{\bigstar}$) is the left adjoint of $f^{\dagger}$ (resp. $\int_f$).*

**Theorem 3.2.16.** *There exists a morphism of functors*

$$\int_{f!} \longrightarrow \int_f : D^b_h(D_X) \longrightarrow D^b_h(D_Y).$$

*Moreover, if $f$ is proper, then this morphism is an isomorphism.*

*Proof.* By *Hironaka's desingularization theorem* [Hi], there exists a smooth completion $\widetilde{X}$ of $X$. Since $X$ is quasi-projective, a desingularization $\widetilde{X}$ of the Zariski closure $\overline{X}$ of $X$ in the projective space is such a completion (even if $X$ is not quasi-projective, there exists a smooth completion by a theorem due to Nagata). Therefore, the map $f : X \to Y$ factorizes as

$$X \xrightarrow{g} X \times Y \xrightarrow{j} \widetilde{X} \times Y \xrightarrow{p} Y,$$

where $g$ is the graph embedding associated to $f$ and $p = \mathrm{pr}_Y$ is a projection. In this situation, $g$ and $p$ are proper and $j$ is an open embedding. This implies that we can reduce our problem to the cases of proper morphisms and open embeddings. If $f$ is proper, we have an isomorphism

$$\int_{f!} = \mathbb{D}_Y \int_f \mathbb{D}_X \xrightarrow{\sim} \int_f$$

by Theorem 2.7.2. So let us consider the case when $f = j : X \hookrightarrow Y$ is an open embedding. Let $M^{\cdot} \in D^b_h(D_X)$. By Corollary 3.2.15 we have

$$\begin{aligned}\mathrm{Hom}_{D^b_h(D_Y)}\left(\int_{j!} M^{\cdot},\ \int_j M^{\cdot}\right) &\simeq \mathrm{Hom}_{D^b_h(D_X)}\Big(M^{\cdot},\ j^{\dagger}\int_j M^{\cdot}\Big)\\ &\simeq \mathrm{Hom}_{D^b_h(D_X)}(M^{\cdot},\ M^{\cdot}),\end{aligned}$$

and hence we obtain the desired morphism

$$\int_{j!} M^{\cdot} \longrightarrow \int_j M^{\cdot}$$

as the image of $\mathrm{id} \in \mathrm{Hom}_{D^b_h(D_X)}(M^{\cdot},\ M^{\cdot})$. □

## 3.3 Finiteness property

The aim of this section is to show the following.

**Theorem 3.3.1.** *The following conditions on $M^{\cdot} \in D^b_c(D_X)$ are equivalent:*

(i) $M^{\cdot} \in D^b_h(D_X)$.
(ii) *There exists a decreasing sequence*

$$X = X_0 \supset X_1 \supset \cdots \supset X_m \supset X_{m+1} = \emptyset$$

*of closed subsets of $X$ such that $X_r \setminus X_{r+1}$ is smooth and all of the cohomology sheaves $H^k(i_r^{\dagger} M^{\cdot})$ are integrable connections, where $i_r : X_r \setminus X_{r+1} \hookrightarrow X$ denotes the embedding.*
(iii) *For any $x \in X$ all of the cohomology groups $H^k(i_x^{\dagger} M^{\cdot})$ are finite dimensional over $\mathbb{C}$, where $i_x : \{x\} \hookrightarrow X$ denotes the inclusion.*

For the proof we need the following.

**Lemma 3.3.2.** *Let $M$ be a coherent (but not necessarily holonomic) $D_X$-module. Then there exists an open dense subset $U \subset X$ such that $M|_U$ is projective over $\mathcal{O}_U$.*

*Proof.* Take a good filtration $F$ of $M$. Then $\mathrm{gr}^F M$ is coherent over $\pi_*\mathcal{O}_{T^*X}$. It follows from a well-known fact on coherent sheaves that there exists an open dense subset $U \subset X$ such that $(\mathrm{gr}^F M)|_U$ is free over $\pi_*\mathcal{O}_{T^*U}$. By shrinking $U$ if necessary we may assume that $(\mathrm{gr}^F M)|_U$ is free over $\mathcal{O}_U$. This implies that each $(F_i M/F_{i-1}M)|_U$ (and hence each $F_i M|_U$) is projective over $\mathcal{O}_U$. Consequently $M|_U$ is projective over $\mathcal{O}_U$. □

*Proof of Theorem* 3.3.1. (ii) $\Rightarrow$ (i). Set $U_r = X \setminus X_r$. We will show $M^{\cdot}|_{U_r} \in D^b_h(D_{U_r})$ by induction on $r$. Assume $M^{\cdot}|_{U_r} \in D^b_h(D_{U_r})$. Let $j : U_r \to U_{r+1}$, $i : X_r \setminus X_{r+1}$ $(= U_{r+1} \setminus U_r) \to U_{r+1}$ be embeddings. Then we have a distinguished triangle

$$\int_i i^\dagger(M^\cdot|_{U_{r+1}}) \longrightarrow M^\cdot|_{U_{r+1}} \longrightarrow \int_j j^\dagger(M^\cdot|_{U_{r+1}}) \xrightarrow{+1}.$$

By $i^\dagger(M^\cdot|_{U_{r+1}}) = i_r^\dagger M^\cdot \in D_h^b(D_{X_r\setminus X_{r+1}})$ we have $\int_i i^\dagger(M^\cdot|_{U_{r+1}}) \in D_h^b(D_{U_{r+1}})$. On the other hand by $j^\dagger(M^\cdot|_{U_{r+1}}) = M^\cdot|U_r \in D_h^b(D_{U_r})$ we have $\int_j j^\dagger(M^\cdot|_{U_{r+1}}) \in D_h^b(D_{U_{r+1}})$. Hence the above distinguished triangle implies $M^\cdot|_{U_{r+1}} \in D_h^b(D_{U_{r+1}})$.

(i) $\Rightarrow$ (iii). By Theorem 3.2.3 we have $i_x^\dagger M^\cdot \in D_h^b(D_{\{x\}})$. Note that $D_{\{x\}} \simeq \mathbb{C}$. Hence the desired result follows from the fact that objects of $D_h^b(D_{\{x\}}) = D_c^b(D_{\{x\}}) = D_c^b(\mathrm{Mod}(\mathbb{C}))$ are just complexes of vector spaces whose cohomology groups are finite dimensional.

(iii) $\Rightarrow$ (ii). It is sufficient to show that for any closed subset $Y$ of $X$ satisfying $Y \supset \mathrm{supp}(M^\cdot) := \bigcup_k \mathrm{supp}(H^k(M^\cdot))$ there exists a decreasing sequence

$$Y = Y_0 \supset Y_1 \supset \cdots \supset Y_m \supset Y_{m+1} = \emptyset$$

of closed subsets of $Y$ such that $Y_r \setminus Y_{r+1}$ is smooth and all of the cohomology sheaves $H^k(j_r^\dagger M^\cdot)$ are integrable connections, where $j_r : Y_r \setminus Y_{r+1} \hookrightarrow X$ denotes the embedding. We will prove this statement by induction on $\dim Y$. Take an open dense smooth subset $V$ of $Y$, and let $i : V \to X$ denote the embedding. By Kashiwara's equivalence we have $i^\dagger M^\cdot \in D_c^b(D_V)$. Hence by Lemma 3.3.2 there exists an open dense subset $V'$ of $V$ such that each cohomology sheaf $H^k(i^\dagger M^\cdot)|_{V'}$ is projective over $\mathcal{O}_{V'}$. Therefore, by shrinking $V$ if necessary we may assume from the beginning that each cohomology sheaf $H^k(i^\dagger M^\cdot)$ is coherent over $D_V$ and projective over $\mathcal{O}_V$. We first show that $H^k(i^\dagger M^\cdot)$ is an integrable connection. Take $x \in V$ and denote by $j_x : \{x\} \hookrightarrow V$ the embedding. Then we have

$$\mathbb{C} \otimes_{\mathcal{O}_{V,x}} H^k(i^\dagger M^\cdot)_x \simeq H^{k+d_V}(j_x^\dagger i^\dagger M^\cdot) \simeq H^{k+d_V}(i_x^\dagger M^\cdot),$$

where the first isomorphism follows from the fact that $H^k(i^\dagger M^\cdot)_x$ is projective over $\mathcal{O}_{V,x}$. Hence the finite-dimensionality of $H^{k+d_V}(i_x^\dagger M^\cdot)$ implies that the rank of the projective $\mathcal{O}_{V,x}$-module $H^k(i^\dagger M^\cdot)_x$ is finite. It follows that $H^k(i^\dagger M^\cdot)$ is coherent over $\mathcal{O}_V$, hence an integrable connection. Now take an open subset $U$ of $X$ such that $V = Y \cap U$, and let $j : U \to X$ be the embedding. Define $N^\cdot$ by the distinguished triangle

$$N^\cdot \longrightarrow M^\cdot \longrightarrow \int_j j^\dagger M^\cdot \xrightarrow{+1}.$$

We easily see that $\int_j j^\dagger M^\cdot \simeq \int_i i^\dagger M^\cdot \in D_h^b(D_X)$, and hence the above distinguished triangle implies $N^\cdot \in D_c^b(D_X)$. We also easily see that $\mathrm{supp}(N^\cdot) \subset Y \setminus V$. Moreover, for any locally closed smooth subset $Z$ of $Y \setminus V$ we have $i_Z^\dagger M^\cdot \simeq i_Z^\dagger N^\cdot$, where $i_Z : Z \to X$ denotes the embedding. Indeed, we have $i_Z^\dagger \int_j = 0$ by Proposition 1.7.1 (ii). In particular, for any $x \in Y \setminus V$ we have $H^*(i_x^\dagger M^\cdot) \simeq H^*(i_x^\dagger N^\cdot)$. Hence by applying the hypothesis of induction to $N^\cdot$ there exists a decreasing sequence

$$Y \setminus V = Y_1 \supset \cdots \supset Y_m \supset Y_{m+1} = \emptyset$$

of closed subsets of $Y \setminus V$ such that $Y_r \setminus Y_{r+1}$ is smooth and all of the cohomology sheaves $H^k(j_r^\dagger M^\cdot)$ are integrable connections, where $j_r : Y_r \setminus Y_{r+1} \hookrightarrow X$ denotes the embedding. Then the decreasing sequence

$$Y = Y_0 \supset Y_1 \supset \cdots \supset Y_m \supset Y_{m+1} = \emptyset$$

satisfies the desired property. □

## 3.4 Minimal extensions

A non-zero coherent $D$-module $M$ is called simple if it contains no coherent $D$-submodules other than $M$ or 0. Proposition 3.1.2 implies that for any holonomic $D$-module $M$ there exists a finite sequence

$$M = M_0 \supset M_1 \supset \cdots \supset M_r \supset M_{r+1} = 0$$

of holonomic $D$-submodules such that $M_i/M_{i+1}$ is simple for each $i$ (Jordan–Hölder series of $M$). In this section we will give a classification of simple holonomic $D$-modules. More precisely, we will construct simple holonomic $D$-modules from integrable connections on locally closed smooth subvarieties using functors introduced in earlier sections, and show that any simple holonomic $D$-module is of this type. This construction corresponds via the Riemann–Hilbert correspondence to the minimal extension (Deligne–Goresky–MacPherson extension) in the category of perverse sheaves.

Let $Y$ be a (locally closed) smooth subvariety of a smooth algebraic variety $X$. Assume that the inclusion map $i : Y \hookrightarrow X$ is affine. Then $D_{X\leftarrow Y}$ is locally free over $D_Y$ and $Ri_* = i_*$ (higher cohomology groups vanish). Therefore, for a holonomic $D_Y$-module $M$ we have $H^j \int_i M = H^j \int_{i!} M = 0$ for $^\forall j \neq 0$. Namely, we may regard $\int_i M$ and $\int_{i!} M$ as $D_X$-modules. These $D_X$-modules are holonomic by Theorem 3.2.3. By Theorem 3.2.16 we have a morphism

$$\int_{i!} M \longrightarrow \int_i M$$

in $\mathrm{Mod}_h(D_X)$.

**Definition 3.4.1.** We call the image $L(Y,\ M)$ of the canonical morphism $\int_{i!} M \longrightarrow \int_i M$ the *minimal extension* of $M$.

By Proposition 3.1.2 the minimal extension $L(Y,\ M)$ is a holonomic $D_X$-module.

**Theorem 3.4.2.**

(i) *Let $Y$ be a locally closed smooth connected subvariety of $X$ such that $i : Y \to X$ is affine, and let $M$ be a simple holonomic $D_Y$-module. Then the minimal extension $L(Y,\ M)$ is also simple, and it is characterized as the unique simple submodule (resp. unique simple quotient module) of $\int_i M$ (resp. of $\int_{i!} M$).*

(ii) *Any simple holonomic $D_X$-module is isomorphic to the minimal extension $L(Y, M)$ for some pair $(Y, M)$, where $Y$ is as in* (i) *and $M$ is a simple integrable connection on $Y$.*

(iii) *Let $(Y, M)$ be as in* (ii), *and let $(Y', M')$ be another such pair. Then we have $L(Y, M) \simeq L(Y', M')$ if and only if $\overline{Y} = \overline{Y'}$ and $M|_U \simeq M'|_U$ for an open dense subset $U$ of $Y \cap Y'$.*

*Proof.* (i) We choose an open subset $U \subset X$ containing $Y$ such that $k : Y \hookrightarrow U$ is a closed embedding. Let $j : U \hookrightarrow X$ be the embedding, and let $\mathrm{Mod}_{qc}^{\overline{Y}} D_X$ denote the category of $\mathcal{O}_X$-quasi-coherent $D_X$-modules whose support is contained in $\overline{Y}$. We first show the following four results:

(a) For any $E \in \mathrm{Mod}_{qc}^{\overline{Y}}(D_X)$ we have $H^l i^\dagger E = 0$ $(l \neq 0)$. Hence $H^0 i^\dagger = i^\natural : \mathrm{Mod}_{qc}^{\overline{Y}}(D_X) \to \mathrm{Mod}_{qc}(D_Y)$ is an exact functor.

(b) For any non-zero holonomic submodule $N$ of $\int_i M$, we have $i^\dagger N \simeq M$.

(c) $\int_i M$ (resp. $\int_{i!} M$) has a unique simple holonomic submodule (resp. simple holonomic quotient module).

(d) For a sequence $0 \neq N_1 \subset N_2 \subset \int_i M$ of holonomic submodules of $\int_i M$, we have $i^\dagger(N_2/N_1) = 0$.

For $E \in \mathrm{Mod}_{qc}^{\overline{Y}}(D_X)$ we have $i^\dagger E = k^\dagger j^\dagger E = k^\dagger j^{-1} E$ and $\mathrm{supp}\, j^{-1}E \subset Y$. Hence (a) is a consequence of Kashiwara's equivalence.

Let $N$ be as in (b). By Corollary 3.2.15 we have

$$\begin{aligned}\mathrm{Hom}_{D_X}\Big(N, \int_i M\Big) &= \mathrm{Hom}_{D_X}\Big(N, \int_j \int_k M\Big)\\ &\simeq \mathrm{Hom}_{D_U}\Big(j^\bigstar N, \int_k M\Big).\end{aligned}$$

Since $j$ is an open embedding, we have $j^\bigstar = j^\dagger = j^{-1}$. Therefore, the inclusion $N \hookrightarrow \int_i M$ induces a non-zero morphism $\varphi : j^\dagger N \to \int_k M$. Since $\int_k M$ is a simple holonomic $D_U$-module by Kashiwara's equivalence, $\varphi$ is surjective. Applying $k^\dagger$ to it, we obtain a surjective morphism $i^\dagger N \twoheadrightarrow k^\dagger \int_k M \simeq M$. On the other hand, we have an injective morphism $i^\dagger N \to i^\dagger \int_i M = M$ because $i^\dagger$ is exact by (a). Hence we must have $i^\dagger N \simeq M$, and (b) is proved.

Suppose there exist two simple holonomic submodules $L \neq L'$ of $\int_i M$. Set $N = L + L' = L \oplus L'$. Then by (b) we have

$$M \simeq i^\dagger N = i^\dagger L \oplus i^\dagger L' = M \oplus M,$$

which is a contradiction. The assertion (c) for $\int_i M$ is proved. Another assertion for $\int_{i!} M$ is easily proved using the duality functor.

By (a) we have

$$i^\dagger N_1 \subset i^\dagger N_2 \subset i^\dagger \int_i M = M, \qquad i^\dagger N_2 / i^\dagger N_1 \simeq i^\dagger(N_2/N_1).$$

Hence (b) implies $i^\dagger(N_2/N_1) = 0$, and (d) is proved.

Now let us finish the proof of (i). By (c) there exists a unique simple holonomic submodule $L$ of $\int_i M$. By Corollary 3.2.15 there exist two isomorphisms

$$\mathrm{Hom}_{D_X}\Big(\int_{i!} M,\ L\Big) \simeq \mathrm{Hom}_{D_Y}(M,\ i^\dagger L) \overset{\text{(b)}}{\simeq} \mathrm{Hom}_{D_Y}(M,\ M),$$
$$\mathrm{Hom}_{D_X}\Big(\int_{i!} M,\ \int_i M\Big) \simeq \mathrm{Hom}_{D_Y}\Big(M,\ i^\dagger \int_i M\Big) \simeq \mathrm{Hom}_{D_Y}(M,\ M),$$

from which we see that the canonical morphism $\int_{i!} M \to \int_i M$ is non-zero and factorizes as $\int_{i!} M \to L \hookrightarrow \int_i M$. Since $L$ is a simple module, the image of this morphism should be $L$. This completes the proof of (i).

(ii) Assume that $L$ is a simple holonomic $D_X$-module. We take an affine open dense subset $Y$ ($i : Y \hookrightarrow X$) of an irreducible component of $\mathrm{supp}\, L$ so that $i^\dagger L$ is an integrable connection on $Y$ (this is possible by Proposition 3.1.6). Set $M = i^\dagger L$. We easily see by Proposition 3.1.7 that $M$ is simple. Moreover, by Corollary 3.2.15 we get an isomorphism

$$\mathrm{Hom}_{D_X}\Big(\int_{i!} M,\ L\Big) \simeq \mathrm{Hom}_{D_Y}(M,\ i^\dagger L) \simeq \mathrm{Hom}_{D_Y}(M,\ M) \neq 0,$$

from which we see that there exists a non-zero surjective morphism $\int_{i!} M \to L$. Namely, $L$ is a simple holonomic quotient module of $\int_{i!} M$. Hence we obtain $L = L(Y,\ M)$ by (i). The assertion (ii) is proved.

The proof for the last part (iii) is easy and left to the readers. □

**Proposition 3.4.3.** *Let $Y$ be a locally closed smooth subvariety of $X$ such that $i : Y \to X$ is affine, and let $M$ be an integrable connection on $Y$. Then we have*

$$\mathbb{D}_X L(Y,\ M) \simeq L(Y,\ \mathbb{D}_Y M).$$

*Proof.* By the exactness of the duality functor we obtain

$$\begin{aligned}\mathbb{D}_X L(Y,\ M) &\simeq \mathrm{Im}(\mathbb{D}_X \int_i M \to \mathbb{D}_X \int_{i!} M) \simeq \mathrm{Im}(\int_{i!} \mathbb{D}_Y M \to \int_i \mathbb{D}_Y M)\\ &= L(Y,\ \mathbb{D}_Y M).\end{aligned}$$

The proof is complete. □

# 4

# Analytic $D$-Modules and the de Rham Functor

Although our objectives in this book are algebraic $D$-modules ($D$-modules on smooth algebraic varieties), we have to consider the corresponding analytic $D$-modules ($D$-modules on the underlying complex manifolds with classical topology) in defining their solution (and de Rham) complexes. In this chapter after giving a brief survey of the general theory of analytic $D$-modules which are partially parallel to the theory of algebraic $D$-modules given in earlier chapters we present fundamental properties on the solution and the de Rham complexes. In particular, we give a proof of Kashiwara's constructibility theorem for analytic holonomic $D$-modules. We note that we also include another shorter proof of this important result in the special case of algebraic holonomic $D$-modules due to Beilinson–Bernstein. Therefore, readers who are interested only in the theory of algebraic $D$-modules can skip reading Sections 4.4 and 4.6 of this chapter.

## 4.1 Analytic $D$-modules

The aim of this section is to give a brief account of the theory of $D$-modules on complex manifolds. The proofs are occasionally similar to the algebraic cases and are omitted. Readers can refer to the standard textbooks such as Björk [Bj2] and Kashiwara [Kas18] for details.

Let $X$ be a complex manifold. It is regarded as a topological space via the classical topology, and its dimension is denoted by $d_X$. We denote by $\mathcal{O}_X$ the sheaf of holomorphic functions on $X$, and by $\Theta_X$, $\Omega_X^p$ the sheaves of $\mathcal{O}_X$-modules consisting of holomorphic vector fields and holomorphic differential forms of degree $p$, respectively ($0 \leq p \leq d_X$). We also set $\Omega_X = \Omega_X^{d_X}$. The sheaf $D_X$ of holomorphic differential operators on $X$ is defined as the subring of $\mathcal{E}nd_{\mathbb{C}}(\mathcal{O}_X)$ generated by $\mathcal{O}_X$ and $\Theta_X$. In terms of a local coordinate $\{x_i\}_{1\leq i\leq n}$ on a open subset $U$ of $X$ we have

$$D_X|_U = \bigoplus_{\alpha\in\mathbb{N}^n} \mathcal{O}_U \partial^\alpha,$$

where

$$\partial_i = \frac{\partial}{\partial x_i} \quad (1 \leq i \leq n), \qquad \partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} \quad (\alpha = (\alpha_1, \ldots, \alpha_n)).$$

We have the order filtration $F = \{F_l D_X\}_{l \geq 0}$ of $D_X$ given by

$$F_l D_X|_U = \sum_{|\alpha| \leq l} \mathcal{O}_U \partial_x^\alpha \qquad \big(|\alpha| = \sum_i \alpha_i\big),$$

where $U$ and $\{x_i\}$ are as above. It satisfies properties parallel to those in Proposition 1.1.3, and $D_X$ turns out to be a filtered ring. The associated graded ring $\mathrm{gr}\, D_X$ is a sheaf of commutative algebras over $\mathcal{O}_X$, which is canonically regarded as a subalgebra of $\pi_* \mathcal{O}_{T^*X}$, where $\pi : T^*X \to X$ denotes the cotangent bundle of $X$.

Note that we have obvious analogies of the contents of Section 1.2, 1.3. In particular, we have an equivalence

$$\Omega_X \otimes_{\mathcal{O}_X} (\bullet) : \mathrm{Mod}(D_X) \longrightarrow \mathrm{Mod}(D_X^{\mathrm{op}})$$

between the categories $\mathrm{Mod}(D_X)$, $\mathrm{Mod}(D_X^{\mathrm{op}})$ of left and right $D_X$-modules, respectively. Moreover, for a morphism $f : X \to Y$ of complex manifolds we have a $(D_X, f^{-1}D_Y)$-bimodule $D_{X \to Y} = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}D_Y$ and an $(f^{-1}D_Y, D_X)$-bimodule $D_{Y \leftarrow X} = \Omega_X \otimes_{\mathcal{O}_X} D_{X \to Y} \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\Omega_Y^{\otimes -1}$. We say that a $D_X$-module is an integrable connection on $X$ if it is locally free over $\mathcal{O}_X$ of finite rank.

**Notation 4.1.1.** We denote by $\mathrm{Conn}(X)$ the category of integrable connections on the complex manifold $X$.

We have an analogy of Theorem 1.4.10. In particular, $\mathrm{Conn}(X)$ is an abelian category.

The following result is fundamental in the theory of analytic $D$-modules.

**Theorem 4.1.2.**

(i) *$D_X$ is a coherent sheaf of rings.*

(ii) *For any $x \in X$ the stalk $D_{X,x}$ is a noetherian ring with left and right global dimensions* $\dim X$.

The statement (i) follows from the corresponding fact for $\mathcal{O}_X$ due to Oka, and (ii) is proved similarly to the algebraic case.

We can define the notion of a good filtration on a coherent $D_X$-module as in Section 2.1. We remark that in our analytic situation a good filtration on a coherent $D_X$-module exists only locally. In fact, there is an example of a coherent $D_X$-module which does not admit a global good filtration. Nevertheless, this local existence of a good filtration is sufficient for many purposes. For example, we can define the characteristic variety $\mathrm{Ch}(M)$ of a coherent $D_X$-module $M$ as follows. For an open subset $U$ of $X$ such that $M|_U$ admits a good filtration $F$ we have a coherent $\mathcal{O}_{T^*U}$-module $\widetilde{\mathrm{gr}^F(M|_U)} := \mathcal{O}_{T^*U} \otimes_{\pi_U^{-1}\,\mathrm{gr}\, D_U} \pi_U^{-1}\,\mathrm{gr}^F M|_U$, where $\pi_U : T^*U \to U$ denotes the projection. Then the characteristic variety $\mathrm{Ch}(M)$ is defined to be the closed subvariety of $T^*X$ such that $\mathrm{Ch}(M) \cap T^*U = \mathrm{supp}(\widetilde{\mathrm{gr}^F(M|_U)})$ for any $U$ and $F$ as above. It is shown to be well defined by Proposition D.1.3.

As in the algebraic case we have the following.

**Theorem 4.1.3.** *For any coherent $D_X$-module $M$ its characteristic variety* $\mathrm{Ch}(M)$ *is involutive with respect to the canonical symplectic structure of the cotangent bundle $T^*X$. In particular, for any irreducible component $\Lambda$ of* $\mathrm{Ch}(M)$, *we have that* $\dim \Lambda \geq \dim X$.

We say that a coherent $D_X$-module $M$ is holonomic if it satisfies

$$\dim \mathrm{Ch}(M) \leq \dim X.$$

**Notation 4.1.4.**

(i) We denote by $\mathrm{Mod}_c(D_X)$ (resp. $\mathrm{Mod}_h(D_X)$) the category of coherent (resp. holonomic) $D_X$-modules.

(ii) We denote by $D^b_c(D_X)$ (resp. $D^b_h(D_X)$) the subcategory of $D^b(D_X)$ consisting of $M^\cdot \in D^b(D_X)$ satisfying $H^i(M^\cdot) \in \mathrm{Mod}_c(D_X)$ (resp. $\mathrm{Mod}_h(D_X)$) for any $i$.

As in Section 2.6 we can define the duality functor $\mathbb{D}_X : D^b_c(D_X) \to D^b_c(D_X)^{\mathrm{op}}$ satisfying $\mathbb{D}_X^2 \simeq \mathrm{Id}$ by

$$\mathbb{D}_X M^\cdot = R\mathcal{H}om_{D_X}(M^\cdot, D_X \otimes_{\mathcal{O}_X} \Omega_X^{\otimes -1}[d_X]).$$

All of the arguments in Section 2.6 are also valid for analytic $D$-modules. In particular, $\mathbb{D}_X$ induces $\mathbb{D}_X : \mathrm{Mod}_h(D_X) \to \mathrm{Mod}_h(D_X)^{\mathrm{op}}$.

Let $f : X \to Y$ be a morphism of complex manifolds. The functors

$$Lf^* : D^b(D_Y) \to D^b(D_X) \qquad (M^\cdot \mapsto D_{X\to Y} \otimes^L_{f^{-1}D_Y} f^{-1}M^\cdot),$$
$$f^\dagger : D^b(D_Y) \to D^b(D_X) \qquad (M^\cdot \mapsto Lf^*M^\cdot[d_X - d_Y])$$

are called the inverse image functors. Note that the boundedness of $Lf^*M^\cdot$ follows from Theorem 4.1.2 (ii). The notion that $f : X \to Y$ is non-characteristic with respect to a coherent $D_Y$-module $M$ is defined similarly to the algebraic case, and we have the following analogy of Theorems 2.4.6 and 2.7.1.

**Theorem 4.1.5.** *Let $f : X \to Y$ be a morphism of complex manifolds and let $M$ be a coherent $D_Y$-module. Assume that $f$ is non-characteristic with respect to $M$.*

(i) $H^j(Lf^*M) = 0$ *for* $^\forall j \neq 0$.
(ii) $H^0(Lf^*M)$ *is a coherent $D_Y$-module.*
(iii) $\mathrm{Ch}(H^0(Lf^*M)) \subset \rho_f \varpi_f^{-1}(\mathrm{Ch}(M))$.
(iv) $\mathbb{D}_X(Lf^*M) \simeq Lf^*(\mathbb{D}_Y M)$.

*Here, $\rho_f : X \times_Y T^*Y \to T^*X$ and $\varpi_f : X \times_Y T^*Y \to T^*Y$ are the canonical morphisms.*

The proof is more or less the same as that for Theorem 2.4.6, 2.7.1.

For a morphism $f : X \to Y$ of complex manifolds we can also define the direct image functor

$$\int_f : D^b(D_X) \to D^b(D_Y) \qquad (M^\cdot \mapsto Rf_*(D_{Y\leftarrow X} \otimes^L_{D_X} M^\cdot)).$$

The fact that $\int_f$ preserves the boundedness can be proved as follows. By decomposing $f$ into a composite of a closed embedding and a projection we may assume that $f$ is either a closed embedding or a projection. The case of a closed embedding is easy. Assume that $f : X = Y \times Z \to Y$ is a projection. We may assume that $M^{\cdot} = M \in \mathrm{Mod}(D_X)$. As in the algebraic case we have $\int_f M = Rf_*(DR_{X/Y}M)$, where $DR_{X/Y}M$ is the relative de Rham complex defined similarly to the algebraic case. Then the assertion follows from the well-known fact that $R^i f_*(K) = 0$ unless $0 \le i \le 2 \dim Z$ for any sheaf $K$ on $X$ (see, e.g., [KS2, Proposition 3.2.2]).

We have the following analogy of Theorem 2.5.1, Theorem 2.7.2.

**Theorem 4.1.6.** *Let $f : X \to Y$ be a proper morphism of complex manifolds. Assume that a coherent $D_X$-module $M$ admits a good filtration locally on $Y$.*

(i) $\int_f M \in D^b_c(D_Y)$.
(ii) $\int_f \mathbb{D}_X M \simeq \mathbb{D}_Y \int_f M$.

The proof of this result is rather involved and omitted (see Kashiwara [Kas18]). In the situation where $f : X \to Y$ comes from a proper morphism of smooth algebraic varieties and $M$ is associated to an algebraic coherent $D$-module (in the sense of Section 4.7 below) the statements (i) and (ii) in Theorem 4.1.6 follow from Theorem 2.5.1, Theorem 2.7.2, respectively, in view of Proposition 4.7.2 (ii) below. We also point out that if $f$ is a projective morphism of complex manifolds, the proof of Theorem 4.1.6 is more or less the same as that of Theorem 2.5.1, 2.7.2.

In the algebraic case holonomicity is preserved under the inverse and direct images; however, in our analytic situation this is true for inverse images but not for general direct images.

**Theorem 4.1.7.** *Let $f : X \to Y$ be a morphism of complex manifolds, and let $M$ be a holonomic $D_Y$-module. Then we have $Lf^*M \in D^b_h(D_X)$.*

**Theorem 4.1.8.** *Let $f : X \to Y$ be a proper morphism of complex manifolds. Assume that a holonomic $D_X$-module $M$ admits a good filtration locally on $Y$. Then we have $\int_f M \in D^b_h(D_Y)$.*

Theorem 4.1.7 is proved using the theory of $b$-functions (see Kashiwara [Kas7]), and Theorem 4.1.8 can be proved using $\mathrm{Ch}(\int_f M) \subset \varpi_f \rho_f^{-1}(\mathrm{Ch}(M))$ and some results from symplectic geometry. The proofs are omitted. We note that in both theorems if we only consider the situation where $f$ comes from a morphism of smooth algebraic varieties and $M$ is associated to an algebraic holonomic $D$-module, then they are consequences of the corresponding facts on algebraic $D$-modules in view of Proposition 4.7.2 below.

**Example 4.1.9.** Let us give an example so that the holonomicity is not preserved by the direct image with respect to a non-proper morphism of complex manifolds even if it comes from a morphism of smooth algebraic varieties. Set $X = \mathbb{C} \setminus \{0\}$, $Y = \mathbb{C}$ and let $x$ be the canonical coordinate of $Y = \mathbb{C}$. Let $j : X \to Y$ be the embedding. We regard it as a morphism of algebraic varieties. If we regard it as a

morphism of complex manifolds, we denote it by $j^{\mathrm{an}} : X^{\mathrm{an}} \to Y^{\mathrm{an}}$. Then we have $H^0(\int_j \mathcal{O}_X) = j_*\mathcal{O}_X$ and $H^0(\int_{j^{\mathrm{an}}} \mathcal{O}_{X^{\mathrm{an}}}) = j^{\mathrm{an}}_*\mathcal{O}_{X^{\mathrm{an}}}$. Note that $j_*\mathcal{O}_X = \mathcal{O}_Y[x^{-1}]$ is holonomic, while $j^{\mathrm{an}}_*\mathcal{O}_{X^{\mathrm{an}}}$ contains non-meromorphic functions like $\exp(x^{-1})$ and is much larger than $\mathcal{O}_{Y^{\mathrm{an}}}[x^{-1}]$. The $D_{Y^{\mathrm{an}}}$-module $\mathcal{O}_{Y^{\mathrm{an}}}[x^{-1}]$ is holonomic; however, $j^{\mathrm{an}}_*\mathcal{O}_{X^{\mathrm{an}}}$ is not even a coherent $D_{Y^{\mathrm{an}}}$-module.

For a closed submanifold $X$ of a complex manifold $Y$ we denote by $\mathrm{Mod}^X_c(D_Y)$ (resp. $\mathrm{Mod}^X_h(D_Y)$) the category of coherent (resp. holonomic) $D_Y$-modules whose support is contained in $X$. Kashiwara's equivalence also holds in the analytic situation.

**Theorem 4.1.10.** *Let $i : X \hookrightarrow Y$ be a closed embedding of complex manifolds. Then the functor $\int_i$ induces equivalences*

$$\mathrm{Mod}_c(D_X) \xrightarrow{\sim} \mathrm{Mod}^X_c(D_Y),$$
$$\mathrm{Mod}_h(D_X) \xrightarrow{\sim} \mathrm{Mod}^X_h(D_Y)$$

*of categories.*

The proof is more or less the same as that of the corresponding result on algebraic $D$-modules.

## 4.2 Solution complexes and de Rham functors

Let $X$ be a complex manifold. For $M^{\cdot} \in D^b(D_X)$ we set

$$\begin{cases} DR_X\, M^{\cdot} := \Omega_X \otimes^L_{D_X} M^{\cdot} \\ \mathrm{Sol}_X\, M^{\cdot} := R\mathcal{H}om_{D_X}(M^{\cdot}, \mathcal{O}_X). \end{cases}$$

We call $DR_X\, M^{\cdot} \in D^b(\mathbb{C}_X)$ (resp. $\mathrm{Sol}_X\, M^{\cdot} \in D^b(\mathbb{C}_X)$) the *de Rham complex* (resp. the *solution complex*) of $M^{\cdot} \in D^b(D_X)$. Then $DR_X(\bullet)$ and $\mathrm{Sol}_X(\bullet)$ define functors

$$DR_X : D^b(D_X) \longrightarrow D^b(\mathbb{C}_X),$$
$$\mathrm{Sol}_X : D^b(D_X) \longrightarrow D^b(\mathbb{C}_X)^{\mathrm{op}}.$$

As we have explained in the introduction, a motivation for introducing the solution complexes $\mathrm{Sol}_X\, M^{\cdot} = R\mathcal{H}om_{D_X}(M^{\cdot}, \mathcal{O}_X)$ came from the theory of linear partial differential equations. In fact, for a coherent $D_X$-module $M$ the sheaf $\mathcal{H}om_{D_X}(M, \mathcal{O}_X)$ (on $X$) is the sheaf of holomorphic solutions to the system of linear PDEs corresponding to $M$.

By (an analogue in the analytic situation of) Proposition 2.6.14 we have the following.

**Proposition 4.2.1.** *For $M^{\cdot} \in D^b_c(D_X)$ we have*

$$DR_X(M^{\cdot}) \simeq R\mathcal{H}om_{D_X}(\mathcal{O}_X, M^{\cdot})[d_X] \simeq \mathrm{Sol}_X(\mathbb{D}_X M^{\cdot})[d_X].$$

Hence properties of $\mathrm{Sol}_X$ can be deduced from those of $DR_X$. The functor $DR_X$ has the advantage that it can be computed using a resolution of the right $D_X$-module $\Omega_X$. In fact, similar to Lemma 1.5.27 we have a locally free resolution

$$0 \to \Omega_X^0 \otimes_{\mathcal{O}_X} D_X \to \cdots \to \Omega_X^{d_X} \otimes_{\mathcal{O}_X} D_X \to \Omega_X \to 0$$

of the right $D_X$-module $\Omega_X$. It follows that for $M \in \mathrm{Mod}(D_X)$ the object $DR_X(M)[-d_X]$ of the derived category is represented by the complex

$$\Omega_X^{\cdot} \otimes_{\mathcal{O}_X} M = \left[\Omega_X^0 \otimes_{\mathcal{O}_X} M \to \cdots \to \Omega_X^{d_X} \otimes_{\mathcal{O}_X} M\right],$$

where

$$d^p : \Omega_X^p \otimes_{\mathcal{O}_X} M \longrightarrow \Omega_X^{p+1} \otimes_{\mathcal{O}_X} M$$

is given by

$$d^p(\omega \otimes s) = d\omega \otimes s + \sum_i dx_i \wedge \omega \otimes \partial_i s \qquad (\omega \in \Omega_X^p, s \in M)$$

($\{x_i, \partial_i\}$ is a local coordinate system of $X$).

Let us consider the case where $M$ is an integrable connection of rank $m$ (a coherent $D_X$-module which is locally free of rank $m$ over $\mathcal{O}_X$). In this case the 0th cohomology sheaf $L := H^0(\Omega_X^{\cdot} \otimes_{\mathcal{O}_X} M) \simeq \mathcal{H}om_{D_X}(\mathcal{O}_X, M)$ of $\Omega_X^{\cdot} \otimes_{\mathcal{O}_X} M$ coincides with the kernel of the sheaf homomorphism

$$d^0 = \nabla : M \simeq \Omega_X^0 \otimes_{\mathcal{O}_X} M \longrightarrow \Omega_X^1 \otimes_{\mathcal{O}_X} M,$$

which is the sheaf

$$M^{\nabla} = \{s \in M \mid \nabla s = 0\} = \{s \in M \mid \Theta_X s = 0\}$$

of *horizontal sections* of the integrable connection $M$. It is a locally free $\mathbb{C}_X$-module of rank $m$ by the classical Frobenius theorem.

**Definition 4.2.2.** We call a locally free $\mathbb{C}_X$-module of finite rank a *local system* on $X$.

**Notation 4.2.3.** We denote by $\mathrm{Loc}(X)$ the category of local systems on $X$.

Using the local system $L = M^{\nabla}$ we have a $D_X$-linear isomorphism $\mathcal{O}_X \otimes_{\mathbb{C}_X} L \simeq M$. Conversely, for a local system $L$ we can define an integrable connection $M$ by $M = \mathcal{O}_X \otimes_{\mathbb{C}_X} L$ and $\nabla = d \otimes \mathrm{id}_L : \mathcal{O}_X \otimes_{\mathbb{C}_X} L \simeq \Omega_X^0 \otimes_{\mathcal{O}_X} M \to \Omega_X^1 \otimes_{\mathbb{C}_X} L \simeq \Omega_X^1 \otimes_{\mathcal{O}_X} M$ such that $M^{\nabla} \simeq L$. As a result, the category of integrable connections on $X$ is equivalent to that of local systems on $X$.

$$\boxed{\text{integrable connections on } X} \longleftrightarrow \boxed{\text{local systems on } X}$$

Under the identification $\mathcal{O}_X \otimes_{\mathbb{C}_X} L \simeq M$, the differentials in the complex $\Omega_X^{\cdot} \otimes_{\mathcal{O}_X} M$ are written explicitly by

$$d \otimes \mathrm{id}_L : \Omega_X^p \otimes_{\mathbb{C}_X} L \longrightarrow \Omega_X^{p+1} \otimes_{\mathbb{C}_X} L.$$

Therefore, the higher cohomology groups $H^i(\Omega_X^{\cdot} \otimes_{\mathcal{O}_X} M)$ $(i \geq 1)$ of the complex $\Omega_X^{\cdot} \otimes_{\mathcal{O}_X} M$ vanish by the holomorphic Poincaré lemma, and we get finally a quasi-isomorphism $\Omega_X^{\cdot} \otimes_{\mathcal{O}_X} M \simeq L = M^{\nabla}$ for an integrable connection $M$. We have obtained the following.

**Theorem 4.2.4.** *Let $M$ be an integrable connection of rank $m$ on a complex manifold $X$. Then $H^i(DR_X(M)) = 0$ for $i \neq -d_X$, and $H^{-d_X}(DR_X(M))$ is a local system on $X$. Moreover, we have an equivalence*

$$H^{-d_X}(DR_X(\bullet)) : \mathrm{Conn}(X) \xrightarrow{\sim} \mathrm{Loc}(X)$$

*of categories.*

**Theorem 4.2.5.** *Let $f : X \to Y$ be a morphism of complex manifolds. For $M^{\cdot} \in D^b(D_X)$ we have an isomorphism*

$$Rf_* DR_X\ M^{\cdot} \simeq DR_Y \int_f M^{\cdot}$$

*in $D^b(\mathbb{C}_Y)$*

*Proof.* By

$$\begin{aligned} DR_Y \int_f M^{\cdot} &= \Omega_Y \otimes^L_{D_Y} Rf_*(D_{Y\leftarrow X} \otimes^L_{D_X} M^{\cdot}) \\ &\simeq Rf_*\big(f^{-1}\Omega_Y \otimes^L_{f^{-1}D_Y} D_{Y\leftarrow X} \otimes^L_{D_X} M^{\cdot}\big), \\ Rf_* DR_X\, M &= Rf_*(\Omega_X \otimes^L_{D_X} M). \end{aligned}$$

It is sufficient to show $\Omega_X \simeq f^{-1}\Omega_Y \otimes^L_{f^{-1}D_Y} D_{Y\leftarrow X}$. This follows easily from Lemma 1.3.4. □

## 4.3 Cauchy–Kowalevski–Kashiwara theorem

The following classical theorem due to Cauchy–Kowalevski is one of the most fundamental results in the theory of PDEs.

**Theorem 4.3.1 (Cauchy–Kowalevski).** *Let $X$ be an open subset of $\mathbb{C}^n$ with a local coordinate $\{z_i, \partial_i\}_{1\leq i\leq n}$, and let $Y$ be the hypersurface of $X$ defined by $Y = \{z_1 = 0\}$. Let $P \in D_X$ be a differential operator of order $m \geq 0$ on $X$ such that $Y$ is non-characteristic with respect to $P$ (this notion is defined similarly to the algebraic case, see Example* 2.4.4*). Then for any holomorphic function $v \in \mathcal{O}_X$ defined on an open neighborhood of $Y$ and any $m$-tuple $(u_0, \ldots, u_{m-1}) \in \mathcal{O}_Y^{\oplus m}$ of holomorphic*

*functions on $Y$, there exists a unique holomorphic solution $u \in \mathcal{O}_X$ defined on an open neighborhood of $Y$ to the Cauchy problem*

$$\begin{cases} Pu = v, \\ \partial_1^j u|_Y = u_j \quad (j = 0, 1, \dots, m-1). \end{cases}$$

For $X, Y, P$ as in Theorem 4.3.1 let $f : Y \to X$ be the inclusion and set $M = D_X/D_X P$. By Theorem 4.1.5 we have $H^i(Lf^*M) = 0$ for $i \neq 0$. Set $M_Y = H^0(Lf^*M)$. Then Theorem 4.3.1 implies in particular that the natural morphism

$$\begin{aligned} &f^{-1}\mathcal{H}om_{D_X}(M, \mathcal{O}_X) \\ &\quad \simeq \{u \in \mathcal{O}_X|_Y \mid Pu = 0\} \longrightarrow \mathcal{O}_Y^{\oplus m} \simeq \mathcal{H}om_{D_Y}(M_Y, \mathcal{O}_Y). \\ &\qquad \cup \qquad\qquad\qquad\quad \cup \\ &\qquad u \qquad\qquad\quad \longmapsto (u|_Y, \partial_1 u|_Y, \dots, \partial_1^{m-1} u|_Y) \end{aligned}$$

obtained by taking the first $m$-traces of $u \in \mathcal{O}_X|_Y$ is an isomorphism (see Example 2.4.4).

In this section we will give a generalization of this result due to Kashiwara. We first note that results in Section 2.4 for algebraic $D$-modules can be formulated in the framework of analytic $D$-modules and proved similarly to the algebraic case. Let $f : Y \to X$ be a morphism of complex manifolds. For any coherent $D_X$-module $M$ we can construct a canonical morphism

$$\begin{aligned} f^{-1}\mathcal{H}om_{D_X}(M, \mathcal{O}_X) &\longrightarrow \mathcal{H}om_{f^{-1}D_X}(f^{-1}M, f^{-1}\mathcal{O}_X) \\ &\longrightarrow \mathcal{H}om_{D_Y}(\mathcal{O}_Y \otimes_{f^{-1}\mathcal{O}_X} f^{-1}M, \mathcal{O}_Y \otimes_{f^{-1}\mathcal{O}_X} f^{-1}\mathcal{O}_X) \\ &\simeq \mathcal{H}om_{D_Y}(\mathcal{O}_Y \otimes_{f^{-1}\mathcal{O}_X} f^{-1}M, \mathcal{O}_Y), \end{aligned}$$

which extends the above trace map in the classical case. The corresponding morphism

$$\begin{aligned} &f^{-1}\operatorname{Sol}_X(M) \left(= f^{-1}R\mathcal{H}om_{D_X}(M, \mathcal{O}_X)\right) \\ &\quad \longrightarrow \operatorname{Sol}_Y(Lf^*M) \left(= R\mathcal{H}om_{D_Y}(Lf^*M, \mathcal{O}_Y)\right) \end{aligned}$$

in the derived category $D^b(\mathbb{C}_Y)$ can be also constructed similarly. The following theorem is a vast generalization of the Cauchy–Kowalevski theorem.

**Theorem 4.3.2 (Kashiwara [Kas1]).** *Let $f : Y \to X$ be a morphism of complex manifolds. Assume that $f$ is non-characteristic for a coherent $D_X$-module $M$. Then we have*

$$f^{-1}\operatorname{Sol}_X(M) \xrightarrow{\sim} \operatorname{Sol}_Y(Lf^*M). \tag{4.3.1}$$

*Proof.* As in the proof of Theorem 2.4.6 we can reduce the problem to the case when $Y$ is a hypersurface in $X$. Since the problem is local, we may assume that $X$ and $Y$ are as in Theorem 4.3.1. By an analogue (in the analytic situation) of Lemma 2.4.7 we have an exact sequence of coherent $D_X$-modules

$$0 \longrightarrow K \longrightarrow L \longrightarrow M \longrightarrow 0$$

where $L = \bigoplus_{i=1}^{r} D_X/D_X P_i$ and $Y$ is non-characteristic with respect to each $P_i$. By the classical Cauchy–Kowalevski theorem (Theorem 4.3.1) we have an isomorphism

$$f^{-1}R\mathcal{H}om_{D_X}(L, \mathcal{O}_X) \xrightarrow{\sim} R\mathcal{H}om_{D_Y}(L_Y, \mathcal{O}_Y).$$

for $L$. Now consider the commutative diagram

$$\begin{array}{ccccc}
0 \longrightarrow f^{-1}\mathcal{H}om_{D_X}(M, \mathcal{O}_X) & \longrightarrow & f^{-1}\mathcal{H}om_{D_X}(L, \mathcal{O}_X) & \longrightarrow \\
\Big\downarrow \mathbf{A} & & \Big\downarrow \wr & \\
0 \longrightarrow \mathcal{H}om_{D_Y}(M_Y, \mathcal{O}_Y) & \longrightarrow & \mathcal{H}om_{D_Y}(L_Y, \mathcal{O}_Y) & \longrightarrow
\end{array}$$

$$\begin{array}{ccccc}
f^{-1}\mathcal{H}om_{D_X}(K, \mathcal{O}_X) & \longrightarrow & f^{-1}\mathcal{E}xt^1_{D_X}(M, \mathcal{O}_X) & \longrightarrow & f^{-1}\mathcal{E}xt^1_{D_X}(L, \mathcal{O}_X) \\
\Big\downarrow \mathbf{B} & & \Big\downarrow & & \Big\downarrow \wr \\
\mathcal{H}om_{D_Y}(K_Y, \mathcal{O}_Y) & \longrightarrow & \mathcal{E}xt^1_{D_Y}(M_Y, \mathcal{O}_Y) & \longrightarrow & \mathcal{E}xt^1_{D_Y}(L_Y, \mathcal{O}_Y)
\end{array}$$

with exact rows. We see from this that the morphism **A** is injective. It implies that the canonical morphism

$$f^{-1}\mathcal{H}om_{D_X}(N, \mathcal{O}_X) \to \mathcal{H}om_{D_Y}(N_Y, \mathcal{O}_Y)$$

is injective for any coherent $D_X$-module $N$ with respect to which $Y$ is non-characteristic. In particular, the morphism **B** is injective because $Y$ is non-characteristic with respect to $K$. Hence by the five lemma, the morphism **A** is an isomorphism. Consequently **B** is also an isomorphism by applying the same argument to $K$ instead of $M$. Repeating this argument we finally obtain the quasi-isomorphism

$$f^{-1}R\mathcal{H}om_{D_X}(M, \mathcal{O}_X) \xrightarrow{\sim} R\mathcal{H}om_{D_Y}(M_Y, \mathcal{O}_Y).$$

This completes the proof. □

By Theorem 4.1.5 (iv), Proposition 4.2.1, and Theorem 4.3.2 we have the following.

**Corollary 4.3.3.** *Let $f : Y \to X$ be a morphism of complex manifolds. Assume that $f$ is non-characteristic for a coherent $D_X$-module $M$. Then we have*

$$DR_Y(Lf^*M) \simeq f^{-1}DR_X(M)[d_Y - d_X].$$

## 4.4 Cauchy problems and micro-supports

Theorem 4.3.2 has been extended into several directions. For example, we refer to [DS1], [Is]. Indeed, the methods used in the proof of Theorem 2.4.6 (see also

Theorem 4.1.5) and Theorem 4.3.2 have many interesting applications. We can prove various results for general systems of linear PDEs by reducing the problems to those for single equations. Let us give an example. Denote by $X_{\mathbb{R}}$ the underlying real manifold of $X$. Then we have a natural isomorphism $T^*X_{\mathbb{R}} \simeq (T^*X)_{\mathbb{R}}$. For a point $p \in T^*_x X_{\mathbb{R}}$ take a real-valued $C^1$-function $\phi : X_{\mathbb{R}} \longrightarrow \mathbb{R}$ such that $d\phi(x) = p$ (here $d\phi$ is the real differential of $\phi$) and denote by $\partial\phi(x) \in T^*_x X$ its holomorphic part. Then by this identification $T^*X_{\mathbb{R}} \simeq (T^*X)_{\mathbb{R}}$. The point $p \in T^*_x X_{\mathbb{R}}$ corresponds to $\partial\phi(x) \in T^*_x X$. As for a more intrinsic construction of the isomorphism $T^*X_{\mathbb{R}} \simeq (T^*X)_{\mathbb{R}}$, see Kashiwara–Schapira [KS2, Section 11.1].

**Theorem 4.4.1.** *Let $\phi : X \longrightarrow \mathbb{R}$ be a real-valued $C^\infty$-function on $X$ such that $S = \{z \in X \mid \phi(z) = 0\} \subset X$ is a real smooth hypersurface and $\Omega = \{z \in X \mid \phi(z) < 0\} \subset X$ is Stein. Identifying $T^*X$ with $T^*X_{\mathbb{R}}$ as above, assume that a coherent $D_X$-module $M$ satisfies the condition* $\mathrm{Ch}(M) \cap T^*_S(X_{\mathbb{R}}) \subset T^*_X X$. *Then for $S_+ = \{z \in X \mid \phi(z) \geq 0\}$ we have*

$$\big[\mathrm{R}\Gamma_{S_+} R\mathcal{H}om_{D_X}(M, \mathcal{O}_X)\big]_S \simeq 0.$$

*Proof.* Since we have

$$\big[\mathrm{R}\Gamma_{S_+} R\mathcal{H}om_{D_X}(M, \mathcal{O}_X)\big]_S \simeq R\mathcal{H}om_{D_X}(M, H^1\big[\mathrm{R}\Gamma_{S_+}(\mathcal{O}_X)\big]_S[-1])$$

and $H^1\big[\mathrm{R}\Gamma_{S_+}(\mathcal{O}_X)\big]_S \simeq \big[\Gamma_\Omega(\mathcal{O}_X)/\mathcal{O}_X\big]_S$, the assertion for single equations $M = D_X/D_X P$ is just an interpretation of the classical result in Theorem 4.4.2 below. The general case can be proved by reducing the problem to the case of single equations in the same way as in the proof of Theorem 4.3.2. □

**Theorem 4.4.2.** *Let $\phi : X \longrightarrow \mathbb{R}$ be a real-valued $C^\infty$-function on $X$ such that $S = \{z \in X \mid \phi(z) = 0\} \subset X$ is a real smooth hypersurface and set $\Omega = \{z \in X \mid \phi(z) < 0\}$. For a differential operator $P \in D_X$ assume the condition: $\sigma(P)(z; \partial\phi(z)) \neq 0$ for any $z \in S$. Then we have the following:*

(i) (*Zerner* [Z]) *Let $f$ be a holomorphic function on $\Omega$ such that $Pf$ extends holomorphically across $S = \partial\Omega$ in a neighborhood of $z \in S$. Then $f$ is also holomorphic in a neighborhood of $z$.*

(ii) (*Bony–Schapira* [BS]) *For any $z \in S = \partial\Omega$ the morphism $P : \Gamma_\Omega(\mathcal{O}_X)_z \longrightarrow \Gamma_\Omega(\mathcal{O}_X)_z$ is surjective.*

**Corollary 4.4.3.** *Let $M$, $\phi$, $S$ as in Theorem* 4.4.1. *Then we have an isomorphism*

$$\mathcal{H}om_{D_X}(M, \mathcal{O}_X)_{\bar{\Omega}} \xrightarrow{\sim} \Gamma_\Omega \mathcal{H}om_{D_X}(M, \mathcal{O}_X).$$

*That is, any holomorphic solution to $M$ on $\Omega$ extends across $S$ as a holomorphic solution to $M$.*

*Proof.* Consider the cohomology long exact sequence associated to the distinguished triangle

$$\begin{aligned} \mathrm{R}\Gamma_{S_+}(R\mathcal{H}om_{D_X}(M,\mathcal{O}_X)) &\longrightarrow R\mathcal{H}om_{D_X}(M,\mathcal{O}_X) \\ &\longrightarrow \mathrm{R}\Gamma_{\Omega}(R\mathcal{H}om_{D_X}(M,\mathcal{O}_X)) \overset{+1}{\longrightarrow} \end{aligned}$$

and apply Theorem 4.4.1. □

It is well known that Theorem 4.4.1 is true for arbitrary real-valued $C^\infty$-function $\phi : X \longrightarrow \mathbb{R}$ such that $S = \{z \in X \mid \phi(z) = 0\}$ is smooth. Namely, we do not have to assume that $\Omega = \{z \in X \mid \phi(z) < 0\}$ is Stein. For the proof of this generalization of Theorem 4.4.1, see Kashiwara–Schapira [KS2, Theorem 11.3.3]. This remarkable result was a motivation for introducing the notion of micro-supports in Kashiwara–Schapira [KS1], [KS2].

**Definition 4.4.4.** Let $X$ be a real $C^\infty$-manifold and $F^\cdot \in D^b(\mathbb{C}_X)$. We define a closed $\mathbb{R}_{>0}$-invariant subset $\mathrm{SS}(F^\cdot)$ of $T^*X$ as follows:

$p_0 = (x_0, \xi_0) \notin T^*X$

$\Longleftrightarrow$ There exists an open neighborhood $U$ of $p_0$ in $T^*X$ such that for any $x \in X$ and any $C^\infty$-function $\phi : X \longrightarrow \mathbb{R}$ satisfying $\phi(x) = 0$ and $(x, \operatorname{grad}\phi(x)) \in U$ we have $\mathrm{R}\Gamma_{\{\phi \geq 0\}}(F^\cdot)_x \simeq 0$.

We call $\mathrm{SS}(F^\cdot)$ the *micro-support* of $F^\cdot$.

Note that the notion of micro-supports was recently generalized to that of truncated micro-supports in [KFS]. As we see in the next theorem, using micro-supports we can reconstruct the characteristic variety of a coherent $D_X$-module $M$ from its solution complex $R\mathcal{H}om_{D_X}(M, \mathcal{O}_X)$.

**Theorem 4.4.5 (Kashiwara–Schapira [KS1]).** *Let $X$ be a complex manifold and $M$ a coherent $D_X$-module. Then under the natural identification $(T^*X)_{\mathbb{R}} \simeq T^*X_{\mathbb{R}}$, we have*

$$\mathrm{Ch}(M) = \mathrm{SS}(R\mathcal{H}om_{D_X}(M, \mathcal{O}_X)).$$

The inclusion $\mathrm{Ch}(M) \supset \mathrm{SS}(R\mathcal{H}om_{D_X}(M, \mathcal{O}_X))$ is just an interpretation of Theorem 4.4.1 and its generalization in Kashiwara–Schapira [KS2, Theorem 11.3.3]. The proof of the inverse inclusion is much more difficult and requires the theory of microdifferential operators. See [KS1, Theorem 10.1.1]. Combining Theorem 4.4.1 (or its generalization in [KS2, Theorem 11.3.3]) with Kashiwara's non-characteristic deformation lemma (Theorem C.3.6 in Appendix C), we obtain various global extension theorems for holomorphic solution complexes $R\mathcal{H}om_{D_X}(M, \mathcal{O}_X)$ as in the following theorem.

**Theorem 4.4.6.** *Let $X$ be a complex manifold, $\{\Omega_t\}_{t\in\mathbb{R}}$ a family of relatively compact Stein open subsets of $X$ such that $\partial\Omega_t$ is a $C^\infty$-hypersurface in $X_{\mathbb{R}}$ for any $t \in \mathbb{R}$, and $M$ a coherent $D_X$-module. Identifying $(T^*X)_{\mathbb{R}}$ with $T^*X_{\mathbb{R}}$ assume the following conditions:*

(i) *For any pair $s < t$ of real numbers, $\Omega_s \subset \Omega_t$.*

(ii) *For any* $t \in \mathbb{R}$, $\Omega_t = \bigcup_{s<t} \Omega_s$.
(iii) *For any* $t \in \mathbb{R}$, $\bigcap_{s>t}(\Omega_s \setminus \Omega_t) = \partial\Omega_t$ *and* $\mathrm{Ch}(M) \cap T^*_{\partial\Omega_t}(X_{\mathbb{R}}) \subset T^*_X X$.

*Then we have an isomorphism*

$$\mathrm{R}\Gamma\Big(\bigcup_{s\in\mathbb{R}} \Omega_s,\, R\mathcal{H}om_{D_X}(M, \mathcal{O}_X)\Big) \xrightarrow{\sim} \mathrm{R}\Gamma(\Omega_t,\, R\mathcal{H}om_{D_X}(M, \mathcal{O}_X))$$

*for any* $t \in \mathbb{R}$.

This result will be effectively used in the proof of Kashiwara's constructibility theorem later.

## 4.5 Constructible sheaves

In this section we recall basic facts concerning constructible sheaves on analytic spaces and algebraic varieties. For the details of this subject we refer to Dimca [Di], Goresky–MacPherson [GM2], Kashiwara–Schapira [KS2], Schürmann [Schu], and Verdier [V1].

For a morphism $f : X \to Y$ of analytic spaces we have functors

$$\begin{aligned} f^{-1} &: \mathrm{Mod}(\mathbb{C}_Y) \to \mathrm{Mod}(\mathbb{C}_X), \\ f_* &: \mathrm{Mod}(\mathbb{C}_X) \to \mathrm{Mod}(\mathbb{C}_Y), \\ f_! &: \mathrm{Mod}(\mathbb{C}_X) \to \mathrm{Mod}(\mathbb{C}_Y). \end{aligned}$$

The functor $f^{-1}$ is exact, and the functors $f_*$, $f_!$ are left exact. By taking their derived functors we obtain functors

$$\begin{aligned} f^{-1} &: D^b(\mathbb{C}_Y) \to D^b(\mathbb{C}_X), \\ Rf_* &: D^b(\mathbb{C}_X) \to D^b(\mathbb{C}_Y), \\ Rf_! &: D^b(\mathbb{C}_X) \to D^b(\mathbb{C}_Y) \end{aligned}$$

for derived categories, where $D^b(\mathbb{C}_X) = D^b(\mathrm{Mod}(\mathbb{C}_X))$. We also have a functor

$$f^! : D^b(\mathbb{C}_Y) \to D^b(\mathbb{C}_X)$$

which is right adjoint to $Rf_!$.

Let $X$ be an analytic space. The tensor product gives a functor

$$(\bullet) \otimes_{\mathbb{C}} (\bullet) : D^b(\mathbb{C}_X) \times D^b(\mathbb{C}_X) \to D^b(\mathbb{C}_X)$$

sending $(K^\cdot, L^\cdot)$ to $K^\cdot \otimes_{\mathbb{C}} L^\cdot$.

**Definition 4.5.1.** Let $X$ and $Y$ be analytic spaces. For $K^\cdot \in D^b(\mathbb{C}_X)$ and $L^\cdot \in D^b(\mathbb{C}_Y)$ we define $K^\cdot \boxtimes_{\mathbb{C}} L^\cdot \in D^b(\mathbb{C}_{X\times Y})$ by

$$K^\cdot \boxtimes_{\mathbb{C}} L^\cdot = p_1^{-1}K^\cdot \otimes_{\mathbb{C}_{X\times Y}} p_2^{-1}L^\cdot,$$

where $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ are projections.

**Definition 4.5.2.** For an analytic space $X$ we set

$$\omega_X^{\cdot} = a_X^! \mathbb{C} \in D^b(\mathbb{C}_X),$$

where $a_X : X \to \mathrm{pt}$ is the unique morphism from $X$ to the one-point space pt. We call it the *dualizing complex* of $X$.

When $X$ is a complex manifold, $\omega_X$ is isomorphic to $\mathbb{C}_X[2\dim X]$. The *Verdier dual* $\mathbf{D}_X(F^{\cdot})$ of $F^{\cdot} \in D^b(\mathbb{C}_X)$ is defined by

$$\mathbf{D}_X(F^{\cdot}) := R\mathcal{H}om_{\mathbb{C}_X}(F^{\cdot}, \omega_X{}^{\cdot}) \in D^b(\mathbb{C}_X).$$

It defines a functor

$$\mathbf{D}_X : D^b(\mathbb{C}_X) \to D^b(\mathbb{C}_X)^{\mathrm{op}}.$$

Recall that a locally finite partition $X = \bigsqcup_{\alpha\in A} X_\alpha$ of an analytic space $X$ by locally closed analytic subsets $X_\alpha$ ($\alpha \in A$) is called a *stratification* of $X$ if, for any $\alpha \in A$, $X_\alpha$ is smooth (hence a complex manifold) and $\overline{X}_\alpha = \sqcup_{\beta\in B} X_\beta$ for a subset $B$ of $A$. Each complex manifold $X_\alpha$ for $\alpha \in A$ is called a *stratum* of the stratification $X = \bigsqcup_{\alpha\in A} X_\alpha$.

**Definition 4.5.3.** Let $X$ be an analytic space. A $\mathbb{C}_X$-module $F$ is called a *constructible sheaf* on $X$ if there exists a stratification $X = \bigsqcup_{\alpha\in A} X_\alpha$ of $X$ such that the restriction $F|_{X_\alpha}$ is a local system on $X_\alpha$ for ${}^\forall\alpha \in A$.

**Notation 4.5.4.** For an analytic space $X$ we denote by $D^b_c(X)$ the full subcategory of $D^b(\mathbb{C}_X)$ consisting of bounded complexes of $\mathbb{C}_X$-modules whose cohomology groups are constructible.

**Example 4.5.5.** On the complex plane $X = \mathbb{C}$ let us consider the ordinary differential equation $(x\frac{d}{dx} - \lambda)u = 0$ ($\lambda \in \mathbb{C}$). Denote by $\mathcal{O}_X$ the sheaf of holomorphic functions on $X$ and define a subsheaf $F \subset \mathcal{O}_X$ of holomorphic solutions to this ordinary equation by

$$F = \left\{ u \in \mathcal{O}_X \;\middle|\; \left(x\frac{d}{dx} - \lambda\right)u = 0 \right\}.$$

Then the sheaf $F$ is constructible with respect to the stratification $X = (\mathbb{C} - \{0\}) \sqcup \{0\}$ of $X$. Indeed, the restriction $F|_{\mathbb{C}-\{0\}} \simeq \mathbb{C}x^\lambda$ of $F$ to $\mathbb{C} - \{0\}$ is a locally free sheaf of rank one over $\mathbb{C}_{\mathbb{C}-\{0\}}$ and the stalk at $0 \in X = \mathbb{C}$ is calculated as follows:

$$F_0 \simeq \begin{cases} \mathbb{C} & \lambda = 0, 1, 2, \dots \\ 0 & \text{otherwise.} \end{cases}$$

For an algebraic variety $X$ we denote the underlying analytic space by $X^{\mathrm{an}}$. For a morphism $f : X \to Y$ of algebraic varieties we denote the corresponding morphism for analytic spaces by $f^{\mathrm{an}} : X^{\mathrm{an}} \to Y^{\mathrm{an}}$. A locally finite partition $X = \bigsqcup_{\alpha\in A} X_\alpha$ of an algebraic variety $X$ by locally closed subvarieties $X_\alpha$ ($\alpha \in A$) is called a stratification of $X$ if for any $\alpha \in A$ $X_\alpha$ is smooth and $\overline{X}_\alpha = \sqcup_{\beta\in B} X_\beta$ for a subset $B$ of $A$. A stratification $X = \bigsqcup_{\alpha\in A} X_\alpha$ of an algebraic variety $X$ induces a stratification $X^{\mathrm{an}} = \bigsqcup_{\alpha\in A} X_\alpha^{\mathrm{an}}$ of the corresponding analytic space $X^{\mathrm{an}}$.

**Definition 4.5.6.** Let $X$ be an algebraic variety. A $\mathbb{C}_{X^{\mathrm{an}}}$-module $F$ is called an *algebraically constructible sheaf* if there exists a stratification $X = \bigsqcup_{\alpha\in A} X_\alpha$ of $X$ such that $F|_{X_\alpha^{\mathrm{an}}}$ is a locally constant sheaf on $X_\alpha^{\mathrm{an}}$ for $^\forall\alpha \in A$.

**Notation 4.5.7.**

(i) For an algebraic variety $X$, we denote by $D_c^b(X)$ the full subcategory of $D^b(\mathbb{C}_{X^{\mathrm{an}}})$ consisting of bounded complexes of $\mathbb{C}_{X^{\mathrm{an}}}$-modules whose cohomology groups are algebraically constructible (note that $D_c^b(X)$ is not a subcategory of $D^b(\mathbb{C}_X)$ but of $D^b(\mathbb{C}_{X^{\mathrm{an}}})$).

(ii) For an algebraic variety $X$ we write $\omega_{X^{\mathrm{an}}}^{\cdot}$ and $\mathbf{D}_{X^{\mathrm{an}}} : D^b(\mathbb{C}_{X^{\mathrm{an}}}) \to D^b(\mathbb{C}_{X^{\mathrm{an}}})^{\mathrm{op}}$ simply as $\omega_X^{\cdot}$ and $\mathbf{D}_X$, respectively, by abuse of notation.

(iii) For a morphism $f : X \to Y$ of algebraic varieties we write $(f^{\mathrm{an}})^{-1}$, $(f^{\mathrm{an}})^!$, $Rf_*^{\mathrm{an}}$, $Rf_!^{\mathrm{an}}$ as $f^{-1}$, $f^!$, $Rf_*$, $Rf_!$, respectively.

**Theorem 4.5.8.**

(i) *Let $X$ be an algebraic variety or an analytic space. Then we have $\omega_X^{\cdot} \in D_c^b(X)$. Moreover, the functor $\mathbf{D}_X$ preserves the category $D_c^b(X)$ and $\mathbf{D}_X \circ \mathbf{D}_X \simeq \mathrm{Id}$ on $D_c^b(X)$.*

(ii) *Let $f : X \to Y$ be a morphism of algebraic varieties or analytic spaces. Then the functors $f^{-1}$, and $f^!$ induce*

$$f^{-1},\ f^! : D_c^b(Y) \longrightarrow D_c^b(X),$$

*and we have*

$$f^! = \mathbf{D}_X \circ f^{-1} \circ \mathbf{D}_Y$$

*on $D_c^b(Y)$.*

(iii) *Let $f : X \to Y$ be a morphism of algebraic varieties or analytic spaces. We assume that $f$ is proper in the case where $f$ is a morphism of analytic spaces. Then the functors $Rf_*$, $Rf_!$ induce*

$$Rf_*,\ Rf_! : D_c^b(X) \longrightarrow D_c^b(Y),$$

*and we have*

$$Rf_! = \mathbf{D}_Y \circ Rf_* \circ \mathbf{D}_X$$

*on $D_c^b(X)$.*

(iv) *Let $X$ be an algebraic variety or an analytic space. Then the functor $(\bullet) \otimes_{\mathbb{C}} (\bullet)$ induces*

$$(\bullet) \otimes_{\mathbb{C}} (\bullet) : D_c^b(X) \times D_c^b(X) \longrightarrow D_c^b(X).$$

**Proposition 4.5.9.**

(i) *Let $f_i : X_i \to Y_i$ $(i = 1, 2)$ be a morphism of algebraic varieties or analytic spaces. Then we have*

$$(f_1 \times f_2)^{-1}(L_1^{\cdot} \boxtimes_{\mathbb{C}} L_2^{\cdot}) \simeq f_1^{-1}L_1^{\cdot} \boxtimes_{\mathbb{C}} f_2^{-1}L_2^{\cdot} \qquad (L_i^{\cdot} \in D^b(Y_i)), \tag{4.5.1}$$

$$(f_1 \times f_2)^!(L_1^{\cdot} \boxtimes_{\mathbb{C}} L_2^{\cdot}) \simeq f_1^!L_1^{\cdot} \boxtimes_{\mathbb{C}} f_2^!L_2^{\cdot} \qquad (L_i^{\cdot} \in D_c^b(Y_i)). \tag{4.5.2}$$

(ii) *Let $f_i : X_i \to Y_i$ $(i = 1, 2)$ be a morphism of algebraic varieties or analytic spaces. We assume that $f$ is proper in the case where $f$ is a morphism of analytic spaces. Then we have*

$$R(f_1 \times f_2)_!(K_1^{\cdot} \boxtimes_{\mathbb{C}} K_2^{\cdot}) \simeq Rf_{1!}K_1^{\cdot} \boxtimes_{\mathbb{C}} Rf_{2!}K_2^{\cdot} \qquad (K_i^{\cdot} \in D^b(X_i)), \quad (4.5.3)$$

$$R(f_1 \times f_2)_*(K_1^{\cdot} \boxtimes_{\mathbb{C}} K_2^{\cdot}) \simeq Rf_{1*}K_1^{\cdot} \boxtimes_{\mathbb{C}} Rf_{2*}K_2^{\cdot} \qquad (K_i^{\cdot} \in D_c^b(X_i)). \quad (4.5.4)$$

(iii) *Let $X_1$, $X_2$ be analytic spaces. Then we have*

$$\mathbf{D}_{X_1 \times X_2}(K_1^{\cdot} \boxtimes_{\mathbb{C}} K_2^{\cdot}) \simeq \mathbf{D}_{X_1}(K_1^{\cdot}) \boxtimes_{\mathbb{C}} \mathbf{D}_{X_2}(K_2^{\cdot}) \qquad (K_i^{\cdot} \in D_c^b(X_i)).$$

*Proof.* Note that (4.5.1) follows easily from the definition, and (4.5.3) is a consequence of the projection formula (see Proposition C.2.6). Hence in view of Theorem 4.5.8 we have only to show (iii). Let $p_i : X_1 \times X_2 \to X_i$ $(i = 1, 2)$ be the projections. Then we have

$$\begin{aligned}
\mathbf{D}_{X_1 \times X_2}(K_1^{\cdot} \boxtimes_{\mathbb{C}} K_2^{\cdot}) &\simeq R\mathcal{H}om(p_1^{-1}K_1^{\cdot} \otimes_{\mathbb{C}} p_2^{-1}K_2^{\cdot}, \omega_{X_1 \times X_2}^{\cdot}) \\
&\simeq R\mathcal{H}om(p_1^{-1}K_1^{\cdot}, R\mathcal{H}om(p_2^{-1}K_2^{\cdot}, \omega_{X_1 \times X_2}^{\cdot})) \\
&\simeq R\mathcal{H}om(p_1^{-1}K_1^{\cdot}, \mathbf{D}_{X_1 \times X_2}p_2^{-1}K_2^{\cdot}) \\
&\simeq R\mathcal{H}om(p_1^{-1}K_1^{\cdot}, p_2^{!}\mathbf{D}_{X_2}K_2^{\cdot}) \\
&\simeq \mathbf{D}_{X_1}K_1^{\cdot} \boxtimes_{\mathbb{C}} \mathbf{D}_{X_2}K_2^{\cdot},
\end{aligned}$$

where the last isomorphism is a consequence of [KS1, Proposition 3.4.4]. □

**Definition 4.5.10.** Let $X$ be an algebraic variety or an analytic space. An object $F^{\cdot} \in D_c^b(X)$ is called a *perverse sheaf* if we have

$$\dim \operatorname{supp}(H^j(F^{\cdot})) \leq -j, \qquad \dim \operatorname{supp}(H^j(\mathbf{D}_X F^{\cdot})) \leq -j$$

for any $j \in \mathbb{Z}$. We denote by $\operatorname{Perv}(\mathbb{C}_X)$ the full subcategory of $D_c^b(X)$ consisting of perverse sheaves.

We will present a detailed account of the theory of perverse sheaves in Chapter 8.

## 4.6 Kashiwara's constructibility theorem

In this section we prove some basic properties of holomorphic solutions to holonomic $D$-modules. If $M$ is a holonomic $D_X$-module on a complex manifold $X$, its holomorphic solution complex $\operatorname{Sol}_X(M) = R\mathcal{H}om_{D_X}(M, \mathcal{O}_X)$ possesses very rigid structures. Namely, all the cohomology groups of $\operatorname{Sol}_X(M)$ are constructible sheaves on $X$. In other words, we have $\operatorname{Sol}_X(M) \in D_c^b(\mathbb{C}_X) = D_c^b(X)$. This is the famous *constructibility theorem*, due to Kashiwara [Kas3]. In particular, we obtain

$$\dim H^j \operatorname{Sol}_X(M)_z < +\infty$$

for $^\forall j \in \mathbb{Z}$ and $^\forall z \in X$. Moreover, in his Ph.D. thesis [Kas3], Kashiwara essentially proved that $\mathrm{Sol}_X(M)[d_X]$ satisfies the conditions of perverse sheaves, although the theory of perverse sheaves did not exist at that time. Let us give a typical example. Let $Y$ be a complex submanifold of $X$ with codimension $d = d_X - d_Y$. Then for the holonomic $D_X$-module $M = \mathcal{B}_{Y|X}$ (see Example 1.6.4), the complex

$$\mathrm{Sol}_X(M)[d_X] \simeq (\mathbb{C}_Y[-d])[d_X] = \mathbb{C}_Y[d_Y]$$

is a perverse sheaf on $X$. Before giving the proof of Kashiwara's results, let us recall the following fact. It was shown by Kashiwara that for any holonomic $D_X$-module there exists a Whitney stratification $X = \bigsqcup_{\alpha\in A} X_\alpha$ of $X$ such that $\mathrm{Ch}(M) \subset \bigsqcup_{\alpha\in A} T^*_{X_\alpha}X$. This follows from the geometric fact that $\mathrm{Ch}(M)$ is a $\mathbb{C}^\times$-invariant Lagrangian analytic subset of $T^*X$ (see Theorem E.3.9). Let us fix such a stratification $X = \bigsqcup_{\alpha\in A} X_\alpha$ for a holonomic system $M$.

**Proposition 4.6.1.** *Set $F^\cdot = R\mathcal{H}om_{D_X}(M, \mathcal{O}_X) \in D^b(\mathbb{C}_X) = D^b(X)$. Then for $^\forall j \in \mathbb{Z}$ and $^\forall \alpha \in A$, $H^j(F^\cdot)|_{X_\alpha}$ is a locally constant sheaf on $X_\alpha$.*

*Proof.* Let us fix a stratum $X_{\alpha_0}$. The problem being local, we may assume

$$X_{\alpha_0} = \mathbb{C}^{n-d} = \{z_1 = \cdots = z_d = 0\} \subset X = \mathbb{C}^n_z.$$

It is enough to show that for $^\forall j \in \mathbb{Z}$ and $z_0 \in X_{\alpha_0}$ there exists a small open ball $B(z_0;\varepsilon)$ in $X_{\alpha_0}$ centered at $z_0$ such that the restriction map

$$\Gamma(\overline{B(z_0;\varepsilon)}, H^j(F^\cdot)) \longrightarrow H^j(F^\cdot)_z$$

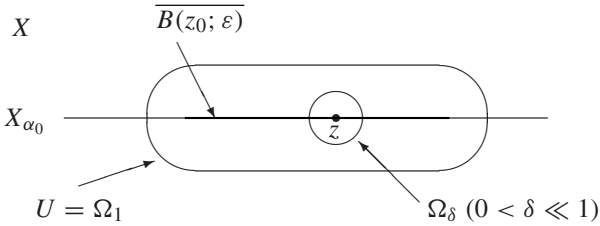
is an isomorphism for $^\forall z \in B(z_0;\varepsilon)$. First, let us treat the case when $j = 0$. Since the geometric normal structure of the Whitney stratification $X = \bigsqcup_{\alpha\in A} X_\alpha$ is locally constant along the stratum $X_{\alpha_0}$ (the Whitney condition (b)), by Theorem 4.4.6 for each $z \in B(z_0;\varepsilon)$ we can choose a sufficiently small open neighborhood $U$ of $\overline{B(z_0;\varepsilon)}$ in $X$ so that we have a quasi-isomorphism

$$\mathrm{R}\Gamma(U, F^\cdot) \longrightarrow F^\cdot_z. \tag{4.6.1}$$

Indeed, by $\mathrm{SS}(F^\cdot) = \mathrm{Ch}(M) \subset \bigsqcup_{\alpha\in A} T^*_{X_\alpha}X$ and the Whitney condition (b) (see Definition E.3.7) we can find a family of increasing open subsets $\{\Omega_t \subset X\}_{t\in(0,1]}$ of $X$ such that

$$\begin{cases} \text{(i) } \Omega_1 = U, \quad \bigcap_{t\in(0,1]} \Omega_t = \{z\} \\ \text{(ii) } \partial\Omega_t \text{ is a real } C^\infty\text{-hypersurface in } X \text{ and } T^*_{\partial\Omega_t}(X) \cap \mathrm{Ch}(M) \subset T^*_X X \end{cases}$$

(see the figure below).



Since $H^j(F^\bullet) = 0$ for $j < 0$, it follows from the quasi-isomorphism (4.6.1) that

$$\Gamma(U, H^0(F^\bullet)) \xrightarrow{\sim} H^0(F^\bullet)_z.$$

If we take an inductive limit of the left-hand side by shrinking $U$, we get the desired isomorphism

$$\Gamma(\overline{B(z_0;\varepsilon)}, H^0(F^\bullet)) \xrightarrow{\sim} H^0(F^\bullet)_z.$$

This shows that $H^0(F^\bullet)|_{X_{\alpha_0}}$ is a locally constant sheaf on $X_{\alpha_0}$ in a neighborhood of $z_0 \in X_{\alpha_0}$. To prove the corresponding assertion for $H^1(F^\bullet)|_{X_{\alpha_0}}$ at the given point $z_0 \in X_\alpha$, first choose a sufficiently small open ball $B(z_0;\varepsilon)$ in $X_{\alpha_0}$ centered at $z_0$ so that we have a quasi-isomorphism

$$\mathrm{R}\Gamma(\overline{B(z_0;\varepsilon)}, F^\bullet) \xrightarrow{\sim} F_z^\bullet$$

for $^\forall z \in B(z_0;\varepsilon)$. Next setting $K = \overline{B(z_0;\varepsilon)}$ and fixing $z \in B(z_0;\varepsilon)$ consider the morphism of distinguished triangles

$$\begin{array}{ccccccc}
\mathrm{R}\Gamma(K, H^0(F^\bullet)) & \longrightarrow & \mathrm{R}\Gamma(K, F^\bullet) & \longrightarrow & \mathrm{R}\Gamma(K, \tau^{\geqslant 1}F^\bullet) & \xrightarrow{+1} & \\
\downarrow & & \wr\downarrow & & \downarrow & & \\
H^0(F^\bullet)_z & \longrightarrow & F_z^\bullet & \longrightarrow & \tau^{\geqslant 1}F_z^\bullet & \xrightarrow{+1} & .
\end{array}$$

Then the leftmost vertical arrow is a quasi-isomorphism, because $H^0(F^\bullet)|_{X_{\alpha_0}}$ is a locally constant sheaf on $X_{\alpha_0}$ and $K$ is contractible. Therefore, the rightmost vertical arrow is also a quasi-isomorphism:

$$\mathrm{R}\Gamma(\overline{B(z_0;\varepsilon)}, \tau^{\geqslant 1}F^\bullet) \xrightarrow{\sim} \tau^{\geqslant 1}F_z^\bullet.$$

Taking $H^1(\bullet)$ of both sides, we finally get

$$\Gamma(\overline{B(z_0;\varepsilon)}, H^1(F^\bullet)) \xrightarrow{\sim} H^1(F^\bullet)_z.$$

By repeating this argument, we can finally show that for all $j \in \mathbb{Z}$, $H^j(F^\bullet)|_{X_{\alpha_0}}$ is a locally constant sheaf on $X_\alpha$ for $^\forall\alpha \in A$. This completes the proof. □

**Proposition 4.6.2.** *Let $M$ be a holonomic $D_X$-module. Then for $^\forall j \in \mathbb{Z}$ and $^\forall z \in X$ the stalk $H^j[R\mathcal{H}om_{D_X}(M, \mathcal{O}_X)]_z$ at $z$ is a finite-dimensional vector space over $\mathbb{C}$.*

*Proof.* Let $X = \bigsqcup_{\alpha\in A} X_\alpha$ be a Whitney stratification of $X$ such that $\mathrm{Ch}(M) \subset \bigsqcup_{\alpha\in A} T^*_{X_\alpha}X$. Let us prove our assertion for $z \in X_\alpha$. By the Whitney condition (b) of the stratification $\bigsqcup_{\alpha\in A} X_\alpha$ we can take a small positive number $\delta > 0$ such that

$$T^*_{\partial(B(z;\varepsilon))}X \cap \mathrm{Ch}(M) \subset T^*_X X$$

for $0 < {}^\forall\varepsilon < \delta$. Here $B(z;\varepsilon)$ is an open ball in $X$ centered at $z$ with radius $\varepsilon$. Therefore, by the non-characteristic deformation lemma (see Theorem 4.4.6) we have

$$\mathrm{R}\Gamma(B(z;\varepsilon_1), R\mathcal{H}om_{D_X}(M,\mathcal{O}_X)) \xrightarrow{\sim} \mathrm{R}\Gamma(B(z;\varepsilon_2), R\mathcal{H}om_{D_X}(M,\mathcal{O}_X))$$

for $0 < {}^{\forall}\varepsilon_2 < {}^{\forall}\varepsilon_1 < \delta$. Since the open balls $B(z;\varepsilon_i)$ $(i = 1, 2)$ are Stein, this quasi-isomorphism can be represented by the morphism

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}_X(B(z;\varepsilon_1))^{N_0} & \xrightarrow{P_1\times} & \mathcal{O}_X(B(z;\varepsilon_1))^{N_1} & \xrightarrow{P_2\times} & \cdots \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_X(B(z;\varepsilon_2))^{N_0} & \xrightarrow{P_1\times} & \mathcal{O}_X(B(z;\varepsilon_2))^{N_1} & \xrightarrow{P_2\times} & \cdots
\end{array}$$

between complexes, where $P_i$ is an $N_i \times N_{i-1}$ matrix of differential operators. Since the vertical arrows are compact maps of Fréchet spaces, the resulting cohomology groups

$$H^i(B(z;\varepsilon_1), R\mathcal{H}om_{D_X}(M,\mathcal{O}_X)) \xrightarrow{\sim} H^i(B(z;\varepsilon_2), R\mathcal{H}om_{D_X}(M,\mathcal{O}_X))$$

are finite dimensional by a standard result in functional analysis. □

By Proposition 4.2.1, 4.6.1 and 4.6.2 we obtain Kashiwara's constructibility theorem:

**Theorem 4.6.3.** *Let $M$ be a holonomic $D$-module on a complex manifold $X$. Then $\mathrm{Sol}_X(M) = R\mathcal{H}om_{D_X}(M,\mathcal{O}_X)$ and $DR_X(M) = \Omega_X \otimes^L_{D_X} M$ are objects in the category $D^b_c(X)$.*

For a holonomic $D_X$-module $M$ we saw that $\mathrm{Sol}_X(M)[d_X]$ and $DR_X(M)$ were constructible sheaves on $X$. Next we will prove moreover that they are dual to each other:

$$DR_X(M) \xrightarrow{\sim} \mathbf{D}_X(\mathrm{Sol}_X(M)[d_X]),$$

where $\mathbf{D}_X : D^b_c(X) \xrightarrow{\sim} D^b_c(X)$ is the Verdier duality functor. For this purpose, recall that for a point $z \in X$ the complex $\mathrm{R}\Gamma_{\{z\}}(\mathcal{O}_X)|_z$ satisfies

$$H^j\big(\mathrm{R}\Gamma_{\{z\}}(\mathcal{O}_X)|_z\big) \simeq 0 \quad \text{for } {}^{\forall} j \neq d_X$$

and that $\mathcal{B}^{\infty}_{\{z\}|X} = H^{d_X}(\mathrm{R}\Gamma_{\{z\}}(\mathcal{O}_X)|_z)$ is an (FS) type (Fréchet–Schwartz type) topological vector space. Regarding $\mathcal{B}^{\infty}_{\{z\}|X}$ as the space of Sato's hyperfunctions supported by the point $z \in X$, we see that the (DFS) type (dual F-S type) topological vector space $(\mathcal{O}_X)_z$ is a topological dual of $\mathcal{B}^{\infty}_{\{z\}|X}$. The following results were also obtained in Kashiwara [Kas3].

**Proposition 4.6.4.** *Let $M$ be a holonomic $D_X$-module. Then*

(i) *For ${}^{\forall}z \in X$ and ${}^{\forall}i \in \mathbb{Z}$, $\mathcal{E}xt^i_{D_X}(M, \mathcal{B}^{\infty}_{\{z\}|X})$ is a finite-dimensional vector space over $\mathbb{C}$.*

(ii) *For ${}^{\forall}z \in X$ and ${}^{\forall}i \in \mathbb{Z}$, the vector spaces $H^{-i}(DR_X(M)_z)$ and $\mathcal{E}xt^i_{D_X}(M, \mathcal{B}^{\infty}_{\{z\}|X})$ are dual to each other.*

(iii) $DR_X(M) \xrightarrow{\sim} \mathbf{D}_X(\mathrm{Sol}_X(M)[d_X])$.

*Proof.* (i) Since $R\mathcal{H}om_{D_X}(M, \mathcal{B}^\infty_{\{z\}|X}) = \mathrm{R}\Gamma_{\{z\}} R\mathcal{H}om_{D_X}(M, \mathcal{O}_X)[d_X]$ and the functor $\mathrm{R}\Gamma_{\{z\}}(\bullet)$ preserves the constructibility, the result follows.

(ii) Let us take a locally free resolution

$$0 \longrightarrow D_X^{N_k} \longrightarrow \cdots \xrightarrow{\times P_2} D_X^{N_1} \xrightarrow{\times P_1} D_X^{N_0} \longrightarrow M \longrightarrow 0$$

of $M$ on an open neighborhood of $z \in X$, where $P_i$ is a $N_i \times N_{i-1}$ matrix of differential operators acting on the right of $D_X^{N_i}$. Then we get

$$\begin{aligned} DR_X(M) &= \big[0 \longrightarrow (\Omega_X)^{N_k} \longrightarrow \cdots \xrightarrow{\times P_2} (\Omega_X)^{N_1} \xrightarrow{\times P_1} (\Omega_X)^{N_0} \longrightarrow 0\big], \\ R\mathcal{H}om_{D_X}(M, \mathcal{B}^\infty_{\{z\}|X}) & \\ &= \big[0 \longrightarrow {\mathcal{B}^\infty_{\{z\}|X}}^{N_0} \xrightarrow{P_1\times} {\mathcal{B}^\infty_{\{z\}|X}}^{N_1} \xrightarrow{P_2\times} \cdots \longrightarrow {\mathcal{B}^\infty_{\{z\}|X}}^{N_k} \longrightarrow 0\big]. \end{aligned}$$

Taking a local coordinate and identifying $\Omega_X$ with $\mathcal{O}_X$, we also have

$$DR_X(M)_z = \big[ \quad \cdots \xrightarrow{P_2^*\times} (\mathcal{O}_X)_z^{N_1} \xrightarrow{P_1^*\times} (\mathcal{O}_X)_z^{N_0} \longrightarrow 0\big],$$

where $P_i^*$ is a formal adjoint of $P_i$. Because $(\mathcal{O}_X)_z$ and $\mathcal{B}^\infty_{\{z\}|X}$ are topological dual to each other, and both $DR_X(M)_z$ and $R\mathcal{H}om_{D_X}(M, \mathcal{B}^\infty_{\{z\}|X})$ have finite-dimensional cohomology groups, we obtain the duality isomorphism

$$\big[H^{-i}(DR_X(M)_z)\big]^* \simeq \mathcal{E}xt^i_{D_X}(M, \mathcal{B}^\infty_{\{z\}|X}).$$

(iii) Since $\mathbb{C}_X \simeq R\mathcal{H}om_{D_X}(\mathcal{O}_X, \mathcal{O}_X)$, we have a natural morphism

$$\begin{aligned} DR_X(M) &= R\mathcal{H}om_{D_X}(\mathcal{O}_X, M)[d_X] \\ &\xrightarrow{\exists} R\mathcal{H}om_{\mathbb{C}_X}\big(R\mathcal{H}om_{D_X}(M, \mathcal{O}_X),\ R\mathcal{H}om_{D_X}(\mathcal{O}_X, \mathcal{O}_X)\big)[d_X] \\ &\simeq R\mathcal{H}om_{\mathbb{C}_X}\big(\mathrm{Sol}_X(M)[d_X],\ \mathbb{C}_X\big)[2d_X] \\ &\simeq \mathbf{D}_X(\mathrm{Sol}_X(M)[d_X]). \end{aligned}$$

Our task is to prove $DR_X(M)_z \simeq \mathbf{D}_X(\mathrm{Sol}_X(M)[d_X])_z$ for $^\forall z \in X$. Indeed, by (ii), we get the following chain of isomorphisms for $i_{\{z\}} : \{z\} \hookrightarrow X$:

$$\begin{aligned} \mathbf{D}_X(\mathrm{Sol}_X(M)[d_X])_z &= i_{\{z\}}^{-1}\mathbf{D}_X(\mathrm{Sol}_X(M)[d_X]) \\ &\simeq \mathbf{D}_{\{\mathrm{pt}\}} i^!_{\{z\}}(\mathrm{Sol}_X(M)[d_X]) \\ &\simeq \big[R\mathcal{H}om_{D_X}(M,\ \mathrm{R}\Gamma_{\{z\}}(\mathcal{O}_X)[d_X])\big]^* \\ &\simeq \big[R\mathcal{H}om_{D_X}(M,\ \mathcal{B}^\infty_{\{z\}|X})\big]^* \\ &\simeq DR_X(M)_z. \end{aligned}$$

This completes the proof. □

**Corollary 4.6.5.** *Let $M$ be a holonomic $D_X$-module and $\mathbb{D}_X M$ its dual. Then we have isomorphisms*

$$\begin{cases} \mathbf{D}_X(DR_X(M)) \simeq DR_X(\mathbb{D}_X M) \\ \mathbf{D}_X(\mathrm{Sol}_X(M)[d_X]) \simeq \mathrm{Sol}_X(\mathbb{D}_X M)[d_X]. \end{cases}$$

*Proof.* The results follow immediately from Proposition 4.6.4 and the formula $DR_X(\mathbb{D}_X M) \simeq \mathrm{Sol}_X(M)[d_X]$. □

**Theorem 4.6.6.** *Let $X$ be a complex manifold and $M$ a holonomic $D$-module on it. Then $\mathrm{Sol}_X(M)[d_X] = R\mathcal{H}om_{D_X}(M, \mathcal{O}_X)[d_X]$ and $DR_X(M) = \Omega_X \otimes^L_{D_X} M$ are perverse sheaves on $X$.*

*Proof.* By $DR_X(M) \simeq \mathrm{Sol}_X(\mathbb{D}_X M)[d_X]$, it is sufficient to prove that $F^\cdot = \mathrm{Sol}_X(M)[d_X]$ is a perverse sheaf for any holonomic $D_X$-module $M$. Moreover, since we have $\mathbf{D}_X(\mathrm{Sol}_X(M)[d_X]) \simeq \mathrm{Sol}_X(\mathbb{D}_X M)[d_X]$ by Corollary 4.6.5, we have only to prove that $\dim \mathrm{supp}(H^j(F^\cdot)) \leq -j$ for $^\forall j \in \mathbb{Z}$. Let us take a Whitney stratification $X = \bigsqcup_{\alpha\in A} X_\alpha$ of $X$ such that $\mathrm{Ch}(M) \subset \bigsqcup_{\alpha\in A} T^*_{X_\alpha} X$ and set $i_{X_\alpha} : X_\alpha \hookrightarrow X$ for $\alpha \in A$. Then by Proposition 4.6.1 the complex $i^{-1}_{X_\alpha} F^\cdot$ of sheaves on $X_\alpha$ has locally constant cohomology groups for $^\forall \alpha \in A$. For $j \in \mathbb{Z}$ set $Z = \mathrm{supp}\, H^j(F^\cdot)$. Then $Z$ is a union of connected components of strata $X_\alpha$'s. We need to prove $\dim Z = d_Z \leq -j$. Choose a smooth point $z$ of $Z$ contained in a stratum $X_\alpha$ such that $\dim X_\alpha = \dim Z$ and take a germ of complex submanifold $Y$ of $X$ at $z$ which intersects with $Z$ transversally at $z \in Z$ ($\dim Y = d_Y = d_X - d_Z$). We can choose the pair $(z, Y)$ so that $Y$ is non-characteristic for $M$, because for the Whitney stratification $X = \bigsqcup_{\alpha\in A} X_\alpha$ we have the estimate $\mathrm{Ch}(M) \subset \bigsqcup_{\alpha\in A} T^*_{X_\alpha} X$. Therefore, by the Cauchy–Kowalevski–Kashiwara theorem (Theorem 4.3.2), we obtain

$$\begin{aligned} F^\cdot|_Y &= R\mathcal{H}om_{D_X}(M, \mathcal{O}_X)|_Y[d_X] \\ &\simeq R\mathcal{H}om_{D_Y}(M_Y, \mathcal{O}_Y)[d_X]. \end{aligned}$$

Our assumption $H^j(F^\cdot)_z \neq 0$ implies $\mathcal{E}xt^{j+d_X}_{D_Y}(M_Y, \mathcal{O}_Y)_z \neq 0$. On the other hand, by Theorem 4.1.2 and

$$R\mathcal{H}om_{D_Y}(M_Y, \mathcal{O}_Y) \simeq R\mathcal{H}om_{D_Y}(M_Y, D_Y) \otimes^L_{D_Y} \mathcal{O}_Y,$$

we have $\mathcal{E}xt^i_{D_Y}(M_Y, \mathcal{O}_Y) = 0$ for $^\forall i > d_Y$. Hence we must have $j + d_X \leq d_Y \Longleftrightarrow d_Z = d_X - d_Y \leq -j$. This completes the proof. □

Let $M$ be a holonomic $D_X$-module as before. Then Kashiwara's constructibility theorem implies that for any point $x \in X$ the *local Euler–Poincaré index*

$$\chi_x[\mathrm{Sol}_X(M)] := \sum_{i\in\mathbb{Z}} (-1)^i \dim \mathcal{E}xt^i_{D_X}(M, \mathcal{O}_X)_x$$

of $\mathrm{Sol}_X(M)$ at $x$ is a finite number (an integer). An important problem is to express this local Euler–Poincaré index in terms of geometric invariants of $M$. This problem

was solved by Kashiwara [Kas8] and its solution has many applications in various fields of mathematics.

Let us briefly explain this result. First take a Whitney stratification $X = \bigsqcup_{\alpha\in A} X_\alpha$ of $X$ such that $\mathrm{Ch}(M) \subset \bigsqcup_{\alpha\in A} T^*_{X_\alpha} X$. Next denote by $m_\alpha \in \mathbb{Z}_{\geq 0}$ the multiplicity of the coherent $\mathcal{O}_{T^*X}$-module $\mathrm{gr}^F M = \mathcal{O}_{T^*X} \otimes_{\pi^{-1}\mathrm{gr}^F D_X} \pi^{-1}(\mathrm{gr}^F M)$ along $T^*_{X_\alpha} X$, where $F$ is a good filtration of $M$ and $\pi : T^*X \to X$ is the projection. Then the characteristic cycle $\mathbf{CC}(M)$ of the (analytic) holonomic $D_X$-module $M$ is defined by

$$\mathbf{CC}(M) := \sum_{\alpha\in A} m_\alpha [T^*_{X_\alpha} X].$$

This is a Lagrangian cycle in $T^*X$. Finally, for an analytic subset $S \subset X$ denote by $Eu_S : S \to \mathbb{Z}$ the *Euler obstruction* of S, which is introduced by Kashiwara [Kas2], [Kas8] and MacPherson [Mac] independently. Recall that for any Whitney stratification of $S$ the Euler obstruction $Eu_S$ is a locally constant function on each stratum (and on the regular part of $S$, the value of $Eu_S$ is one). Then we have

**Theorem 4.6.7 (Kashiwara [Kas2], [Kas8]).** *For any $x \in X$ the local Euler–Poincaré index $\chi_x[\mathrm{Sol}_X(M)]$ of the solution complex $\mathrm{Sol}_X(M)$ at $x$ is given by*

$$\chi_x[\mathrm{Sol}_X(M)] = \sum_{x\in\overline{X_\alpha}} (-1)^{c_\alpha} m_\alpha \cdot Eu_{\overline{X_\alpha}}(x),$$

*where $c_\alpha$ is the codimension of the stratum $X_\alpha$ in $X$.*

*Kashiwara's local index theorem* for holonomic $D$-modules was a starting point of intensive activities in the last decades. The global index theorem was obtained by Dubson [Du] and its generalization to real constructible sheaves was proved by Kashiwara [Kas11] (see also Kashiwara–Schapira [KS2] for the details). As for further developments of the theory of index theorems, see, for example, [BMM], [Gi], [Gui], [SS], [SV], [Tk1]. Note also that Euler obstructions play a central role in the study of characteristic classes of singular varieties (see [Mac], [Sab1]).

## 4.7 Analytic $D$-modules associated to algebraic $D$-modules

Recall that for an algebraic variety $X$ we denote by $X^{\mathrm{an}}$ the corresponding analytic space. We have a morphism $\iota = \iota_X : X^{\mathrm{an}} \to X$ of topological spaces, and a morphism $\iota^{-1}\mathcal{O}_X \to \mathcal{O}_{X^{\mathrm{an}}}$ of sheaves of rings. In other words we have a morphism $(X^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}}) \to (X, \mathcal{O}_X)$ of ringed spaces.

Assume that $X$ is a smooth algebraic variety. Then $X^{\mathrm{an}}$ is a complex manifold, and we have a canonical morphism

$$\iota^{-1} D_X \to D_{X^{\mathrm{an}}}$$

of sheaves of rings satisfying

$$D_{X^{\mathrm{an}}} \simeq \mathcal{O}_{X^{\mathrm{an}}} \otimes_{\iota^{-1}\mathcal{O}_X} \iota^{-1} D_X \simeq \iota^{-1} D_X \otimes_{\iota^{-1}\mathcal{O}_X} \mathcal{O}_{X^{\mathrm{an}}}.$$

Hence we obtain a functor

$$(\bullet)^{\mathrm{an}} : \mathrm{Mod}(D_X) \to \mathrm{Mod}(D_{X^{\mathrm{an}}})$$

sending $M \in \mathrm{Mod}(D_X)$ to $M^{\mathrm{an}} := D_{X^{\mathrm{an}}} \otimes_{\iota^{-1}D_X} \iota^{-1} M \in \mathrm{Mod}(D_{X^{\mathrm{an}}})$. Since $D_{X^{\mathrm{an}}}$ is faithfully flat over $\iota^{-1}D_X$, this functor is exact and extends to a functor

$$(\bullet)^{\mathrm{an}} : D^b(D_X) \to D^b(D_{X^{\mathrm{an}}})$$

between derived categories. Note that $(\bullet)^{\mathrm{an}}$ induces

$$(\bullet)^{\mathrm{an}} : \mathrm{Mod}_c(D_X) \to \mathrm{Mod}_c(D_{X^{\mathrm{an}}}), \qquad (\bullet)^{\mathrm{an}} : D^b(D_X) \to D^b(D_{X^{\mathrm{an}}}).$$

We will sometimes write $(M^{\cdot})^{\mathrm{an}} = D_{X^{\mathrm{an}}} \otimes_{D_X} M^{\cdot}$ by abuse of notation.

The following is easily verified.

**Proposition 4.7.1.** *For $M^{\cdot} \in D^b_c(D_X)$ we have $(\mathbb{D}_X M^{\cdot})^{\mathrm{an}} \simeq \mathbb{D}_{X^{\mathrm{an}}}(M^{\cdot})^{\mathrm{an}}$.*

**Proposition 4.7.2.** *Let $f : X \to Y$ be a morphism of smooth algebraic varieties.*

(i) *For $M^{\cdot} \in D^b(D_Y)$ we have $(f^{\dagger} M^{\cdot})^{\mathrm{an}} \simeq (f^{\mathrm{an}})^{\dagger}(M^{\cdot})^{\mathrm{an}}$.*
(ii) *For $M^{\cdot} \in D^b(D_X)$ we have a canonical morphism $(\int_f M^{\cdot})^{\mathrm{an}} \to \int_{f^{\mathrm{an}}} (M^{\cdot})^{\mathrm{an}}$. This morphism is an isomorphism if $f$ is proper and $M^{\cdot} \in D^b_c(D_X)$.*

*Proof.* The proof of (i) is easy and omitted.

Let us show (ii). First note that there exists a canonical morphism

$$(f^{\mathrm{an}})^{-1} D_{Y^{\mathrm{an}}} \otimes_{(f^{\mathrm{an}})^{-1}\iota_Y^{-1} D_Y} \iota_X^{-1} D_{Y \leftarrow X} \to D_{Y^{\mathrm{an}} \leftarrow X^{\mathrm{an}}}.$$

Indeed, it is obtained by applying the side-changing operation to

$$\begin{aligned}
&\iota_X^{-1} D_{X \to Y} \otimes_{(f^{\mathrm{an}})^{-1}\iota_Y^{-1} D_Y} (f^{\mathrm{an}})^{-1} D_{Y^{\mathrm{an}}} \\
&\quad = \iota_X^{-1} \mathcal{O}_X \otimes_{(f^{\mathrm{an}})^{-1}\iota_Y^{-1}\mathcal{O}_Y} (f^{\mathrm{an}})^{-1} D_{Y^{\mathrm{an}}} \\
&\quad \simeq \left( \iota_X^{-1} \mathcal{O}_X \otimes_{(f^{\mathrm{an}})^{-1}\iota_Y^{-1}\mathcal{O}_Y} (f^{\mathrm{an}})^{-1} \mathcal{O}_{Y^{\mathrm{an}}} \right) \otimes_{(f^{\mathrm{an}})^{-1}\mathcal{O}_{Y^{\mathrm{an}}}} (f^{\mathrm{an}})^{-1} D_{Y^{\mathrm{an}}} \\
&\quad \to \mathcal{O}_{X^{\mathrm{an}}} \otimes_{(f^{\mathrm{an}})^{-1}\mathcal{O}_{Y^{\mathrm{an}}}} (f^{\mathrm{an}})^{-1} D_{Y^{\mathrm{an}}} \\
&\quad = D_{X^{\mathrm{an}} \to Y^{\mathrm{an}}}
\end{aligned}$$

(note $\iota_Y \circ f^{\mathrm{an}} = f \circ \iota_X$). Next note that there exists a canonical morphism

$$\iota_Y^{-1} R f_* K^{\cdot} \to R f_*^{\mathrm{an}} \iota_X^{-1} K^{\cdot}$$

for any $K^{\cdot} \in D^b(f^{-1} D_Y)$. Indeed, it is obtained as the image of Id for

$$\mathrm{Hom}_{\iota_X^{-1} f^{-1} D_Y}(\iota_X^{-1} K^{\cdot}, \iota_X^{-1} K^{\cdot}) \simeq \mathrm{Hom}_{f^{-1} D_Y}(K^{\cdot}, R\iota_{X*} \iota_X^{-1} K^{\cdot})$$

$$\begin{aligned}
&\to \mathrm{Hom}_{D_Y}(Rf_*K^{\cdot}, Rf_*R\iota_{X*}\iota_X^{-1}K^{\cdot}) \\
&\simeq \mathrm{Hom}_{D_Y}(Rf_*K^{\cdot}, R\iota_{Y*}Rf_*^{\mathrm{an}}\iota_X^{-1}K^{\cdot}) \\
&\simeq \mathrm{Hom}_{\iota_Y^{-1}D_Y}(\iota_Y^{-1}Rf_*K^{\cdot}, Rf_*^{\mathrm{an}}\iota_X^{-1}K^{\cdot}).
\end{aligned}$$

Then we obtain

$$\begin{aligned}
\left(\int_f M^{\cdot}\right)^{\mathrm{an}} &= D_{Y^{\mathrm{an}}} \otimes_{\iota_Y^{-1}D_Y}^{L} \iota_Y^{-1}Rf_*(D_{Y\leftarrow X} \otimes_{D_X}^{L} M^{\cdot}) \\
&\to D_{Y^{\mathrm{an}}} \otimes_{\iota_Y^{-1}D_Y}^{L} Rf_*^{\mathrm{an}}\iota_X^{-1}(D_{Y\leftarrow X} \otimes_{D_X}^{L} M^{\cdot}) \\
&\to Rf_*^{\mathrm{an}}\left((f^{\mathrm{an}})^{-1}D_{Y^{\mathrm{an}}} \otimes_{(f^{\mathrm{an}})^{-1}\iota_Y^{-1}D_Y}^{L} \iota_X^{-1}D_{Y\leftarrow X} \otimes_{\iota_X^{-1}D_X}^{L} \iota_X^{-1}M^{\cdot}\right) \\
&\to Rf_*^{\mathrm{an}}\left(D_{Y^{\mathrm{an}}\leftarrow X^{\mathrm{an}}} \otimes_{\iota_X^{-1}D_X}^{L} \iota_X^{-1}M^{\cdot}\right) \\
&\to Rf_*^{\mathrm{an}}\left(D_{Y^{\mathrm{an}}\leftarrow X^{\mathrm{an}}} \otimes_{D_{X^{\mathrm{an}}}}^{L} D_{X^{\mathrm{an}}} \otimes_{\iota_X^{-1}D_X}^{L} \iota_X^{-1}M^{\cdot}\right) \\
&= \int_{f^{\mathrm{an}}} (M^{\cdot})^{\mathrm{an}}.
\end{aligned}$$

It remains to show that $(\int_f M^{\cdot})^{\mathrm{an}} \to \int_{f^{\mathrm{an}}} (M^{\cdot})^{\mathrm{an}}$ is an isomorphism if $f$ is proper. We may assume that $f$ is either a closed embedding or a projection $f : X = Y \times \mathbb{P}^n \to Y$. The case of a closed embedding is easy and omitted. Assume that $f$ is a projection $f : X = Y \times \mathbb{P}^n \to Y$. We may also assume that $M^{\cdot} = M \in \mathrm{Mod}_c(D_X)$. In this case we have

$$\begin{aligned}
\left(\int_f M\right)^{\mathrm{an}} &= \mathcal{O}_{Y^{\mathrm{an}}} \otimes_{\iota_Y^{-1}\mathcal{O}_Y}^{L} Rf_*(DR_{X/Y}(M)), \\
\int_{f^{\mathrm{an}}} M^{\mathrm{an}} &= Rf_*^{\mathrm{an}}(DR_{X^{\mathrm{an}}/Y^{\mathrm{an}}}(M^{\mathrm{an}})),
\end{aligned}$$

and hence it is sufficient to show that

$$\mathcal{O}_{Y^{\mathrm{an}}} \otimes_{\iota_Y^{-1}\mathcal{O}_Y}^{L} Rf_*(DR_{X/Y}(M)^k) \simeq Rf_*^{\mathrm{an}}(DR_{X^{\mathrm{an}}/Y^{\mathrm{an}}}(M^{\mathrm{an}})^k)$$

for each $k$. Since $DR_{X/Y}(M)^k$ is a quasi-coherent $\mathcal{O}_X$-module satisfying

$$\mathcal{O}_{X^{\mathrm{an}}} \otimes_{\iota_X^{-1}\mathcal{O}_X}^{L} DR_{X/Y}(M)^k \simeq DR_{X^{\mathrm{an}}/Y^{\mathrm{an}}}(M^{\mathrm{an}})^k,$$

this follows from the *GAGA-principle*. □

For a smooth algebraic variety $X$ we define functors

$$\begin{aligned}
DR_X &: D^b(D_X) \longrightarrow D^b(\mathbb{C}_{X^{\mathrm{an}}}), \\
\mathrm{Sol}_X &: D^b(D_X) \longrightarrow D^b(\mathbb{C}_{X^{\mathrm{an}}})^{\mathrm{op}}
\end{aligned}$$

by

$$DR_X(M^{\cdot}) := DR_{X^{\mathrm{an}}}((M^{\cdot})^{\mathrm{an}}) = \Omega_{X^{\mathrm{an}}} \otimes^L_{D_{X^{\mathrm{an}}}} (M^{\cdot})^{\mathrm{an}},$$
$$\mathrm{Sol}_X(M^{\cdot}) := \mathrm{Sol}_{X^{\mathrm{an}}}((M^{\cdot})^{\mathrm{an}}) = R\mathcal{H}om_{D_{X^{\mathrm{an}}}}((M^{\cdot})^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}}).$$

**Remark 4.7.3.** It is not a good idea to consider $\Omega_X \otimes^L_{D_X} M^{\cdot}$ and $R\mathcal{H}om_{D_X}(\mathcal{O}_X, M^{\cdot})$ for a smooth algebraic variety $X$ as the following example suggests. Regard $X = \mathbb{C}$ as an algebraic variety, and set $M = D_X/D_X(\frac{d}{dx} - \lambda)$ for $\lambda \in \mathbb{C} \setminus \mathbb{Z}$. Then we easily see that $DR_X(M) \simeq \mathbb{C}_{X^{\mathrm{an}}}[1]$ and $\mathrm{Sol}_X(M) \simeq \mathbb{C}_{X^{\mathrm{an}}}$, while $\Omega_X \otimes^L_{D_X} M = R\mathcal{H}om_{D_X}(\mathcal{O}_X, M)[1] = 0$. This comes from the fact that the differential equation $\frac{du}{dx} = \lambda u$ has a holomorphic solution $\exp(\lambda x) \in \mathcal{O}_{X^{\mathrm{an}}}$ which does not belong to $\mathcal{O}_X$.

By Proposition 4.2.1 and Proposition 4.7.1 we have the following.

**Proposition 4.7.4.** *Let $X$ be a smooth algebraic variety. For $M^{\cdot} \in D^b_c(D_X)$ we have*

$$DR_X(M^{\cdot}) \simeq R\mathcal{H}om_{D_{X^{\mathrm{an}}}}(\mathcal{O}_{X^{\mathrm{an}}}, (M^{\cdot})^{\mathrm{an}})[d_X] \simeq \mathrm{Sol}_X(\mathbb{D}_X M^{\cdot})[d_X].$$

By Theorem 4.2.5 and Proposition 4.7.2 we have the following.

**Proposition 4.7.5.** *Let $f : X \to Y$ be a morphism of smooth algebraic varieties. For $M^{\cdot} \in D^b_c(D_X)$ there exists a canonical morphism*

$$DR_Y(\int_f M) \to Rf_*(DR_X(M^{\cdot})).$$

*This morphism is an isomorphism if $f$ is proper.*

By Corollary 4.3.3 and Proposition 4.7.2 we have the following.

**Proposition 4.7.6.** *Let $f : X \to Y$ be a morphism of smooth algebraic varieties. Assume that $f$ is non-characteristic for a coherent $D_Y$-module $M$. Then we have*

$$DR_X(Lf^*M) \simeq f^{-1} DR_Y(M)[d_X - d_Y].$$

The following is a special case of Kashiwara's constructibility theorem for analytic holonomic $D$-modules. Here we present another proof following Bernstein [Ber3] for the convenience of readers who wants a shortcut for algebraic $D$-modules.

**Theorem 4.7.7.** *For $M^{\cdot} \in D^b_h(D_X)$ we have $DR_X(M^{\cdot}), \mathrm{Sol}_X(M^{\cdot}) \in D^b_c(X)$.*

*Proof.* By Proposition 4.7.4 we have only to show the assertion on $DR_X(M^{\cdot})$. Moreover, we may assume that $M^{\cdot} = M \in \mathrm{Mod}_h(D_X)$. By Proposition 3.1.6 $M$ is generically an integrable connection, and hence $DR_X(M)$ is generically a local system up to a shift of degrees. Hence there exists an open dense subset $U$ of $X$ such that $DR_U(M|_U) \in D^b_c(U)$. Therefore, it is sufficient to show the following.

**Claim.** Let $M \in \mathrm{Mod}_h(D_X)$, and assume that $DR_U(M|_U) \in D^b_c(U)$ for an open dense subset $U$ of $X$. Then there exists an open dense subset $Y$ of $X \setminus U$ such that $DR_{U \cup Y}(M|_{U \cup Y}) \in D^b_c(U \cup Y)$.

For each irreducible component $Z$ of $X \setminus U$ there exists an étale morphism $f$ from an open subset $V$ of $X$ onto an open subset $V'$ of $\mathbb{A}^n$ such that $V \cap (X \setminus U)$ (resp. $V' \cap \mathbb{A}^{n-k}$) is an open dense subset of $Z$ (resp. $\mathbb{A}^{n-k}$) and $f^{-1}(V' \cap \mathbb{A}^{n-k}) = V \cap (X \setminus U)$, where $0 < k \leq n$ and $\mathbb{A}^{n-k}$ is identified with the subset $\{0\} \times \mathbb{A}^{n-k}$ of $\mathbb{A}^n$ (see Theorem A.5.3). Since $f$ is an étale morphism, $DR_V(M|_V) \in D^b_c(V)$ if and only if $f_*(DR_V(M|_V)) \in D^b_c(V')$. Moreover, we have $f_*(DR_V(M|_V)) = DR_{V'}(\int^0_f (M|_V))$ by Proposition 4.7.5. Hence we may assume from the beginning that $X$ is an open subset of $\mathbb{A}^n$, $X \setminus U = X \cap \mathbb{A}^{n-k}$, and $X \cap \mathbb{A}^{n-k}$ is dense in $\mathbb{A}^{n-k}$. Set $T = X \cap \mathbb{A}^{n-k}$. By shrinking $X$ if necessary we may assume that $X$ is an open subset of $\mathbb{A}^k \times T$.

Now we regard $\mathbb{A}^k$ as an open subset of $\mathbb{P}^k$. Set $S = (\mathbb{P}^k \times T) \setminus X$. Then we have $\mathbb{P}^k \times T = S \sqcup U \sqcup T$, $X = U \sqcup T$, and $S$ and $T$ are closed subsets of $\mathbb{P}^k \times T$. Let $p : \mathbb{P}^k \times T \to T$ be the projection and let $j_X : X \to \mathbb{P}^k \times T$, $j_U : U \to \mathbb{P}^k \times T$, $j_S : S \to \mathbb{P}^k \times T$, $j_T : T \to \mathbb{P}^k \times T$ be the embeddings. Set $N^{\cdot} = \int_{j_X} M$, $K^{\cdot} = DR_{\mathbb{P}^k \times T}(N^{\cdot})$. By applying $Rp_*(= Rp_!)$ to the distinguished triangle

$$j_{U!}j_U^{-1}K^{\cdot} \longrightarrow K^{\cdot} \longrightarrow j_{S!}j_S^{-1}K^{\cdot} \oplus j_{T!}j_T^{-1}K^{\cdot} \xrightarrow{+1}$$

we obtain a distinguished triangle

$$R(p \circ j_U)_! j_U^{-1}K^{\cdot} \longrightarrow Rp_*K^{\cdot} \longrightarrow R(p \circ j_S)_! j_S^{-1}K^{\cdot} \oplus R(p \circ j_T)_! j_T^{-1}K^{\cdot} \xrightarrow{+1}.$$

By $j_U^{-1}K^{\cdot} \simeq DR_U(M|_U) \in D^b_c(U)$ we have $R(p \circ j_U)_! j_U^{-1}K^{\cdot} \in D^b_c(T)$. By Proposition 4.7.5 we have

$$Rp_*K^{\cdot} = Rp_* DR_{\mathbb{P}^k \times T}(N^{\cdot}) \simeq DR_T(\int_p N^{\cdot}).$$

By $\int_p N^{\cdot} \in D^b_h(D_T)$ there exists an open dense subset $Y$ of $T$ such that $Rp_*K^{\cdot}|_{Y^{\mathrm{an}}} \in D^b_c(Y)$. It follows from the above distinguished triangle that $R(p \circ j_T)_! j_T^{-1}K^{\cdot}|_{Y^{\mathrm{an}}} \in D^b_c(Y)$. By $p \circ j_T = \mathrm{id}$ we have $R(p \circ j_T)_! j_T^{-1}K^{\cdot} \simeq i^{-1}DR_X(M)$, where $i : T \to X$ is the embedding. Thus $i^{-1}DR_X(M)|_{Y^{\mathrm{an}}} \in D^b_c(Y)$. Hence we have $DR_{U \cup Y}(M|_{U \cup Y}) \in D^b_c(U \cup Y)$. The proof is complete. □

The technique used in the proof of Theorem 4.7.7 also allows us to prove the following results.

**Proposition 4.7.8.** *Let $X$ and $Y$ be smooth algebraic varieties. For $M^{\cdot} \in D^b_c(D_X)$ and $N^{\cdot} \in D^b_c(D_Y)$ we have a canonical morphism*

$$DR_X(M^{\cdot}) \boxtimes_{\mathbb{C}} DR_Y(N^{\cdot}) \to DR_{X \times Y}(M^{\cdot} \boxtimes N^{\cdot}).$$

*This morphism is an isomorphism if $M^{\cdot} \in D^b_h(D_X)$ or $N^{\cdot} \in D^b_h(D_Y)$.*

**Proposition 4.7.9.** *Let $X$ be a smooth algebraic variety. For $M^{\cdot} \in D^b_c(D_X)$ we have canonical morphisms*

$$DR_X(\mathbb{D}_X M^{\cdot}) \to \mathbf{D}_X(DR_X(M^{\cdot})),$$
$$\mathrm{Sol}_X(\mathbb{D}_X M^{\cdot})[d_X] \to \mathbf{D}_X(\mathrm{Sol}_X(M^{\cdot})[d_X]).$$

*These morphisms are isomorphisms if* $M^{\cdot} \in D^b_h(D_X)$.

*Proof of Proposition* 4.7.8. Let $M^{\cdot} \in D^b_c(D_X)$ and $N^{\cdot} \in D^b_c(D_X)$. By

$$(M^{\cdot} \boxtimes N^{\cdot})^{\mathrm{an}} \simeq D_{X^{\mathrm{an}} \times Y^{\mathrm{an}}} \otimes^L_{D_{X^{\mathrm{an}}} \boxtimes_{\mathbb{C}} D_{Y^{\mathrm{an}}}} ((M^{\cdot})^{\mathrm{an}} \boxtimes_{\mathbb{C}} (N^{\cdot})^{\mathrm{an}})$$

we have

$$DR_{X\times Y}(M^{\cdot} \boxtimes N^{\cdot}) \simeq \Omega_{X^{\mathrm{an}} \times Y^{\mathrm{an}}} \otimes^L_{D_{X^{\mathrm{an}}} \boxtimes_{\mathbb{C}} D_{Y^{\mathrm{an}}}} ((M^{\cdot})^{\mathrm{an}} \boxtimes_{\mathbb{C}} (N^{\cdot})^{\mathrm{an}}).$$

On the other hand we have

$$\begin{aligned} DR_X(M^{\cdot}) \boxtimes_{\mathbb{C}} DR_Y(N^{\cdot}) &\simeq (\Omega_{X^{\mathrm{an}}} \otimes^L_{D_{X^{\mathrm{an}}}} (M^{\cdot})^{\mathrm{an}}) \boxtimes_{\mathbb{C}} (\Omega_{Y^{\mathrm{an}}} \otimes^L_{D_{Y^{\mathrm{an}}}} (N^{\cdot})^{\mathrm{an}}) \\ &\simeq (\Omega_{X^{\mathrm{an}}} \boxtimes_{\mathbb{C}} \Omega_{Y^{\mathrm{an}}}) \otimes^L_{D_{X^{\mathrm{an}}} \boxtimes_{\mathbb{C}} D_{Y^{\mathrm{an}}}} ((M^{\cdot})^{\mathrm{an}} \boxtimes_{\mathbb{C}} (N^{\cdot})^{\mathrm{an}}). \end{aligned}$$

Hence the canonical morphism $\Omega_{X^{\mathrm{an}}} \boxtimes_{\mathbb{C}} \Omega_{Y^{\mathrm{an}}} \to \Omega_{X^{\mathrm{an}} \times Y^{\mathrm{an}}}$ induces a canonical morphism

$$DR_X(M^{\cdot}) \boxtimes_{\mathbb{C}} DR_Y(N^{\cdot}) \to DR_{X\times Y}(M^{\cdot} \boxtimes N^{\cdot}).$$

Let us show that this morphism is an isomorphism if either $M^{\cdot} \in D^b_h(D_X)$ or $N^{\cdot} \in D^b_h(D_Y)$. By symmetry we can only deal with the case $M^{\cdot} \in D^b_h(D_X)$.

We first show it when $M^{\cdot}$ is an integrable connection. In this case we have $(M^{\cdot})^{\mathrm{an}} \simeq \mathcal{O}_{X^{\mathrm{an}}} \otimes_{\mathbb{C}_{X^{\mathrm{an}}}} K$ for a local system $K$ on $X^{\mathrm{an}}$ and we have $DR_X(M^{\cdot}) \simeq K[d_X]$. Then we have

$$(M^{\cdot} \boxtimes N^{\cdot})^{\mathrm{an}} \simeq p_1^{-1}K \otimes_{\mathbb{C}_{X^{\mathrm{an}} \times Y^{\mathrm{an}}}} (\mathcal{O}_X \boxtimes N^{\cdot})^{\mathrm{an}} \simeq p_1^{-1}K \otimes_{\mathbb{C}_{X^{\mathrm{an}} \times Y^{\mathrm{an}}}} (p_2^* N^{\cdot})^{\mathrm{an}},$$

where $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ are projections. Hence we have

$$\begin{aligned} DR_{X\times Y}(M^{\cdot} \boxtimes N^{\cdot}) &\simeq p_1^{-1}K \otimes_{\mathbb{C}_{X^{\mathrm{an}} \times Y^{\mathrm{an}}}} DR_{X\times Y}(p_2^* N^{\cdot}) \\ &\simeq p_1^{-1}K \otimes_{\mathbb{C}_{X^{\mathrm{an}} \times Y^{\mathrm{an}}}} p_2^{-1} DR_Y(N^{\cdot})[d_X] \\ &\simeq DR_X(M^{\cdot}) \boxtimes_{\mathbb{C}} DR_Y(N^{\cdot}) \end{aligned}$$

by Proposition 4.7.6.

Finally, we consider the general case. We may assume that $M^{\cdot} = M \in \mathrm{Mod}_h(D_X)$. Since $M$ is generically an integrable connection, there exists an open subset $U$ of $X$ such that the canonical morphism

$$DR_U(M|_U) \boxtimes_{\mathbb{C}} DR_Y(N^{\cdot}) \to DR_{U\times Y}((M|_U) \boxtimes N^{\cdot})$$

is an isomorphism. Therefore, it is sufficient to show the following.

**Claim.** Assume that the canonical morphism

$$DR_U(M|_U) \boxtimes_{\mathbb{C}} DR_Y(N^{\cdot}) \to DR_{U\times Y}((M|_U) \boxtimes N^{\cdot})$$

is an isomorphism for an open dense subset $U$ of $X$. Then there exists an open dense subset $Z$ of $X \setminus U$ such that

$$DR_{U\cup Z}(M|_{U\cup Z}) \boxtimes_{\mathbb{C}} DR_Y(N^{\cdot}) \to DR_{(U\cup Z)\times Y}((M|_{U\cup Z}) \boxtimes N^{\cdot})$$

is an isomorphism.

This can be proved similarly to the claim in Theorem 4.7.7. The details are omitted. □

*Proof of Proposition* 4.7.9. By Proposition 4.7.4 it is sufficient to show that there exists a canonical morphism

$$\mathrm{Sol}_X(M^{\cdot}) \to \mathbf{D}_X(DR_X(M^{\cdot}))[-d_X] \qquad (M^{\cdot} \in D^b_c(D_X)),$$

which turns out to be an isomorphism for $M^{\cdot} \in D^b_h(D_X)$.

Let $M^{\cdot} \in D^b_c(D_X)$. Then we have a canonical morphism

$$\begin{aligned} &R\mathcal{H}om_{D_{X^{\mathrm{an}}}}(\mathcal{O}_{X^{\mathrm{an}}}, (M^{\cdot})^{\mathrm{an}}) \otimes_{\mathbb{C}_{X^{\mathrm{an}}}} R\mathcal{H}om_{D_{X^{\mathrm{an}}}}((M^{\cdot})^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}}) \\ &\quad \to R\mathcal{H}om_{D_{X^{\mathrm{an}}}}(\mathcal{O}_{X^{\mathrm{an}}}, \mathcal{O}_{X^{\mathrm{an}}}). \end{aligned}$$

By

$$\begin{aligned} R\mathcal{H}om_{D_{X^{\mathrm{an}}}}(\mathcal{O}_{X^{\mathrm{an}}}, (M^{\cdot})^{\mathrm{an}}) &\simeq DR_X(M)[-d_X], \\ R\mathcal{H}om_{D_{X^{\mathrm{an}}}}((M^{\cdot})^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}}) &\simeq \mathrm{Sol}_X(M), \\ R\mathcal{H}om_{D_{X^{\mathrm{an}}}}(\mathcal{O}_{X^{\mathrm{an}}}, \mathcal{O}_{X^{\mathrm{an}}}) &\simeq \mathbb{C}_{X^{\mathrm{an}}}, \end{aligned}$$

we obtain

$$DR_X(M) \otimes_{\mathbb{C}_{X^{\mathrm{an}}}} \mathrm{Sol}_X(M) \to \mathbb{C}_{X^{\mathrm{an}}}[d_X].$$

Hence there exists a canonical morphism

$$\mathrm{Sol}_X(M) \to R\mathcal{H}om_{\mathbb{C}_{X^{\mathrm{an}}}}(DR_X(M), \mathbb{C}_{X^{\mathrm{an}}}[d_X]) (\simeq \mathbf{D}_X(DR_X(M^{\cdot}))[-d_X]).$$

Let us show that this morphism is an isomorphism for $M^{\cdot} \in D^b_h(D_X)$. We may assume that $M^{\cdot} = M \in \mathrm{Mod}_h(D_X)$. If $M$ is an integrable connection, then we have $M^{\mathrm{an}} \simeq \mathcal{O}_{X^{\mathrm{an}}} \otimes_{\mathbb{C}_{X^{\mathrm{an}}}} K$ for a local system $K$ on $X^{\mathrm{an}}$, and $\mathrm{Sol}_X(M) \simeq \mathcal{H}om_{\mathbb{C}_{X^{\mathrm{an}}}}(K, \mathbb{C}_{X^{\mathrm{an}}})$, $DR_X(M) \simeq K[d_X]$. Hence the assertion is obvious in this case. Let us consider the general case $M \in \mathrm{Mod}_h(D_X)$. Since $M$ is generically an integrable connection, there exists an open subset $U$ of $X$ such that the canonical morphism

$$\mathrm{Sol}_U(M^{\cdot}|_U) \to \mathbf{D}_U(DR_U(M^{\cdot}|_U))[-d_X]$$

is an isomorphism. Therefore, it is sufficient to show the following.

**Claim.** Assume that the canonical morphism

$$\mathrm{Sol}_U(M^{\cdot}|_U) \to \mathbf{D}_U(DR_U(M^{\cdot}|_U))[-d_X]$$

is an isomorphism for an open dense subset $U$ of $X$. Then there exists an open dense subset $Y$ of $X \setminus U$ such that

$$\mathrm{Sol}_{U\cup Y}(M^{\cdot}|_{U\cup Y}) \to \mathbf{D}_{U\cup Y}(DR_{U\cup Y}(M^{\cdot}|_{U\cup Y}))[-d_X]$$

is an isomorphism.

This can be proved similarly to the claim in Theorem 4.7.7. Details are omitted. □

# 5

# Theory of Meromorphic Connections

In this chapter we present several important results on meromorphic connections such as the Riemann–Hilbert correspondence for regular meromorphic connections due to Deligne. In subsequent chapters these results will be effectively used to establish various properties of regular holonomic systems.

## 5.1 Meromorphic connections in the one-dimensional case

### 5.1.1 Systems of ODEs and meromorphic connections

We start from the classical theory of ordinary differential equations (we call them *ODEs* for short). We always consider the problem in an open neighborhood of $x = 0 \in \mathbb{C}$. Here the complex plane $\mathbb{C}$ is considered as a complex manifold and we use only the classical topology in this subsection and the next. Set $\mathcal{O} = (\mathcal{O}_{\mathbb{C}})_0$ and denote by $K$ its quotient field. Then $K$ is the field of meromorphic functions with possible poles at $x = 0$. Note that $\mathcal{O}$ and $K$ are identified with the ring of convergent power series $\mathbb{C}\{\{x\}\}$ at $x = 0$ and its quotient field $\mathbb{C}\{\{x\}\}[x^{-1}]$, respectively.

For a matrix $A(x) = (a_{ij}(x)) \in M_n(K)$ let us consider the system of ODEs

$$\frac{d}{dx}\vec{u}(x) = A(x)\vec{u}(x), \tag{5.1.1}$$

where $\vec{u}(x) = {}^t(u_1(x), u_2(x), \ldots, u_n(x))$ is a column vector of unknown functions. Setting $\vec{v}(x) = T^{-1}\vec{u}(x)$ for an invertible matrix $T = T(x) \in GL_n(K)$ (5.1.1) is rewritten as

$$\frac{d}{dx}\vec{v}(x) = \Big(T^{-1}AT - T^{-1}\frac{d}{dx}T\Big)\vec{v}(x).$$

Therefore, we say that two systems

$$\frac{d}{dx}\vec{u}(x) = A(x)\vec{u}(x) \qquad (A(x) \in M_n(K))$$

and

$$\frac{d}{dx}\vec{v}(x) = B(x)\vec{v}(x) \qquad (B(x) \in M_n(K))$$

are equivalent if there exists $T \in GL_n(K)$ such that $B = T^{-1}AT - T^{-1}\frac{d}{dx}T$.

As solutions to (5.1.1) we consider holomorphic (but possibly multivalued) solutions on a punctured disk $B_\varepsilon^* = \{x \in \mathbb{C} \mid 0 < |x| < \varepsilon\}$, where $\varepsilon$ is a sufficiently small positive number. Namely, let $\tilde{K}$ denote the ring consisting of possibly multivalued holomorphic functions defined on a punctured disk $B_\varepsilon^*$ for a sufficiently small $\varepsilon > 0$. Then we say that $\vec{u}(x) = {}^t(u_1(x), u_2(x), \ldots, u_n(x))$ is a solution to (5.1.1) if it belongs to $\tilde{K}^n$ and satisfies (5.1.1).

Let us now reformulate these classical notions by the modern language of meromorphic connections.

**Definition 5.1.1.**

(i) Let $M$ be a finite-dimensional vector space $M$ over $K$ endowed with a $\mathbb{C}$-linear map $\nabla : M \to M$. Then $M$ (or more precisely the pair $(M, \nabla)$) is called a *meromorphic connection* (at $x = 0$) if it satisfies the condition

$$\nabla(fu) = \frac{df}{dx}u + f\nabla u \quad (f \in K,\ u \in M). \tag{5.1.2}$$

(ii) Let $(M, \nabla)$ and $(N, \nabla)$ be meromorphic connections. A $K$-linear map $\varphi : M \to N$ is called a morphism of meromorphic connections if it satisfies $\varphi \circ \nabla = \nabla \circ \varphi$. In this case we write $\varphi : (M, \nabla) \to (N, \nabla)$.

**Remark 5.1.2.** The condition (5.1.2) can be replaced with the weaker one

$$\nabla(fu) = \frac{df}{dx}u + f\nabla u \quad (f \in \mathcal{O},\ u \in M). \tag{5.1.3}$$

Indeed, if the condition (5.1.3) holds, then for $f \in \mathcal{O} \setminus \{0\}$, $g \in \mathcal{O}$, $u \in M$, we have

$$\nabla(gu) = \nabla(ff^{-1}gu) = f'f^{-1}gu + f\nabla(f^{-1}gu)$$

and hence

$$\begin{aligned}\nabla(f^{-1}gu) &= -f^{-2}f'gu + f^{-1}\nabla(gu) = (-f^{-2}f'g + f^{-1}g')u + f^{-1}g\nabla u \\ &= (f^{-1}g)'u + f^{-1}g\nabla u.\end{aligned}$$

Meromorphic connections naturally form an abelian category. Note that for a meromorphic connection $(M, \nabla)$ the vector space $M$ is a left $(D_{\mathbb{C}})_0$-module by the action $\frac{d}{dx}u = \nabla u$ $(u \in M)$. Note also that $\nabla$ is uniquely extended to an element of $\mathrm{End}_{\mathbb{C}}(\tilde{K} \otimes_K M)$ satisfying

$$\nabla(fu) = \frac{df}{dx}u + f\nabla u \quad (f \in \tilde{K},\ u \in M),$$

and $\tilde{K} \otimes_K M$ is also a left $(D_{\mathbb{C}})_0$-module. We say that $u \in \tilde{K} \otimes_K M$ is a *horizontal section* of $(M, \nabla)$ if it satisfies $\nabla u = 0$.

Let $(M, \nabla)$ be a meromorphic connection and choose a $K$-basis $\{e_i\}_{1\le i\le n}$ of $M$. Then the matrix $A(x) = (a_{ij}(x)) \in M_n(K)$ defined by

$$\nabla e_j = -\sum_{i=1}^{n} a_{ij}(x) e_i \tag{5.1.4}$$

is called the *connection matrix* of $(M, \nabla)$ with respect to the basis $\{e_i\}_{1\le i\le n}$. In terms of this basis the action of $\nabla$ is described by

$$\nabla\left(\sum_{i=1}^{n} u_i e_i\right) = \sum_{i=1}^{n}\Big(\frac{du_i}{dx} - \sum_{j=1}^{n} a_{ij}u_j\Big)e_i.$$

Hence the condition $\nabla u = 0$ for $u = \sum_{i=1}^{n} u_i e_i \in \tilde{K} \otimes_K M$ is equivalent to the equation

$$\frac{d}{dx}\vec{u}(x) = A(x)\vec{u}(x) \tag{5.1.5}$$

for $\vec{u}(x) = {}^t(u_1(x), \dots, u_n(x)) \in \tilde{K}^n$. We have seen that to each meromorphic connection $(M, \nabla)$ endowed with a $K$-basis $\{e_i\}_{1\le i\le n}$ of $M$ we can associate a system (5.1.5) of ODEs and that the horizontal sections of $(M, \nabla)$ correspond to solutions of (5.1.5). Conversely, to any $A = (a_{ij}(x)) \in M_n(K)$ we can associate a meromorphic connection $(M_A, \nabla_A)$ given by

$$M_A = \bigoplus_{i=1}^{n} K e_i, \qquad \nabla e_j = -\sum_{i=1}^{n} a_{ij}(x) e_i.$$

Under this correspondence we easily see the following.

**Lemma 5.1.3.** *Two systems of ODEs*

$$\frac{d}{dx}\vec{u}(x) = A_1(x)\vec{u}(x) \qquad (A_1(x) \in M_n(K))$$

*and*

$$\frac{d}{dx}\vec{v}(x) = A_2(x)\vec{v}(x) \qquad (A_2(x) \in M_n(K))$$

*are equivalent if and only if the associated meromorphic connections* $(M_{A_1}, \nabla_{A_1})$ *and* $(M_{A_2}, \nabla_{A_2})$ *are isomorphic.*

Let $(M_1, \nabla_1)$, $(M_2, \nabla_2)$ be meromorphic connections. Then $M_1 \otimes_K M_2$ and $\mathrm{Hom}_K(M_1, M_2)$ are endowed with structures of meromorphic connections by

$$\begin{cases} \nabla(u_1 \otimes u_2) = \nabla_1 u_1 \otimes u_2 + u_1 \otimes \nabla_2 u_2 \\ (\nabla\phi)(u_1) = \nabla_2(\phi(u_1)) - \phi(\nabla_1 u_1) \end{cases}$$

($\phi \in \mathrm{Hom}_K(M_1, M_2)$, $u_i \in M_i$).

Note that the one-dimensional $K$-module $K$ is naturally endowed with a structure of a meromorphic connection by $\nabla f = \frac{df}{dx}$. In particular, for a meromorphic connection $(M, \nabla)$, the dual space $M^* = \mathrm{Hom}_K(M, K)$ is endowed with a structure of meromorphic connection by

$$\langle \nabla\phi, u\rangle = \frac{d}{dx}\langle \phi, u\rangle - \langle \phi, \nabla u\rangle \quad (\phi \in M^*, u \in M).$$

If $A = (a_{ij}(x)) \in M_n(K)$ is the connection matrix of $M$ with respect to a $K$-basis $\{e_1, e_2, \ldots, e_n\}$ of $M$, then the connection matrix $A^*$ of $M^*$ with respect to the dual basis $\{e_1^*, e_2^*, \ldots, e_n^*\}$ is given by $A^* = -{}^tA$.

### 5.1.2 Meromorphic connections with regular singularities

For an open interval $(a, b) \subset \mathbb{R}$ and $\epsilon > 0$ we set

$$S^{\varepsilon}_{(a,b)} = \{x \mid 0 < |x| < \varepsilon,\ \arg(x) \in (a, b)\}.$$

It is a subset of (the universal covering of) $\mathbb{C} \setminus \{0\}$ called an open angular sector. We say that a function $f \in \tilde{K}$ is said to have *moderate growth* (or to be in the Nilsson class) at $x = 0$ if it satisfies the following condition:

$$\begin{cases} \text{For any open interval } (a, b) \subset \mathbb{R} \text{ and } \epsilon > 0 \text{ such that } f \text{ is defined on } S^{\varepsilon}_{(a,b)} \\ \text{there exist } C > 0 \text{ and } N \gg 0 \text{ such that } |f(x)| \leq C\,|x|^{-N} \text{ for } {}^\forall x \in S^{\varepsilon}_{(a,b)}. \end{cases}$$

We denote $\tilde{K}^{\mathrm{mod}}$ the set of $f \in \tilde{K}$ which have moderate growth at $x = 0$. Note that in the case where $f$ is single-valued $f$ has moderate growth if and only if it is meromorphic.

Let us consider a system of ODEs:

$$\frac{d}{dx}\vec{u}(x) = A(x)\vec{u}(x) \tag{5.1.6}$$

for $A(x) = (a_{ij}(x)) \in M_n(K)$. It is well known in the theory of linear ODEs that the set of solutions $\vec{u} \in \tilde{K}^n$ to (5.1.6) forms a vector space of dimension $n$ over $\mathbb{C}$. Let us take $n$ linearly independent solutions $\vec{u}_1(x), \vec{u}_2(x), \ldots, \vec{u}_n(x)$ to this equation. Then the matrix $S(x) = (\vec{u}_1(x), \vec{u}_2(x), \ldots, \vec{u}_n(x))$ is called a *fundamental solution matrix* of (5.1.6). Since the analytic continuation of $S(x)$ along a circle around $x = 0 \in \mathbb{C}$ is again a solution matrix of (5.1.6), there exists an invertible matrix $G \in GL_n(\mathbb{C})$ such that

$$\lim_{t \to 2\pi} S(e^{\sqrt{-1}t}x) = S(x)G.$$

The matrix $G$ is called the *monodromy matrix* of the equation (5.1.6). Let us take a matrix $\Gamma \in M_n(\mathbb{C})$ such that $\exp(2\pi\sqrt{-1}\Gamma) = G$ and set $T(x) := S(x)\exp(-\Gamma\log(x))$. Then we can easily check that the entries of $T(x)$ are single-valued functions. Thus we obtained a decomposition $S(x) = T(x)\exp(\Gamma\log(x))$ of $S(x)$, in which the last part $\exp(\Gamma\log(x))$ has the same monodromy as that of $S(x)$.

The following well-known fact is fundamental. We present its proof for the sake of the reader's convenience.

**Theorem 5.1.4.** *The following three conditions on the system* (5.1.6) *are equivalent:*

(i) *The system* (5.1.6) *is equivalent to the system*

$$\frac{d}{dx}\vec{v}(x) = \frac{\Gamma(x)}{x}\vec{v}(x)$$

*for some* $\Gamma(x) \in M_n(\mathcal{O})$.

(ii) *The system* (5.1.6) *is equivalent to the system*

$$\frac{d}{dx}\vec{v}(x) = \frac{\Gamma}{x}\vec{v}(x)$$

*for some* $\Gamma \in M_n(\mathbb{C})$.

(iii) *All solutions to* (5.1.6) *in* $\tilde{K}^n$ *belong to* $(\tilde{K}^{\mathrm{mod}})^n$.

*Proof.* First, let us prove the part (iii) $\Rightarrow$ (ii). Since the entries of a fundamental solution matrix $S(x) = T(x)\exp(\Gamma\log(x))$ and $\exp(-\Gamma\log(x))$ have moderate growth at $x = 0$, the product matrix $S(x)\exp(-\Gamma\log(x)) = T(x)$ has the same property. Therefore, the entries of $T(x)$ must be meromorphic functions, i.e., $T(x) \in GL_n(K)$. If we set $v(x) = T^{-1}(x)u(x)$, then we can easily verify that the system (5.1.6) is equivalent to

$$\frac{d}{dx}\vec{v}(x) = \frac{\Gamma}{x}\vec{v}(x).$$

The part (ii) $\Rightarrow$ (i) is trivial. Finally, let us prove (i) $\Rightarrow$ (iii). We prove that a holomorphic solution $\vec{v}(x) = {}^t(v_1(x), v_2(x), \ldots, v_n(x))$ to the system

$$\frac{d}{dx}\vec{v}(x) = \frac{\Gamma(x)}{x}\vec{v}(x) \quad (\Gamma(x) \in M_n(\mathcal{O}))$$

has moderate growth at $x = 0$. Set $x = re^{i\theta} \in \mathbb{C}^\times$ ($r \geq 0, \theta \in \mathbb{R}$). Then there exists $C > 0$ such that we have

$$\left|\frac{\partial}{\partial r}v_i(re^{i\theta})\right| = \left|\frac{dv_i}{dx}(x)\right| \leq \frac{C}{r}\|\vec{v}(x)\|$$

for each $i = 1, 2, \ldots, n$. Here for $\vec{a} = {}^t(a_1, a_2, \ldots, a_n) \in \mathbb{C}$ we set $\|\vec{a}\| := \sqrt{\sum_{i=1}^n |a_i|^2}$. Let $r_0 > 0$ be a fixed positive real number. For $0 <^{\forall} r < r_0$ and $^{\forall}\theta \in \mathbb{R}$ we have

$$\vec{v}(r_0e^{i\theta}) - \vec{v}(re^{i\theta}) = \int_r^{r_0} \frac{\partial}{\partial s}\vec{v}(se^{i\theta})\,ds,$$

and hence

$$\|\vec{v}(re^{i\theta})\| \leq \|\vec{v}(r_0e^{i\theta})\| + \left\|\int_r^{r_0} \frac{\partial}{\partial s}\vec{v}(se^{i\theta})\,ds\right\|$$

$$\leq \left\| \vec{v}(r_0 e^{i\theta}) \right\| + \sqrt{\sum_{i=1}^{n} \left( \int_r^{r_0} \left| \frac{\partial}{\partial s} v_i(s e^{i\theta}) \right| ds \right)^2}$$

$$\leq \left\| \vec{v}(r_0 e^{i\theta}) \right\| + \sqrt{n} \int_r^{r_0} \frac{C}{s} \left\| \vec{v}(s e^{i\theta}) \right\| ds.$$

Therefore, by *Gronwall's inequality* there exists $C_1, C_2 > 0$ and $N \gg 0$ such that

$$\left\| \vec{v}(r e^{i\theta}) \right\| \leq C_1 \left( \frac{r_0}{r} \right)^{\sqrt{n}C} \leq C_2 |x|^{-N},$$

which implies that $v_i(x)$ $(i = 1, 2, \ldots, n)$ have moderate growth at $x = 0$. □

On a neighborhood of $x = 0$ in $\mathbb{C}$ consider an ordinary differential equation

$$P(x, \partial)\, u = 0 \qquad \left( P(x, \partial) = \sum_{i=0}^{n} a_i(x) \partial^i,\ \partial = \frac{d}{dx} \right),$$

where $a_i(x)$ is holomorphic on a neighborhood of $x = 0$ ($\Leftrightarrow a_i \in \mathcal{O}$) and $a_n(x)$ is not identically zero (i.e., the order of $P(x, \partial)$ is $n$). We can rewrite $P$ in the form

$$P(x, \partial) = \sum_{i=0}^{n} b_i(x) \theta^i,\ b_i(x) \in K$$

with $b_n(x) \neq 0$, where $\theta = x\partial$. Recall the following classical result.

**Theorem 5.1.5 (Fuchs, 1866).** *For $P$ as above the following conditions are equivalent:*

(i) *All solutions to the ODE*

$$P(x, \partial)\, u = 0 \tag{5.1.7}$$

*belong to $\tilde{K}^{\mathrm{mod}}$.*

(ii) *We have* $\mathrm{ord}_{x=0}(a_i/a_n) \geq -(n-i)$ *for* $0 \leq \forall i \leq n$, *where* $\mathrm{ord}_{x=0}$ *denotes the order of zeros at* $x = 0$.

(iii) $b_i/b_n$ *are holomorphic for* $0 \leq \forall i \leq n$.

This fact will not be used in the rest of this book. Here we only show (iii) ⇒ (i) by using Theorem 5.1.4 (The equivalence of (ii) and (iii) is easy. For the proof of (i) ⇒ (iii), see, e.g., [Bor3, Chapter III]). We associate to (5.1.7) a system of ODEs

$$\frac{d}{dx} \vec{u}(x) = \frac{1}{x} \Gamma(x) \vec{u}(x), \tag{5.1.8}$$

for

$$\Gamma(x) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\frac{b_0}{b_n} & -\frac{b_1}{b_n} & -\frac{b_2}{b_n} & \cdots & -\frac{b_{n-1}}{b_n} \end{pmatrix} \in M_n(K).$$

Then $u$ is a solution to (5.1.7) if and only if $\vec{u} = {}^t(u, \theta u, \theta^2 u, \dots, \theta^{n-1}u)$ is a solution to (5.1.8). Since $b_i/b_n$ are holomorphic, we see by Theorem 5.1.4 that any solution $u$ to (5.1.7) belongs to $\tilde{K}^{\mathrm{mod}}$. The proof of (iii) $\Rightarrow$ (i) is complete.

**Definition 5.1.6.** We say that a meromorphic connection $(M, \nabla)$ at $x = 0$ is *regular* if there exists a finitely generated $\mathcal{O}$-submodule $L \subset M$ which is stable by the action of $\theta = x\nabla$ (i.e., $\theta L \subset L$) and generates $M$ over $K$. We call such an $\mathcal{O}$-submodule $L$ an *$\mathcal{O}$-lattice* of $(M, \nabla)$.

**Lemma 5.1.7.** *Let $(M, \nabla)$ be a regular meromorphic connection. Then any $\mathcal{O}$-lattice $L$ of $(M, \nabla)$ is a free $\mathcal{O}$-module of rank $\dim_K M$.*

*Proof.* Since $L$ is a torsion free finitely generated module over the principal ideal domain $\mathcal{O}$, it is free of finite-rank. Hence it is sufficient to show that the canonical homomorphism $K \otimes_{\mathcal{O}} L \to M$ is an isomorphism. The surjectivity is clear. To show the injectivity take a free basis $\{e_i\}_{1 \le i \le n}$ of $L$. It is sufficient to show that $\{e_i\}_{1 \le i \le n}$ is linearly independent over $K$. Assume $\sum_{i=1}^n f_i e_i = 0$ for $f_i \in K$. For $N \gg 0$ we have $a_i := x^N f_i \in \mathcal{O}$ for any $i = 1, \dots, n$. Then from $\sum_{i=1}^n a_i e_i = 0$ we obtain $a_i = 0$, and hence $f_i = 0$. □

By this lemma we easily see that a meromorphic connection $(M, \nabla)$ is regular if and only if there exists a $K$-basis $\{e_i\}_{1 \le i \le n}$ of $M$ such that the associated system of ODEs is of the form

$$\frac{d}{dx}\vec{u}(x) = \frac{\Gamma(x)}{x}\vec{u}(x) \quad (\Gamma(x) \in M_n(\mathcal{O})).$$

In particular, we have the following by Theorem 5.1.4.

**Proposition 5.1.8.** *A meromorphic connection $(M, \nabla)$ is regular if and only if all of its horizontal sections belong to $\tilde{K}^{\mathrm{mod}} \otimes_K M$.*

**Proposition 5.1.9.** *For a meromorphic connection $(M, \nabla)$ at $x = 0$, the following three conditions are equivalent:*

(i) *$(M, \nabla)$ is regular.*

(ii) *For any $u \in M$ there exists a finitely generated $\mathcal{O}$-submodule $L$ of $M$ such that $u \in L$ and $\theta L \subset L$, i.e., $M$ is a union of $\theta$-stable finitely generated $\mathcal{O}$-submodules.*

(iii) *For any $u \in M$ there exists a polynomial*

$$F(t) = t^m + a_1 t^{m-1} + \cdots + a_m \in \mathcal{O}[t]$$

*such that $F(\theta)u = 0$.*

*Proof.* (i) $\Rightarrow$ (ii): Let $L \subset M$ be an $\mathcal{O}$-lattice of $(M, \nabla)$. Then $M = \bigcup_{N \ge 0} x^{-N} L$ and each $x^{-N}L$ is a $\theta$-stable finitely generated $\mathcal{O}$-submodule of $M$.

(ii) $\Rightarrow$ (i): Take a $K$-basis $e_1, e_2, \dots, e_n$ of $M$ and choose a family of $\theta$-stable finitely generated $\mathcal{O}$-submodules $L_i$ of $M$ such that $e_i \in L_i$. Then the sum $L = \sum_{i=1}^n L_i \subset M$ is an $\mathcal{O}$-lattice of $(M, \nabla)$.

(ii) $\Rightarrow$ (iii): Take a $\theta$-stable finitely generated $\mathcal{O}$-submodule $L \subset M$ such that $u \in L$, and set

$$L_i = \mathcal{O}u + \mathcal{O}\theta u + \cdots + \mathcal{O}\theta^{i-1}u.$$

Then $L_1 \subset L_2 \subset \cdots$ is an increasing sequence of $\mathcal{O}$-submodules of $L$. Since $L$ is noetherian over $\mathcal{O}$, there exists $m \gg 0$ such that $\bigcup_{i\geq 1} L_i = L_m$. The condition $L_{m+1} = L_m$ implies that

$$\theta^m u = -\sum_{i=0}^{m-1} a_{m-i}\theta^i u$$

for some $a_i \in \mathcal{O}$.

(iii) $\Rightarrow$ (ii): It is easily seen that

$$L = \mathcal{O}u + \mathcal{O}\theta u + \cdots + \mathcal{O}\theta^{m-1}u \subset M$$

satisfies the desired property. □

**Proposition 5.1.10.** *Let*

$$0 \longrightarrow (M_1, \nabla_1) \longrightarrow (M_2, \nabla_2) \longrightarrow (M_3, \nabla_3) \longrightarrow 0$$

*be an exact sequence of meromorphic connections. Then* $(M_2, \nabla_2)$ *is regular if and only if* $(M_1, \nabla_1)$ *and* $(M_3, \nabla_3)$ *are regular.*

*Proof.* By the condition (iii) of Proposition 5.1.9 $(M_1, \nabla_1)$ and $(M_3, \nabla_3)$ are regular if $(M_2, \nabla_2)$ is regular. Let us prove the converse. For $u \in M_2$ there exist $m \geq 0$ and $a_i \in \mathcal{O}$ $(1 \leq i \leq m)$ such that

$$(\theta^m + a_1\theta^{m-1} + \cdots + a_m)u \in M_1$$

by the regularity of $(M_3, \nabla_3)$. Also by the regularity of $(M_1, \nabla_1)$ there exist $m' \geq 0$ and $b_j \in \mathcal{O}$ $(1 \leq j \leq m')$ such that

$$(\theta^{m'} + b_1\theta^{m'-1} + \cdots + b_{m'})(\theta^m + a_1\theta^{m-1} + \cdots + a_m)u = 0.$$

We can rewrite

$$(\theta^{m'} + b_1\theta^{m'-1} + \cdots + b_{m'})(\theta^m + a_1\theta^{m-1} + \cdots + a_m)$$

in the form

$$\theta^{m+m'} + c_1\theta^{m+m'-1} + \cdots + c_{m+m'}$$

where $c_i \in \mathcal{O}$. Hence $(M_2, \nabla_2)$ is regular. □

The following result can be easily checked by examining the connection matrices.

**Proposition 5.1.11.** *Assume that* $M$ *and* $N$ *are regular meromorphic connections. Then* $\mathrm{Hom}_K(M, N)$ *and* $M \otimes_K N$ *are also regular meromorphic connections.*

### 5.1.3 Regularity of $D$-modules on algebraic curves

We also have the notion of meromorphic connections in the algebraic category. In the algebraic situation the ring $\mathcal{O} = \mathbb{C}\{\{x\}\}$ is replaced by the stalk $\mathcal{O}_{C,p}$, where $C$ is a smooth algebraic curve and $p$ is a point of $C$. We denote by $K_{C,p}$ the quotient field of $\mathcal{O}_{C,p}$. Note that $\mathcal{O}_{C,p}$ is a discrete valuation ring and hence a principal ideal domain.

**Definition 5.1.12.** Let $C$, $p$ be as above.

(i) Let $M$ be a finite-dimensional $K_{C,p}$-module and let $\nabla : M \to \Omega^1_{C,p} \otimes_{\mathcal{O}_{C,p}} M (\simeq (K_{C,p} \otimes_{\mathcal{O}_{C,p}} \Omega^1_{C,p}) \otimes_{K_{C,p}} M)$ be a $\mathbb{C}$-linear map. The pair $(M, \nabla)$ is called an *algebraic meromorphic connection* at $p \in C$ if

$$\nabla(fu) = df \otimes u + f\nabla u \quad (f \in K_{C,p},\ u \in M).$$

(ii) By a morphism $\varphi : (M, \nabla) \to (N, \nabla)$ of algebraic meromorphic connections at $p \in C$ we mean a $K_{C,p}$-linear map $\varphi : M \to N$ satisfying $\nabla \circ \varphi = (\mathrm{id} \otimes \varphi) \circ \nabla$.

Algebraic meromorphic connections at $p \in C$ naturally form an abelian category. Choose a local parameter $x \in \mathcal{O}_{C,p}$ at $p$ and set $\partial = \frac{d}{dx}$. Then we have $K_{C,p} = \mathcal{O}_{C,p}[x^{-1}]$. Identifying $\Omega^1_{C,p}$ with $\mathcal{O}_{C,p}$ by $\mathcal{O}_{C,p} \ni f \leftrightarrow f dx \in \Omega^1_{C,p}$ an algebraic meromorphic connection at $p \in C$ is a finite-dimensional $K_{C,p}$-module endowed with a $\mathbb{C}$-linear map $\nabla : M \to M$ satisfying

$$\nabla(fu) = \frac{df}{dx} u + f\nabla u \quad (f \in K_{C,p},\ u \in M).$$

**Definition 5.1.13.** An algebraic meromorphic connection $(M, \nabla)$ at $p \in C$ is called *regular* if there exists a finitely generated $\mathcal{O}_{C,p}$-submodule $L$ of $M$ such that $M = K_{C,p}L$ and $x\nabla(L) \subset \Omega^1_{C,p} \otimes_{\mathcal{O}_{C,p}} L$ for some (and hence any) local parameter $x$ at $p$. We call such an $\mathcal{O}_{C,p}$-submodule $L$ an $\mathcal{O}_{C,p}$-lattice of $(M, \nabla)$.

Algebraic meromorphic connections share some basic properties with analytic ones discussed in Section 5.1.1. For example, Proposition 5.1.7, 5.1.9 and 5.1.10 remain valid also in the algebraic category.

**Lemma 5.1.14.** *Let $(M, \nabla)$ be an algebraic meromorphic connection at $p \in C$. Choose a local parameter $x$ at $p$, and denote by $(M^{\mathrm{an}}, \nabla)$ the corresponding (analytic) meromorphic connection at $x = 0$, i.e., $M^{\mathrm{an}} = \mathbb{C}\{\{x\}\}[x^{-1}] \otimes_{K_{C,p}} M$. Then $(M, \nabla)$ is regular if and only if $(M^{\mathrm{an}}, \nabla)$ is as well.*

*Proof.* We identify $\Omega^1_{C,p}$ with $\mathcal{O}_{C,p}$ via the local parameter $x$.

Assume that $(M, \nabla)$ is regular. Take an $\mathcal{O}_{C,p}$-lattice $L$ of $(M, \nabla)$. Then we easily see that $\mathbb{C}\{\{x\}\} \otimes_{\mathcal{O}_{C,p}} L$ is an $\mathbb{C}\{\{x\}\}$-lattice of $(M^{\mathrm{an}}, \nabla)$. Hence $(M^{\mathrm{an}}, \nabla)$ is regular.

Assume that $(M^{\mathrm{an}}, \nabla)$ is regular. Let us take a finitely generated $\mathcal{O}_{C,p}$-submodule $L_0$ of $M$ which generates $M$ over $K_{C,p}$. By Proposition 5.1.9 any finitely generated $\mathbb{C}\{\{x\}\}$-submodule of $M^{\mathrm{an}}$ is contained in a $\theta$-stable finitely generated $\mathbb{C}\{\{x\}\}$-module.

Therefore, $L_0$ and hence $L = \mathcal{O}_{C,p}[\theta]L_0$ must be contained in a $\theta$-stable finitely generated $\mathbb{C}\{\{x\}\}$-module. Then $L^{\mathrm{an}} = \mathbb{C}\{\{x\}\} \otimes_{\mathcal{O}} L$ is also finitely generated over $\mathbb{C}\{\{x\}\}$. Since $\mathbb{C}\{\{x\}\}$ is faithfully flat over $\mathcal{O}_{C,p}$, this implies the finiteness of $L$ over $\mathcal{O}_{C,p}$. Hence $(M, \nabla)$ is regular. □

Let us globalize the above definition of regularity. Let $M$ be an integrable connection on an algebraic curve $C$. Take a smooth completion $\overline{C}$ of $C$ and denote by $j : C \hookrightarrow \overline{C}$ the open embedding. Note that $\overline{C}$ is unique up to isomorphisms because $C$ is a curve. Let us consider the $D_{\overline{C}}$-module $j_*M = \int_j M$ (note that $H^i(\int_j M) = 0$ for $i \neq 0$ since $j$ is affine open embedding). Since $M$ is locally free over $\mathcal{O}_C$, it is free on a non-trivial (Zariski) open subset $U = C \setminus V$ of $C$, where $V$ consists of finitely many points. Hence $j_*M|_{\overline{C}\setminus V}$ is also free over $j_*\mathcal{O}_C|_{\overline{C}\setminus V}$. In particular $j_*M$ is locally free over $j_*\mathcal{O}_C$ (of finite rank). Let $p \in \overline{C} \setminus C$. Then the stalk $(j_*M)_p$ is a free module over $K_{\overline{C},p} = (j_*\mathcal{O}_C)_p$. Since $(j_*M)_p$ is a $D_{\overline{C},p}$-module, it is naturally endowed with a structure of an algebraic meromorphic connection at $p \in \overline{C}$ by $\nabla(m) = dx \otimes \partial m$, where $x$ is a local parameter at $p$ and $\partial = \frac{d}{dx}$. We call this $D_{\overline{C}}$-module $j_*M$ the *algebraic meromorphic extension* of $M$.

**Definition 5.1.15.** Let $M$ be an integrable connection on a smooth algebraic curve $C$. For a boundary point $p \in \overline{C} \setminus C$ we say that $M$ has *regular singularity* at $p$ (or $p$ is a regular singular point of $M$) if the algebraic meromorphic connection $((j_*M)_p, \nabla)$ is regular. Moreover, an integrable connection $M$ on $C$ is called *regular* if it has regular singularity at any boundary point $p \in \overline{C} \setminus C$.

The following is easily checked.

**Lemma 5.1.16.** *Let $M$ be an integrable connection on $C$. Then for any open subset $U$ of $C$ the restriction $M|_U$ has regular singularity at any point of $C \setminus U$.*

By Proposition 5.1.10 and Lemma 5.1.14 we easily see the following.

**Lemma 5.1.17.** *Let*

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$$

*be an exact sequence of integrable connections on $C$. Then $M_2$ is regular if and only if $M_1$ and $M_3$ are regular.*

**Lemma 5.1.18.** *Let $M$ and $N$ be regular integrable connections on $C$. Then the integrable connections $M \otimes_{\mathcal{O}_C} N$ and $\mathcal{H}om_{\mathcal{O}_C}(M, N)$ are also regular.*

*Proof.* For $p \in \overline{C} \setminus C$ we have $(j_*(M \otimes_{\mathcal{O}_C} N))_p \simeq (j_*M)_p \otimes_{K_{C,p}} (j_*N)_p$ and $(j_*\mathcal{H}om_{\mathcal{O}_C}(M, N))_p \simeq \mathrm{Hom}_{K_{C,p}}((j_*M)_p, (j_*N)_p)$. Hence the assertion follows from Proposition 5.1.11 and Lemma 5.1.14. □

**Lemma 5.1.19.** *Let $V$ be a subset of $\overline{C} \setminus C$ and set $C' = C \sqcup V$. We denote by $j : C \to C'$ the embedding. For each point $p \in V$ we fix a local parameter $x_p$ and set $\theta_p = x_p \frac{d}{dx_p}$. Then the following three conditions on an integrable connection $M$ on $C$ are equivalent:*

(i) *$M$ has regular singularity at any $p \in V$.*
(ii) *$j_*M$ is a union of coherent $\mathcal{O}_{C'}$-submodules which are stable under the action of $\theta_p$ for any $p \in V$.*
(iii) *There exists a coherent $D_{C'}$-module $M'$ such that $M'|_C \simeq M$ and $M'$ is a union of coherent $\mathcal{O}_{C'}$-submodules which are stable under the action of $\theta_p$ for any $p \in V$.*

*Proof.* (i) $\Rightarrow$ (ii): For $p \in V$ take an $\mathcal{O}_{C',p}$-lattice $L_p$ of $((j_*M)_p, \nabla)$. Then $x_p^{-i}L_p$ is also an $\mathcal{O}_{C',p}$-lattice of $((j_*M)_p, \nabla)$, and we have $(j_*M)_p = \bigcup_i x_p^{-i}L_p$. Note that there exists an open subset $U_p$ of $C \sqcup \{p\}$ containing $p$ such that $x_p^{-i}L_p$ is extended to a coherent $\mathcal{O}_{U_p}$-submodule $L_p^i$ of $j_*M|_{U_p}$ satisfying $L_p^i|_{U_p \cap C} = M|_{U_p \cap C}$. $L_p^i$ for $p \in V$ are patched together and we obtain a coherent $\mathcal{O}_{C'}$-submodule $L^i$ of $j_*M$. Then $L^i$ is stable under the action of $\theta_p$ for any $p \in V$, and we have $j_*M = \bigcup_i L^i$.

(ii) $\Rightarrow$ (iii): This is obvious.

(iii) $\Rightarrow$ (i): For $M'$ as in (iii) we have $(j_*M)_p = K_{C',p} \otimes_{\mathcal{O}_{C',p}} M'_p$ for $p \in V$. Therefore, Proposition 5.1.9 implies that $M$ has regular singularity at $p$. □

**Lemma 5.1.20.** *A coherent $D_C$-module $M$ is holonomic if and only if it is generically an integrable connection.*

*Proof.* The "only if" part follows from Proposition 3.1.6. Assume that $M$ is generically an integrable connection, i.e., there exists an open dense subset $U$ of $C$ such that $M|_U$ is an integrable connection. Note that $V := C \setminus U$ consists of finitely many points. We see from our assumption that the characteristic variety $\mathrm{Ch}(M)$ of $M$ in contained in $T^*_C C \cup (\bigcup_{p \in V}(T^*C)_p)$. By $\dim T^*_C C = \dim(T^*C)_p = 1$ we have $\dim \mathrm{Ch}(M) \leq 1$, and hence $M$ is holonomic. □

**Definition 5.1.21.** A holonomic $D$-module $M$ on an algebraic curve $C$ is said to be *regular* if there exists an open dense subset $C_0$ of $C$ such that $M|_{C_0}$ is a regular integrable connection on $C_0$. An object $M^{\cdot}$ of $D^b_h(D_C)$ is said to be regular if all of the cohomology sheaves $H^*(M^{\cdot})$ are regular.

By definition a holonomic $D_C$-module supported on a finite set is regular.

**Example 5.1.22.** Consider an algebraic ODE $P(x, \partial)\, u = 0$ on $\mathbb{A}^1 = \mathbb{C}$. Then the holonomic $D_{\mathbb{C}}$-module $M := D_{\mathbb{C}}\, u = D_{\mathbb{C}}/D_{\mathbb{C}} P(x, \partial)$ is regular if and only if the ODE $P(x, \partial)\, u = 0$ has a regular singular point (in the classical sense) at any point in $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ (i.e., $P(x, \partial)\, u = 0$ is a Fuchsian ODE).

The following lemma plays a crucial role in defining the regularity of holonomic $D$-modules on higher-dimensional varieties.

**Lemma 5.1.23.** *Let $f : C \to C'$ be a* **dominant morphism** *(i.e.,* Im $f$ *is dense in $C'$) between algebraic curves.*

(i) *$M \in \mathrm{Mod}_h(D_{C'})$ is regular $\Longleftrightarrow$ $f^\dagger M$ is regular.*
(ii) *$N \in \mathrm{Mod}_h(D_C)$ is regular $\Longleftrightarrow$ $\int_f N$ is regular.*

*Proof.* We may assume that $C = \overline{C}$ and $C' = \overline{C'}$. We can take an open subset $C'_0$ of $C'$ such that $f_0 : C_0 := f^{-1}C'_0 \to C'_0$ is étale and $M|_{C'_0}$ and $N|_{C_0}$ are integrable connections. For $p \in C \setminus C_0$ (resp. $p' \in C' \setminus C'_0$) we take a local parameter $x_p$ (resp. $y_{p'}$) at $p$ (resp. $p'$) and set $\theta_p = x_p \frac{d}{dx_p}$ (resp. $\theta_{p'} = y_{p'} \frac{d}{dy_{p'}}$). If $p' = f(p)$ for $p \in C \setminus C_0$, we may assume $\theta_p = m_p \theta_{p'}$ for a positive integer $m_p$. Indeed, we can take local parameters $x_p$ and $y_{p'}$ so that $y_{p'} = x_p^{m_p}$. We denote by $j : C_0 \to C$ and $j' : C'_0 \to C'$ the embeddings.

(i) We may assume that $M = j'_*(M|_{C'_0})$. Note that $Lf^*M|_{C_0} = f_0^*(M|_{C'_0})$ and $f_0^*(M|_{C'_0})$ is an integrable connection. Hence $M$ (resp. $f^\dagger M$) is regular if and only if $(M_{p'}, \nabla)$ (resp. $((j_* f_0^*(M|_{C'_0}))_p, \nabla)$) is a regular algebraic meromorphic connection for any $p' \in C' \setminus C'_0$ (resp. $p \in C \setminus C_0$). Suppose $p \in C \setminus C_0$, $p' = f(p)$. Then we have $\mathcal{O}_{C',p'} \subset \mathcal{O}_{C,p}$. Note

$$\begin{aligned}(j_* f_0^*(M|_{C'_0}))_p &\simeq K_{C,p} \otimes_{\mathcal{O}_{C,p}} (f^*M)_p \simeq K_{C,p} \otimes_{\mathcal{O}_{C',p'}} M_{p'} \\ &\simeq (K_{C,p} \otimes_{\mathcal{O}_{C',p'}} K_{C',p'}) \otimes_{K_{C',p'}} M_{p'} \simeq K_{C,p} \otimes_{K_{C',p'}} M_{p'}.\end{aligned}$$

Therefore, if $M_{p'}$ has a $\theta_{p'}$-stable $\mathcal{O}_{C',p'}$-lattice $L$, then $\mathcal{O}_{C,p} \otimes_{\mathcal{O}_{C',p'}} L$ is a $\theta_p$-stable $\mathcal{O}_{C,p}$-lattice of $K_{C,p} \otimes_{K_{C',p'}} M_{p'}$. This shows ($\Longrightarrow$). Conversely, suppose $K_{C,p} \otimes_{K_{C',p'}} M_{p'}$ is a regular algebraic meromorphic connection at $p$. Then by Proposition 5.1.9, we have $K_{C,p} \otimes_{K_{C',p'}} M_{p'} = \bigcup_i L_i$, where $L_i$ is a $\theta_p$-stable finitely generated $\mathcal{O}_{C,p}$-module. Since $\mathcal{O}_{C,p}$ is finitely generated over $\mathcal{O}_{C',p'}$, $L'_i := L_i \cap (1 \otimes M_{p'})$ is finitely generated over $\mathcal{O}_{C',p'}$. Moreover, by the relation $\theta_p = m_p \theta_{p'}$, we see $L'_i$ is $\theta_{p'}$-stable. Hence it follows from Lemma 5.1.9 that $M_{p'} = 1 \otimes M_{p'}$ is also regular. The proof of ($\Longleftarrow$) is also complete.

(ii) We have $\int_f N|_{C'_0} \simeq f_{0*}(N|_{C_0})$ and $f_{0*}(N|_{C_0})$ is an integrable connection. Moreover, $N$ (resp. $\int_f N$) is regular if and only if $j_*(N|_{C_0})$ (resp. $j'_* f_{0*}(N|_{C_0})$) is a union of coherent $\mathcal{O}_C$(resp. $\mathcal{O}_{C'}$)-modules which are stable under the action of $\theta_p$ (resp. $\theta_{p'}$) for any $p \in C \setminus C_0$ (resp. $p' \in C' \setminus C'_0$). Note that $j'_* f_{0*}(N|_{C_0}) \simeq f_* j_*(N|_{C_0})$. If $j_*(N|_{C_0})$ is a union of coherent $\mathcal{O}_C$-modules $L_i$ which are stable under the action of $\theta_p$ for any $p \in C \setminus C_0$, then $f_* j_*(N|_{C_0})$ is a union of coherent $\mathcal{O}_{C'}$-modules $f_* L_i$ which are stable under the action of $\theta_{p'}$ for any $p' \in C' \setminus C'_0$. This shows ($\Longrightarrow$). Assume that $\int_f N$ is regular. Then $Lf^* \int_f N$ is also regular by (i). The restriction of the canonical morphism $N \to Lf^* \int_f N$ to $C_0$ is given by $N|_{C_0} \to f^* f_*(N|_{C_0})$ and hence a monomorphism. This implies the regularity of $N$. The proof of ($\Longleftarrow$) is also complete. □

Let us give comments on the difference of the notion of regularity in algebraic and analytic situations. Let $C$ be a one-dimensional complex manifold and let $V$ be a finite subset of $C$. We denote by $j : U := C \setminus V \to C$ the embedding. Let $M$ be an integrable connection on $U$. We say that a coherent $D_C$-module $\tilde{M}$ is a meromorphic extension of $M$ if $\tilde{M}|_U \simeq M$ and $\tilde{M}$ is isomorphic as an $\mathcal{O}_C$-module to a locally free $\mathcal{O}_C[V]$-module, where $\mathcal{O}_C[V]$ denotes the sheaf of meromorphic functions on $C$ with possible poles on $V$. The following example shows that in the analytic situation

a meromorphic extension of an integrable connection is not uniquely determined and one cannot define the notion of the regularity of an integrable connection at a boundary point unless its meromorphic extension is specified. Nevertheless, as we see later the uniqueness of a regular meromorphic extension in the analytic situation holds true as a part of the Riemann–Hilbert correspondence.

**Example 5.1.24.** We regard $C = \mathbb{C}$ as an algebraic curve, and let $j : U := \mathbb{C} \setminus \{0\} \to C$ be the embedding. Let us consider two (algebraic) integrable connections $M = D_U/D_U\partial$ and $N = D_U/D_U(x^2\partial - 1)$ on $U$. We have an isomorphism $M^{\mathrm{an}} \simeq N^{\mathrm{an}}$ given by

$$M^{\mathrm{an}} \ni [P \mod D_{U^{\mathrm{an}}}\partial] \longleftrightarrow [P\exp(1/x) \mod D_{U^{\mathrm{an}}}(x^2\partial - 1)] \in N^{\mathrm{an}}.$$

We consider meromorphic extensions $(j_*M)^{\mathrm{an}}$ and $(j_*N)^{\mathrm{an}}$ of $M^{\mathrm{an}}$ and $N^{\mathrm{an}}$, respectively. Let us show that they are not isomorphic. Note that $M$ is regular since it is isomorphic to $\mathcal{O}_U$ as a $D_U$-module. Hence $(j_*M)^{\mathrm{an}}_0$ is a regular meromorphic connection. Therefore, it is sufficient to show that $(j_*N)^{\mathrm{an}}_0$ is not regular as a meromorphic connection. This can be easily shown by checking that its horizontal sections do not have moderate growth. We have verified $(j_*M)^{\mathrm{an}} \not\simeq (j_*N)^{\mathrm{an}}$. We have also shown that $M$ is regular, while $N$ is not regular.

## 5.2 Regular meromorphic connections on complex manifolds

The aim of this section is to give a proof of the Riemann–Hilbert correspondence for regular meromorphic connections on complex manifolds due to Deligne [De1]. We basically follow Malgrange's lecture in [Bor3].

### 5.2.1 Meromorphic connections in higher dimensions

Let $X$ be a complex manifold and $D \subset X$ a *divisor* (complex hypersurface). We denote by $\mathcal{O}_X[D]$ the sheaf of meromorphic functions on $X$ that are holomorphic on $Y := X \setminus D$ and have poles along $D$. For a local defining equation $h \in \mathcal{O}_X$ of $D$ we have $\mathcal{O}_X[D] = \mathcal{O}_X[h^{-1}] \simeq \mathcal{O}_X[t]/\mathcal{O}_X[t](th - 1)$ locally, and hence it is a coherent sheaf of rings.

**Definition 5.2.1.**

(i) Assume that a coherent $\mathcal{O}_X[D]$-module $M$ is endowed with a $\mathbb{C}$-linear morphism

$$\nabla : M \longrightarrow \Omega^1_X \otimes_{\mathcal{O}_X} M$$

satisfying the conditions

$$\nabla(fs) = df \otimes s + f\nabla s \qquad (f \in \mathcal{O}_X[D],\ s \in M), \tag{5.2.1}$$

$$[\nabla_\theta,\ \nabla_{\theta'}] = \nabla_{[\theta,\theta']} \qquad (\theta,\ \theta' \in \Theta_X). \tag{5.2.2}$$

Then we call $M$ (or more precisely the pair $(M,\ \nabla)$) a *meromorphic connection* along the divisor $D$.

(ii) By a morphism $\varphi : (M, \nabla) \to (N, \nabla)$ of meromorphic connections along $D$ we mean an $\mathcal{O}_X[D]$-linear morphism $\varphi : M \to N$ satisfying $\nabla \circ \varphi = (\mathrm{id} \otimes \varphi) \circ \nabla$.
(iii) For a meromorphic connection $(M, \nabla)$ along $D$, we set

$$M^{\nabla} = \{s \in M \mid \nabla s = 0\}.$$

Sections of $M^{\nabla}$ are called *horizontal sections* of $(M, \nabla)$.

**Notation 5.2.2.** We denote by $\mathrm{Conn}(X; D)$ the category of meromorphic connections along $D$.

Note that $\mathrm{Conn}(X; D)$ is an abelian category. Note also that an object of $\mathrm{Conn}(X; D)$ is a $D_X$-module which is isomorphic as an $\mathcal{O}_X$-module to a coherent $\mathcal{O}_X[D]$-module, and a morphism $(M, \nabla) \to (N, \nabla)$ is just a morphism of the corresponding $D_X$-modules. Hence $\mathrm{Conn}(X; D)$ is naturally regarded as a subcategory of $\mathrm{Mod}(D_X)$. For $(M, \nabla) \in \mathrm{Conn}(X; D)$ the restriction $M|_Y$ of $M$ to $Y = X \setminus D$ belongs to $\mathrm{Conn}(X)$; i.e., $M|_Y$ is locally free over $\mathcal{O}_Y$.

**Remark 5.2.3.** One can show that $\mathrm{Conn}(X; D)$ is a subcategory of $\mathrm{Mod}_h(D_X)$ (it is not even obvious that an object of $\mathrm{Conn}(X; D)$ is a coherent $D_X$-module). We do not use this fact in this book.

**Remark 5.2.4.** Assume that $\dim X = 1$. Let $a \in D$, and take a local coordinate $x$ such that $x(a) = 0$. Then the stalk $\mathcal{O}_X[D]_a$ at $a \in D$ is identified with the quotient field $\mathbb{C}\{\{x\}\}[x^{-1}]$ of $\mathcal{O}_{X,a} \simeq \mathbb{C}\{\{x\}\}$. Since $\mathcal{O}_X[D]_a$ is a field, any coherent $\mathcal{O}_X[D]$-module is free on a open neighborhood of $a$. Hence the stalk $(M_a, \nabla_a)$ of $(M, \nabla) \in \mathrm{Conn}(X, D)$ at $a \in D$ turns out to be a meromorphic connection in the sense of Section 5.1.1 by identifying $\Omega^1_X$ with $\mathcal{O}_X$ via $dx \in \Omega^1_X$ (note that the condition (5.2.2) is automatically satisfied in the one-dimensional situation).

For $(M, \nabla), (N, \nabla) \in \mathrm{Conn}(X, D)$ the $\mathcal{O}_X[D]$-modules $M \otimes_{\mathcal{O}_X[D]} N$ and $\mathcal{H}om_{\mathcal{O}_X[D]}(M, N)$ are endowed with structures of meromorphic connections along $D$ by

$$\nabla(s \otimes t) = \sum_i \omega_i \otimes (s_i \otimes t) + \sum_j \omega'_j \otimes (s \otimes t_j),$$
$$(\nabla\varphi)(s) = (\mathrm{id} \otimes \varphi)(\nabla(s)) - \nabla(\varphi(s)),$$

respectively, where $\nabla(s) = \sum_i \omega_i \otimes s_i$ and $\nabla(t) = \sum_j \omega'_j \otimes t_j$. In particular, for $(M, \nabla) \in \mathrm{Conn}(X, D)$ its dual $M^* := \mathcal{H}om_{\mathcal{O}_X[D]}(M, \mathcal{O}_X[D])$ of $M$ is naturally endowed with a structure of a meromorphic connection along $D$.

The following simple observation will be effectively used in proving the classical Riemann–Hilbert correspondence.

**Lemma 5.2.5.** *For $(M, \nabla), (N, \nabla) \in \mathrm{Conn}(X; D)$ we have*

$$\Gamma(X, \mathcal{H}om_{\mathcal{O}_X[D]}(M, N)^{\nabla}) \simeq \mathrm{Hom}_{\mathrm{Conn}(X;D)}((M, \nabla), (N, \nabla)).$$

**Proposition 5.2.6.** *Let $\varphi : M_1 \to M_2$ be a morphism of meromorphic connections along $D$. If $\varphi|_{X\setminus D}$ is an isomorphism, then $\varphi$ is an isomorphism.*

This follows from the following lemma since the kernel and the cokernel of $\varphi$ are coherent $\mathcal{O}_X[D]$-modules supported by $D$.

**Lemma 5.2.7.** *A coherent $\mathcal{O}_X[D]$-module $M$ whose support is contained in $D$ is trivial; $M = 0$.*

*Proof.* Take a local defining equation $h$ of $D$. For a section $s \in M$ whose support is contained in $D$ consider the $\mathcal{O}_X$-coherent submodule $\mathcal{O}_X s \subset M$. Since the support of $\mathcal{O}_X s$ is contained in $D$, we have $h^N s = 0$ ($N \gg 0$) by Hilbert's Nullstellensatz. Therefore, we obtain $s = h^{-N} h^N s = 0$. □

**Corollary 5.2.8.** *Any meromorphic connection $M$ along $D$ is* **reflexive** *in the sense that the canonical morphism $M \to M^{**}$ is an isomorphism.*

Let $f : Z \to X$ be a morphism of complex manifolds such that $f^{-1}D$ is a divisor on $Z$. Then we have

$$\mathcal{O}_Z[f^{-1}D] \simeq \mathcal{O}_Z \otimes_{f^{-1}\mathcal{O}_X} f^{-1}\mathcal{O}_X[D] \simeq \mathcal{O}_Z \otimes^L_{f^{-1}\mathcal{O}_X} f^{-1}\mathcal{O}_X[D].$$

Indeed, since $\mathcal{O}_X[D]$ is flat over $\mathcal{O}_X$, we have $H^i(\mathcal{O}_Z \otimes^L_{f^{-1}\mathcal{O}_X} f^{-1}\mathcal{O}_X[D]) = 0$ for $i \neq 0$. Moreover, for a local defining equation $h = 0$ of $D$ we have $\mathcal{O}_X[D] = \mathcal{O}_X[h^{-1}]$, $\mathcal{O}_Z[f^{-1}D] = \mathcal{O}_Z[h \circ f]$ and hence $\mathcal{O}_Z[f^{-1}D] \simeq \mathcal{O}_Z \otimes_{f^{-1}\mathcal{O}_X} f^{-1}\mathcal{O}_X[D]$. Hence for $M \in \mathrm{Conn}(X; D)$ we have

$$\begin{aligned} Lf^*M &\simeq \mathcal{O}_Z \otimes^L_{f^{-1}\mathcal{O}_X} f^{-1}M \simeq \mathcal{O}_Z[f^{-1}D] \otimes^L_{f^{-1}\mathcal{O}_X[D]} f^{-1}M \\ &\simeq \mathcal{O}_Z[f^{-1}D] \otimes_{f^{-1}\mathcal{O}_X[D]} f^{-1}M. \end{aligned}$$

From this we easily see the following.

**Lemma 5.2.9.** *Let $f$ be as above. For any $M \in \mathrm{Conn}(X; D)$ we have $H^j(Lf^*M) = 0$ for $j \neq 0$ and $H^0(Lf^*M) \in \mathrm{Conn}(Z; f^{-1}D)$. In particular, the inverse image functor for the category of $D$-modules induces an exact functor*

$$f^* : \mathrm{Conn}(X; D) \to \mathrm{Conn}(Z; f^{-1}D).$$

Set

$$B = \{x \in \mathbb{C} \mid |x| < 1\} \qquad \text{(the unit disk)}.$$

For a morphism $i : B \to X$ such that $i^{-1}D = \{0\}$ the stalk $(i^*M)_0$ at $0 \in B$ is a meromorphic connection of one-variable studied in Section 5.1.1.

**Definition 5.2.10.** A meromorphic connection $M$ on $X$ along $D$ is called *regular* if $(i^*M)_0$ is regular in the sense of Section 5.1.2 for any morphism $i : B \to X$ such that $i^{-1}D = \{0\} \subset B$.

**Notation 5.2.11.** We denote by $\mathrm{Conn}^{\mathrm{reg}}(X; D)$ the category of regular meromorphic connections along $D$.

**Proposition 5.2.12.**

(i) *Let*

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

*be a short exact sequence of meromorphic connections along $D$. Then $M_2$ is regular if and only if $M_1$ and $M_3$ are regular.*

(ii) *Assume that $M$ and $N$ are regular meromorphic connections along $D$. Then $M \otimes_{\mathcal{O}_X[D]} N$ and $\mathcal{H}om_{\mathcal{O}_X[D]}(M, N)$ are also regular.*

*Proof.* By definition we can reduce the problem to the case when $X$ is the unit disk $B \subset \mathbb{C}$. Then (i) follows from Proposition 5.1.10. We can prove (ii) by using Lemma 5.1.11. This completes the proof. □

**Definition 5.2.13.** A meromorphic connection on $X$ along $D$ is called *effective* if it is generated as an $\mathcal{O}_X[D]$-module by a coherent $\mathcal{O}_X$-submodule.

We will see later that any regular meromorphic connection is effective (see Corollary 5.2.22 (ii) below).

**Lemma 5.2.14.** *Let $f : X' \to X$ be a proper surjective morphism of complex manifolds such that $D' := f^{-1}D$ is a divisor on $X'$ and $f^{-1}(X' \setminus D') \to X \setminus D$ is an isomorphism. Assume that $N$ is an effective meromorphic connection on $X'$ along $D'$.*

(i) *We have $H^k(\int_f N) = 0$ for $k \neq 0$ and $H^0(\int_f N)$ is an effective meromorphic connection on $X$ along $D$.*

(ii) *If $N$ is regular, then so is $H^0(\int_f N)$.*

*Proof.* We denote by $D_{X'}[D']$ the subalgebra of $\mathcal{E}nd_{\mathbb{C}}(\mathcal{O}_{X'}[D'])$ generated by $D_{X'}$ and $\mathcal{O}_{X'}[D']$. Then we have $D_{X'}[D'] \simeq D_{X'} \otimes_{\mathcal{O}_{X'}} \mathcal{O}_{X'}[D'] \simeq \mathcal{O}_{X'}[D'] \otimes_{\mathcal{O}_{X'}} D_{X'}$.

We first show that

$$D_{X \leftarrow X'} \otimes^L_{D_{X'}} D_{X'}[D'] \simeq D_{X'}[D']. \tag{5.2.3}$$

Note that the canonical morphism $f^{-1}\Omega_X \to \Omega_{X'}$ induces an isomorphism $\mathcal{O}_{X'}[D'] \otimes_{f^{-1}\mathcal{O}_X} f^{-1}\Omega_X \to \mathcal{O}_{X'}[D'] \otimes_{\mathcal{O}_{X'}} \Omega_{X'}$ by Lemma 5.2.7. Hence we have

$$\begin{aligned} D_{X \leftarrow X'} \otimes^L_{D_{X'}} D_{X'}[D'] &\simeq D_{X \leftarrow X'} \otimes^L_{\mathcal{O}_{X'}} \mathcal{O}_{X'}[D'] \\ &\simeq D_{X \leftarrow X'} \otimes_{\mathcal{O}_{X'}} \mathcal{O}_{X'}[D'] \\ &\simeq f^{-1}D_X \otimes_{f^{-1}\mathcal{O}_X} \mathcal{O}_{X'}[D'] \end{aligned}$$

by Lemma 1.3.4. Let us show that the canonical morphism $D_{X'}[D'] \to f^{-1}D_X \otimes_{f^{-1}\mathcal{O}_X} \mathcal{O}_{X'}[D']$ induced by the canonical section $1 \otimes 1$ of the right $D_{X'}[D']$-module $f^{-1}D_X \otimes_{f^{-1}\mathcal{O}_X} \mathcal{O}_{X'}[D']$ is an isomorphism. For this it is sufficient to show

that $F_pD_{X'}[D'] \to f^{-1}F_pD_X \otimes_{f^{-1}\mathcal{O}_X} \mathcal{O}_{X'}[D']$ is an isomorphism for any $p \in \mathbb{Z}$. This follows from Lemma 5.2.7. The assertion (5.2.3) is verified.

Since $N$ is effective, there exists a coherent $\mathcal{O}_{X'}$-submodule $L$ of $N$ such that $N \simeq \mathcal{O}_{X'}[D'] \otimes_{\mathcal{O}_{X'}} L$. Then by (5.2.3) and $f^{-1}\mathcal{O}_X[D] \otimes_{f^{-1}\mathcal{O}_X} \mathcal{O}_{X'} \simeq \mathcal{O}_{X'}[D']$, we have

$$\begin{aligned}\int_f N &= Rf_*(D_{X\leftarrow X'} \otimes^L_{D_{X'}} N) \simeq Rf_*(D_{X\leftarrow X'} \otimes^L_{D_{X'}} D_{X'}[D'] \otimes^L_{D_{X'}[D']} N)\\ &\simeq Rf_*(N) \simeq Rf_*(\mathcal{O}_{X'}[D'] \otimes_{\mathcal{O}_{X'}} L) \simeq Rf_*(f^{-1}\mathcal{O}_X[D] \otimes_{f^{-1}\mathcal{O}_X} L)\\ &\simeq \mathcal{O}_X[D] \otimes_{\mathcal{O}_X} Rf_*(L),\end{aligned}$$

and hence $H^k(\int_f N) \simeq \mathcal{O}_X[D] \otimes_{\mathcal{O}_X} H^k(Rf_*(L))$. Since $H^k(Rf_*(L))$ is coherent over $\mathcal{O}_X$ for any $k$ by the *Grauert direct image theorem*, $H^k(\int_f N)$ is coherent over $\mathcal{O}_X[D]$. Moreover, we have $H^k(\int_f N) = 0$ for $k \neq 0$ by $H^k(Rf_*(L))|_{X\setminus D} = 0$ and Lemma 5.2.7. The statement (i) is proved.

Assume that $N$ is regular. Let $i : B \to X$ be a morphism from the unit disk $B$ satisfying $i^{-1}(D) = \{0\}$. Since $f$ is proper, there exists a lift $j : B \to X'$ satisfying $f \circ j = i$. Then we have

$$i^*H^0\Big(\int_f N\Big) \simeq i^* \int_f N \simeq j^*f^* \int_f N.$$

Since the canonical morphism $N \to f^* \int_f N$ is an isomorphism on $X' \setminus D'$, it is an isomorphism on $X'$ by Proposition 5.2.6. Hence we obtain $i^*H^0(\int_f N) \simeq j^*N$. Therefore, $H^0(\int_f N)$ is regular. The statement (ii) is proved. □

### 5.2.2 Meromorphic connections with logarithmic poles

In this subsection we will consider the case where $D$ is a *normal crossing* divisor on a complex manifold $X$; i.e., we assume that $D$ is locally defined by a function of the form $x_1 \cdots x_r$, where $(x_1, \ldots, x_n)$ is a local coordinate. Let $p \in D$ and fix such a coordinate $(x_1, \ldots, x_n)$ around $p \in D$. For $1 \leq k \leq r$ we denote by $D_k$ the (local) irreducible component of $D$ defined by $x_k$.

The meromorphic connections $M$ on $X$ along $D$ which we will consider in this subsection are also of very special type. First, we assume that there exists a holomorphic vector bundle (locally free $\mathcal{O}_X$-module of finite rank) $L$ on $X$ such that $M = \mathcal{O}_X[D] \otimes_{\mathcal{O}_X} L$ as an $\mathcal{O}_X[D]$-module. Hence taking a local defining equation $x_1x_2 \cdots x_r = 0$ of $D$ and choosing a basis $e_1, e_2, \ldots, e_m$ of $L$ around a point $p \in D$, the associated $\mathbb{C}$-linear morphism $\nabla : M \to \Omega^1_X \otimes_{\mathcal{O}_X} M$ can be expressed as

$$\nabla e_i = \sum_{1\leq k\leq n, 1\leq j\leq m} a^k_{ij} dx_k \otimes e_j, \tag{5.2.4}$$

where $a^k_{ij} \in \mathcal{O}_X[D] = \mathcal{O}_X[x_1^{-1}x_2^{-1} \cdots x_r^{-1}]$. Then we further assume that the functions $x_k a^k_{ij}$ $(1 \leq k \leq r)$, $a^k_{ij}$ $(r < k \leq n)$ are holomorphic. In this case, we say

the meromorphic connection $M$ along the normal crossing divisor $D$ has a *logarithmic pole* with respect to the lattice $L$ at $p$. If this is the case at any $p \in D$, we say that $M$ has a logarithmic pole along $D$ with respect to $L$. Note that this definition does not depend on the choice of the coordinates $\{x_k\}$ and the basis $\{e_i\}$ of $L$.

Let $M$ be a meromorphic connection along $D$ which has a logarithmic pole with respect to the lattice $L$. Take a basis $\{e_1, \dots, e_m\}$ of $L$ and set $A_k = (a_{ij}^k)$ for $1 \le k \le n$, where $a_{ij}^k$ is as in (5.2.4). For $1 \le k \le r$ we also set $B_k = x_k A_k$. Let $1 \le k \le r$. Since $B_k$ belongs to $M_m(\mathcal{O}_X)$, we can consider its restriction $B_k|_{D_k} \in M_m(\mathcal{O}_{D_k})$. Then $B_k|_{D_k}$ defines a canonical section $\mathrm{Res}_{D_k}^L \nabla$ of the vector bundle $\mathcal{E}nd_{\mathcal{O}_{D_k}}(L|_{D_k})$ on $D_k$. Indeed, we can check easily that $B_k|_{D_k} \in M_m(\mathcal{O}_{D_k}) \simeq \mathcal{E}nd_{\mathcal{O}_{D_k}}(L|_{D_k})$ does not depend on the choice of a local coordinate and a basis of $L$. We call $\mathrm{Res}_{D_k}^L \nabla$ the *residue* of $(M, \nabla)$ along $D_k$.

**Proposition 5.2.15.** *Let $M$ be a meromorphic connection along $D$ which has a logarithmic pole with respect to the lattice $L$. We keep the notation as above.*

(i) *On $D_k \cap D_l$ we have*

$$[\mathrm{Res}_{D_k}^L \nabla,\ \mathrm{Res}_{D_l}^L \nabla] = 0.$$

(ii) *The eigenvalues of $(\mathrm{Res}_{D_k}^L \nabla)(a) \in \mathrm{End}_{\mathbb{C}}(L(a))$ do not depend on the choice of $a \in D_k$. Here $L(a)$ denotes the fiber $\mathbb{C} \otimes_{\mathcal{O}_{X,a}} L_a$ of $L$ at $a$.*

*Proof.* (i) By (5.2.2) we have

$$\frac{\partial A^k}{\partial x_l} - \frac{\partial A^l}{\partial x_k} = [A^k,\ A^l].$$

We obtain the desired result from this by developing both sides into the Laurent series with respect to $x_k$, $x_l$ and comparing the coefficients of $(x_k x_l)^{-1}$.

(ii) Let $\overline{A}^p$, $\overline{B}^k$ be the restrictions of the matrices $A^p$, $B^k$ to $D_k$, respectively. Then we have

$$\frac{\partial \overline{A}^i}{\partial x_j} - \frac{\partial \overline{A}^j}{\partial x_i} = [\overline{A}^i,\ \overline{A}^j] \quad (i,\ j \ne k),$$

$$\frac{\partial \overline{B}^k}{\partial x_i} = [\overline{B}^k,\ \overline{A}^i] \quad (i \ne k).$$

The first formula implies that

$$\overline{\nabla} e_i = \sum_{l \ne k} \overline{a}_{ij}^l dx_l \otimes e_j \quad (\overline{A}^l = (\overline{a}_{ij}^l))$$

defines an integrable connection $\overline{\nabla}$ on $L|_{D_k}$. The second one implies that $\overline{B}^k = \mathrm{Res}_{D_k}^L \nabla \in \mathrm{End}_{\mathcal{O}_{D_k}} L|_{D_k}$ is a horizontal section with respect to the connection induced by $\overline{\nabla}$ on $\mathrm{End}_{\mathcal{O}_{D_k}} L|_{D_k}$. From this we can easily check that the values of the matrix $\overline{B}^k$ at two different points of $D_k$ are conjugate to each other. □

**Proposition 5.2.16.** *Let $M$ be a meromorphic connection which has a logarithmic pole along $D$ with respect to a lattice $L$. Then $M$ is regular.*

*Proof.* For any morphism $i : B \to X$ from the unit disk $B$ such that $i^{-1}D = \{0\}$ we easily see that the meromorphic connection $i^*M$ on $B$ has a logarithmic pole along $\{0\}$ with respect to the lattice $i^*L$. Then the stalk $(i^*M)_0$ is regular by Theorem 5.1.4. □

The following construction enables us to extend analytic integrable connections on $Y = X \setminus D$ to regular meromorphic connections along the divisor $D \subset X$.

**Theorem 5.2.17.** *Let $D$ be a normal crossing divisor on $X$ and set $Y := X \setminus D$. We fix a section $\tau : \mathbb{C}/\mathbb{Z} \hookrightarrow \mathbb{C}$ of the projection $\mathbb{C} \to \mathbb{C}/\mathbb{Z}$. Then for an integrable connection $M$ on $Y$ there exists an extension $L_\tau$ of $M$ as a vector bundle on $X$ satisfying the following two conditions:*

(i) *The $\mathbb{C}$-linear morphism $\nabla_M : M \to \Omega^1_Y \otimes_{\mathcal{O}_Y} M$ can be uniquely extended to a $\mathbb{C}$-linear morphism $\nabla : \mathcal{O}_X[D] \otimes_{\mathcal{O}_X} L_\tau \to \Omega^1_X \otimes_{\mathcal{O}_X} (\mathcal{O}_X[D] \otimes_{\mathcal{O}_X} L_\tau)$ so that $(\mathcal{O}_X[D] \otimes_{\mathcal{O}_X} L_\tau, \nabla)$ is a meromorphic connection which has a logarithmic pole along $D$ with respect to $L_\tau$.*

(ii) *For any irreducible component $D'$ of $D$ the eigenvalues of the residue $\mathrm{Res}^{L_\tau}_{D'} \nabla$ of $(\mathcal{O}_X[D] \otimes_{\mathcal{O}_X} L_\tau, \nabla)$ along $D'$ are contained in $\tau(\mathbb{C}/\mathbb{Z}) \subset \mathbb{C}$.*

*Moreover, such an extension is unique up to isomorphisms.*

*Proof.* First we prove the uniqueness of the extension. The problem being local, we may assume $X = B^n$ ($B$ is a unit disk in $\mathbb{C}$) and $Y = (B^*)^r \times B^{n-r}$ ($B^* := B \setminus \{0\}$). We denote by $\{x_i\}_{1 \le i \le n}$ the standard coordinate of $X = B^n$ Let $L$, $L'$ be locally free $\mathcal{O}_X$-modules of rank $m$ satisfying the conditions (i), (ii) for $L_\tau = L$ and $L_\tau = L'$, and let $\nabla : \mathcal{O}_X[D] \otimes_{\mathcal{O}_X} L \to \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{O}_X[D] \otimes_{\mathcal{O}_X} L$ and $\nabla' : \mathcal{O}_X[D] \otimes_{\mathcal{O}_X} L' \to \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{O}_X[D] \otimes_{\mathcal{O}_X} L'$ be the $\mathbb{C}$-linear morphisms satisfying the condition (i). Take a basis $\{e_i\}_{1 \le i \le m}$ of $L$. Then $\nabla$ can be expressed as

$$\nabla e_i = \sum_{k,j} a^k_{ij} dx_k \otimes e_j.$$

Set $\omega = \sum_{1 \le k \le n} A^k dx_k$ ($A^k = (a^k_{ij})$). We also fix a basis $\{e'_i\}$ of $L'$ and define $\omega' = \sum_{1 \le k \le n} A'^k dx_k$ similar to $\omega$. Since $L|_Y$ and $L'|_Y$ are isomorphic, there exists $S \in GL_m(\mathcal{O}_Y)$ such that

$$dS = S\omega - \omega' S. \tag{5.2.5}$$

It is sufficient to verify that $S$ and $S^{-1}$ can be extended to an element of $M_m(\mathcal{O}_X)$ (an $m \times m$ matrix on $X$ whose entries are holomorphic on $X$). By symmetry we have only to show the assertion for $S$. Moreover, by *Hartogs' theorem* it is sufficient to show that $S$ extends holomorphically across $D_k \setminus \bigcup_{l \ne k} D_l$ for each $k$. For simplicity of notation we only consider the case $k = 1$. Write

$$\omega = B^1 \frac{dx_1}{x_1} + \sum_{i>1} A^i dx_i,$$
$$\omega' = B'^1 \frac{dx_1}{x_1} + \sum_{i>1} A'^i dx_i.$$

By our assumption $B^1$, $B'^1$ and $A^i$, $A'^i$ are holomorphic on $x_2 x_3 \cdots x_r \neq 0$. By (5.2.5) we have

$$x_1 \frac{\partial S}{\partial x_1} = SB^1 - B'^1 S. \tag{5.2.6}$$

Taking the matrix norm $\|\bullet\|$ of both sides we obtain an inequality

$$|x_1| \left\| \frac{\partial S}{\partial x_1} \right\| \leq C \|S\| \tag{5.2.7}$$

($C > 0$ is a constant) on a neighborhood of $x_1 = 0$ ($x_2 x_3 \cdots x_r \neq 0$), from which we see that $S$ is meromorphic along $D_1 \setminus \bigcup_{i\neq 1} D_i$ by *Gronwall's inequality*: if $f(t) \geq 0$, $g(t) \geq 0$ are non-negative functions on $0 < t \leq t_0$ such that

$$f(t) \leq a + c \int_t^{t_0} g(s) f(s) ds \quad (a \geq 0,\ c \geq 0),$$

then we have

$$f(t) \leq a \exp(c \int_t^{t_0} g(s) ds) \quad (0 < t < t_0).$$

To show that $S$ is actually holomorphic on $D_1 \setminus \bigcup_{i\neq 1} D_i$ we consider the Laurent expansion

$$S = \sum_{j=p}^{\infty} S_j x_1^j \quad (S_p \neq 0)$$

of $S$ with respect to the variable $x_1$. By (5.2.6) we have

$$(\mathrm{Res}_{D_1} \nabla' + pI) S_p = S_p (\mathrm{Res}_{D_1} \nabla),$$

where $I$ is the unit matrix. By elementary linear algebra, it follows from $S_p \neq 0$ that the matrices $\mathrm{Res}_{D_1} \nabla' + pI$ and $\mathrm{Res}_{D_1} \nabla$ must have at least one common eigenvalue. If $p \neq 0$, then this last result contradicts our assumption that the eigenvalues of $\mathrm{Res}_{D_1} \nabla$ and $\mathrm{Res}_{D_1} \nabla'$ are contained in $\tau(\mathbb{C}/\mathbb{Z})$. Hence we must have $p = 0$, and $S$ is holomorphic on $D_1 \setminus \bigcup_{i\neq 1} D_i$. The proof of the uniqueness is complete.

Next we prove the existence of the extension (as a vector bundle) $L_\tau$ of $M$. If there exists locally such an extension $L_\tau$, we can glue these local extensions to get the global one by the uniqueness of $L_\tau$ proved above. Hence we may assume $Y = (B^*)^r \times B^{n-r} \subset X = B^n$. By Theorem 4.2.4 the integrable connection $M$ on $Y$ is uniquely determined by the monodromy representation

$$\rho : \pi_1(Y) \longrightarrow GL_m(\mathbb{C})$$

defined by the local system corresponding to $M$. Note that for $Y = (B^*)^r \times B^{n-r}$ we have $\pi_1(Y) \simeq \mathbb{Z}^r$, where the element $\gamma_i = (0, 0, \dots, \overset{i}{\overset{\smile}{1}}, \dots, 0) \in \pi_1(Y) \simeq \mathbb{Z}^r$ corresponds to a closed loop turning around the divisor $D_i = \{x_i = 0\}$. Therefore, the monodromy representation $\rho$ is determined by the mutually commuting matrices

$$C_i := \rho(\gamma_i) \in GL_m(\mathbb{C}) \quad (1 \leq i \leq r).$$

Note also that we can take matrices $\Gamma^i \in M_m(\mathbb{C})$ $(1 \leq i \leq r)$ such that

$$\begin{cases} \text{(a) } \exp(2\pi\sqrt{-1}\,\Gamma^i) = C_i, \\ \text{(b) All the eigenvalues of } \Gamma^i \text{ belong to the set } \tau(\mathbb{C}/\mathbb{Z}), \\ \text{(c) } \Gamma^i \ (1 \leq i \leq r) \text{ are mutually commuting matrices.} \end{cases}$$

The proof of the existence of such matrices is left to the reader (in fact, $\Gamma^i$ are uniquely determined). Using these $\Gamma^i = (\Gamma^i_{pq}) \in M_m(\mathbb{C})$, we define a meromorphic connection on $\mathcal{O}_X[D]^m = \mathcal{O}_X[D] \otimes_{\mathcal{O}_X} L_\tau$ $(L_\tau = \mathcal{O}_X^m)$ by

$$\nabla e_q = -\sum_{1 \leq i \leq n, 1 \leq q \leq m} \frac{\Gamma^i_{pq}}{x_i} dx_i \otimes e_q,$$

where $\{e_1, \dots, e_m\}$ is the standard basis of $L_\tau = \mathcal{O}_X^m$. Then the restriction of this meromorphic connection to $Y$ is isomorphic to $M$. Moreover, this meromorphic connection satisfies all of the required conditions. □

### 5.2.3 Deligne's Riemann–Hilbert correspondence

In Theorem 5.2.17 we proved that an integrable connection $M$ defined on the complementary set $Y = X \setminus D$ of a normal crossing divisor $D$ on $X$ can be extended to a meromorphic connection $\widetilde{M}$ $(= \mathcal{O}_X[D] \otimes_{\mathcal{O}_X} L_\tau)$ on $X$, regular along $D$. In this subsection we generalize this result to arbitrary divisors $D$ on $X$.

Let $D$ be a (not necessarily normal crossing) divisor on $X$ and $(N, \nabla)$ a meromorphic connection on $X$ which is meromorphic along $D$. We consider the following condition (R) on $(N, \nabla)$, which is a priori weaker than the regularity along $D$ (in fact, we will prove in Corollary 5.2.22 below that these two conditions are equivalent).

**Condition (R).** There exists an open subset $U$ of the regular part $D_{\mathrm{reg}}$ of $D$ which intersects with each connected component of $D_{\mathrm{reg}}$ and satisfies the condition:

$$\begin{cases} \text{There exist an open neighborhood } \widetilde{U} \text{ of } U \text{ in } X \text{ and an isomorphism} \\ \varphi : B \times U \xrightarrow{\sim} \widetilde{U} \text{ such that } \varphi|_{\{0\}\times U} = \mathrm{Id}_U \text{ and for each } x \in U \text{ the} \\ \text{pull-back } (\varphi_x^* N, \varphi_x^* \nabla) \text{ with respcet to } \varphi_x = \varphi|_{B\times\{x\}} : B \times \{x\} \hookrightarrow X \text{ is} \\ \text{regular along } \{0\} \times \{x\}. \end{cases}$$

Here $B$ is the unit disk centered at 0 in $\mathbb{C}$.

It is clear that if $(N, \nabla)$ is regular along $D$, then it satisfies the condition (R).

**Lemma 5.2.18.** *Assume that a meromorphic connection $(N, \nabla)$ satisfies the condition (R). Then the restriction map*

$$\Gamma(X, N^{\nabla}) \longrightarrow \Gamma(Y, N^{\nabla})$$

*is an isomorphism.*

*Proof.* The injectivity follows from Lemma 5.2.7. Let us show surjectivity. We need to show that any $s \in \Gamma(Y, N^{\nabla})$ can be extended to a section of $N^{\nabla}$ on $X$. By Corollary 5.2.8 it is sufficient to show that for any $p \in D$ and any $u \in N_p^* = \mathcal{H}om_{\mathcal{O}_X[D]}(N, \mathcal{O}_X[D])_p$ the function $g = \langle u, s \rangle$ is meromorphic at $p$. By Hartogs' theorem it is sufficient to show it in the case $p \in D_{\mathrm{reg}}$.

We first deal with the case when $p \in U \subset D_{\mathrm{reg}}$. Here $U$ is as in Condition (R). We take a local coordinate $(x_1, \ldots, x_n)$ around $p$ such that $D$ is defined by $x_1 = 0$ and $p$ corresponds to 0. We may assume that $U = \{0\} \times B^{n-1}$. Let

$$g(x) = \sum_{k \in \mathbb{Z}} g_k(x_2, x_3, \ldots, x_n)\, x_1^k$$

be the Laurent expansion of $g$ with respect to $x_1$. The condition (R) implies that for each $x' = (x_2, x_3, \ldots, x_n) \in B^{n-1}$ the restriction $g|_{B^* \times \{x'\}}$ is meromorphic at the point $\{0\} \times \{x'\}$. This means that for each $x' = (x_2, x_3, \ldots, x_n) \in B^{n-1}$ we have $g_k(x') = 0$ $(k \ll 0)$. For $k \in \mathbb{Z}$ set

$$U_k := \{\, x' = (x_2, x_3, \ldots, x_n) \in B^{n-1} \mid g_i(x') = 0 \ (i \leq k) \,\}.$$

Then we have $B^{n-1} = \bigcup_{k \in \mathbb{Z}} U_k$ and each $U_k$ is a closed analytic subset of $B^{n-1}$. It follows that we have $U_k = B^{n-1}$ for some $k$ and hence $g$ is meromorphic at $p$.

Let us consider the general case $p \in D_{\mathrm{reg}}$. We denote by $K$ the subset of $D_{\mathrm{reg}}$ consisting of $p \in D_{\mathrm{reg}}$ such that $g = \langle u, s \rangle$ is meromorphic at $p$ for any $u \in N_p^*$. We need to show $K = D_{\mathrm{reg}}$. Note that $K$ is an open subset of $D_{\mathrm{reg}}$ containing $U$. In particular, it intersects with any connected component of $D_{\mathrm{reg}}$. Hence it is sufficient to show that $K$ is a closed subset of $D_{\mathrm{reg}}$. Let $q \in \overline{K}$. We take a local coordinate $(x_1, \ldots, x_n)$ around $a$ such that $D$ is defined by $x_1 = 0$ and $q$ corresponds to 0. For $u \in N_p^*$ we consider the expansion

$$g(x) = \sum_{k \in \mathbb{Z}} g_k(x_2, x_3, \ldots, x_n)\, x_1^k$$

of $g = \langle u, s \rangle$. Since $g$ is meromorphic on $K$, there exists some $r$ such that $g_k = 0$ on $K$ for any $k \leq r$. It follows that $g_k = 0$ for any $k \leq r$ on an open neighborhood of $q$. Hence $q \in K$. □

**Lemma 5.2.19.** *Let $(N_i, \nabla_i)$ $(i = 1, 2)$ be two meromorphic connections along $D$ satisfying the condition* (R). *Then the restriction to $Y = X \setminus D$ induces an isomorphism*

$$r : \mathrm{Hom}_{\mathrm{Conn}(X;D)}((N_1, \nabla_1), (N_2, \nabla_2)) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Conn}(Y)}((N_1|_Y, \nabla_1), (N_2|_Y, \nabla_2)).$$

*Proof.* By Lemma 5.2.5 we have

$$\Gamma(X, \mathcal{H}om_{\mathcal{O}_X[D]}(N_1,\ N_2)^{\nabla}) \simeq \mathrm{Hom}_{\mathrm{Conn}(X;D)}((N_1,\ \nabla_1),\ (N_2,\ \nabla_2)).$$

Therefore, it suffices to apply Lemma 5.2.18 to the meromorphic connection $N = \mathcal{H}om_{\mathcal{O}_X[D]}(N_1,\ N_2)$, which satisfies the condition (R) by Proposition 5.2.12 (ii). □

The main result of this section is the following.

**Theorem 5.2.20 (Deligne [De1]).** *Let $X$ be a complex manifold and let $D$ be a (not necessarily normal crossing) divisor on $X$. Set $Y = X \setminus D$. Then the restriction functor $N \longmapsto N|_Y$ induces an equivalence*

$$\mathrm{Conn}^{\mathrm{reg}}(X;\ D) \xrightarrow{\sim} \mathrm{Conn}(Y)$$

*of categories.*

*Proof.* Since a regular meromorphic connection along $D$ satisfies the condition (R), the restriction functor is fully faithful by Lemma 5.2.19. Let us prove the essential surjectivity. We take an integrable connection $M$ on $Y$ and consider the problem of extending $M$ to a regular meromorphic connection on the whole $X$. By Hironaka's theorem there exists a proper surjective morphism $f : X' \to X$ of complex manifolds such that $D' := f^{-1}D$ is a normal crossing divisor on $X'$ and the restriction $g : X' \setminus D' \to X \setminus D = Y$ of $f$ is an isomorphism. By Theorem 5.2.17 we can extend the integrable connection $g^*M$ on $X' \setminus D'$ to a meromorphic connection $N$ on $X'$ along $D'$ which has a logarithmic pole with respect to a lattice $L$. Then $H^0(\int_f N)$ satisfies the desired property by Lemma 5.2.14. □

When $D$ is normal crossing, we proved in Theorem 5.2.17 the uniqueness of the regular meromorphic extension of an integrable connection on $Y$ under an additional condition about the lattice $L_\tau$. Theorem 5.2.20 above asserts that this condition was not really necessary.

By Theorem 4.2.4 this theorem has the following topological interpretation.

**Corollary 5.2.21.** *Let $X$ be a complex manifold and let $D$ be a divisor on $X$. Set $Y = X \setminus D$. Then we have an equivalence*

$$\mathrm{Conn}^{\mathrm{reg}}(X;\ D) \xrightarrow{\sim} \mathrm{Loc}(X \setminus D)$$

*of categories.*

We call this result *Deligne's Riemann–Hilbert correspondence*. This classical Riemann–Hilbert correspondence became the prototype of the Riemann–Hilbert correspondence for analytic regular holonomic $D$-modules (Theorem 7.2.1 below).

**Corollary 5.2.22.** *Let $D$ be a (not necessarily normal crossing) divisor on a complex manifold $X$ and let $N$ be a meromorphic connection along $D$.*

(i) *If $N$ satisfies the condition (R), then $N$ is regular along $D$.*
(ii) *If $N$ is regular along $D$, then $N$ is an effective meromorphic connection along $D$.*

*Proof.* (i) By Theorem 5.2.20 there exists a regular meromorphic connection $N_1$ along $D$ such that $N_1|_Y \xrightarrow{\sim} N|_Y$. Since both $N$ and $N_1$ satisfy the condition (R), this isomorphism $N_1|_Y \xrightarrow{\sim} N|_Y$ can be extended to a morphism $N_1 \to N$ of meromorphic connections on $X$ by Lemma 5.2.19 . This is in fact an isomorphism by Proposition 5.2.6, and hence $N$ is regular along $D$.

(ii) In the proof of Theorem 5.2.20 we explicitly constructed a regular meromorphic extension of $N|_Y$, which is isomorphic to $N$ and has the required property. □

Assume that $D$ is a normal crossing divisor on $X$. We define a subsheaf of the sheaf $\Theta_X$ of holomorphic vector fields on $X$ by

$$\Theta_X\langle D\rangle := \{\, \theta \in \Theta_X \mid \theta\mathcal{I} \subset \mathcal{I} \,\},$$

where $\mathcal{I}$ is the defining ideal of $D$. If $\{x_i, \partial_i\}$ is a local coordinate system of $X$ in which $D$ is defined by $x_1x_2\cdots x_r = 0$, then $\Theta_X\langle D\rangle$ is generated by $x_i\partial_i$ $(1 \le i \le r)$ and $\partial_j$ $(j > r)$ over $\mathcal{O}_X$.

**Corollary 5.2.23.** *Assume that $D$ is a normal crossing divisor on a complex manifold $X$. Under the above notation the following conditions on a meromorphic connection $N$ along $D$ are equivalent:*

(i) *$N$ is regular along $D$.*
(ii) *$N$ is a union of $\Theta_X\langle D\rangle$-stable coherent $\mathcal{O}_X$-submodules.*

*Proof.* (i) $\Longrightarrow$ (ii): We use Theorem 5.2.17. Namely, for a section $\tau : \mathbb{C}/\mathbb{Z} \hookrightarrow \mathbb{C}$ of $\mathbb{C} \to \mathbb{C}/\mathbb{Z}$, we take a locally free $\mathcal{O}_X$-module $L_\tau$ such that $N|_Y \simeq L_\tau|_Y$ and $\mathcal{O}_X[D] \otimes_{\mathcal{O}_X} L_\tau$ is a regular meromorphic connection along $D$. Then by Theorem 5.2.20 we have $N \simeq \mathcal{O}_X[D] \otimes_{\mathcal{O}_X} L_\tau$. Taking a local coordinate system $\{x_i\}$ of $X$ such that $D$ is defined by $g(x) = x_1x_2\cdots x_r = 0$, we have $\mathcal{O}_X[D] \otimes_{\mathcal{O}_X} L_\tau = \bigcup_{k\ge 0} \mathcal{O}_X g^{-k} \otimes_{\mathcal{O}_X} L_\tau$. Since for each $k \ge 0$ the definition of the $\mathcal{O}_X$-coherent subsheaf $\mathcal{O}_X g^{-k} \otimes_{\mathcal{O}_X} L_\tau$ of $\mathcal{O}_X[D] \otimes_{\mathcal{O}_X} L_\tau$ does not depend on the local coordinates $\{x_i\}$ or on the defining equation $g$ of $D$, it is globally defined on the whole $X$. Clearly each $\mathcal{O}_X g^{-k} \otimes_{\mathcal{O}_X} L_\tau$ is $\Theta_X\langle D\rangle$-stable ($L_\tau$ is $\Theta_X\langle D\rangle$-stable).

(ii) $\Longrightarrow$ (i): By Theorem 5.2.22 (i) it suffices to check that $N$ satisfies the condition (R). If we restrict $N$ to a unit disk $B$ in the condition (R), then the restricted meromorphic connection satisfies the condition (ii) on the unit disk $B$. By Proposition 5.1.9 this means that the restriction is a regular meromorphic connection at $0 \in B$. So $N$ satisfies the condition (R). Now the proof is complete. □

**Theorem 5.2.24 (Deligne [De1]).** *Let $D$ be a divisor on a complex manifold $X$ and let $j : Y = X \setminus D \hookrightarrow X$ be the embedding. Let $N$ be a regular meromorphic connection along $D$. Then the natural morphisms*

$$\begin{cases} DR_X(N) \longrightarrow Rj_*j^{-1}\, DR_X(N), \\ R\Gamma(X, DR_X(N)) \longrightarrow R\Gamma(Y, DR_Y(N|_Y)) \end{cases}$$

*are isomorphisms.*

*Proof.* It is enough to prove that $DR_X(N) \to Rj_*j^{-1}\,DR_X(N)$ is an isomorphism.

We first show it in the case where $D$ is normal crossing. The problem being local on $X$, we may assume $X = B^n$ and $Y = (B^*)^r \times B^{n-r}$, where $B$ is the unit disk in $\mathbb{C}$ and $B^* = B \setminus \{0\}$. Recall that the category of meromorphic connections that are regular along $D$ is equivalent to that of local systems on $Y$. Hence by $\pi_1((B^*)^r \times B^{n-r}) \simeq \pi(((\mathbb{C}^\times)^r \times \mathbb{C}^{n-r})^{\mathrm{an}})$ we can assume that $X = (\mathbb{C}^n)^{\mathrm{an}}$ and $Y = ((\mathbb{C}^\times)^r \times \mathbb{C}^{n-r})^{\mathrm{an}}$. Here, $\mathbb{C}^n$, $\mathbb{C}^\times$, $\mathbb{C}^{n-r}$ are regarded as algebraic varieties. We can assume also that $N$ is a simple object in the abelian category of regular meromorphic connections (along $D$). Indeed, let

$$0 \longrightarrow N_1 \longrightarrow N_2 \longrightarrow N_3 \longrightarrow 0$$

be an exact sequence in this category, and denote by $\Phi_i : DR_X(N_i) \longrightarrow Rj_*j^{-1}\,DR_X(N_i)$ $(i = 1, 2, 3)$ the natural morphisms. Then $\Phi_2$ is an isomorphism if $\Phi_1$ and $\Phi_3$ are as well. By

$$\pi_1(Y) = \pi_1((\mathbb{C}^\times)^{\mathrm{an}})^r \times \pi_1((\mathbb{C}^{n-r})^{\mathrm{an}}),$$
$$\pi_1((\mathbb{C}^\times)^{\mathrm{an}}) \simeq \mathbb{Z}, \qquad \pi_1((\mathbb{C}^{n-r})^{\mathrm{an}}) = \{1\},$$

we see that there exist $\lambda_1, \dots, \lambda_r \in \mathbb{C}$ such that

$$N \simeq (N_{\lambda_1} \boxtimes \cdots \boxtimes N_{\lambda_r} \boxtimes \mathcal{O}_{\mathbb{C}^{n-r}})^{\mathrm{an}}. \tag{5.2.8}$$

Here, for $\lambda \in \mathbb{C}$ we denote by $N_\lambda$ the (algebraic) $D_{\mathbb{C}}$-module given by $N_\lambda = D_{\mathbb{C}}/D_{\mathbb{C}}(x\partial - \lambda)$. Hence by Proposition 4.7.8 we obtain

$$DR_X(N) \simeq DR_{\mathbb{C}}(N_{\lambda_1}) \boxtimes_{\mathbb{C}} \cdots \boxtimes_{\mathbb{C}} DR_{\mathbb{C}}(N_{\lambda_1}) \boxtimes_{\mathbb{C}} \mathbb{C}_{(\mathbb{C}^{n-r})^{\mathrm{an}}}.$$

Therefore, it is sufficient to show $DR_{\mathbb{C}}(N_\lambda) \simeq Rk_*k^{-1}\,DR_{\mathbb{C}}(N_\lambda)$, where $k : \mathbb{C}^\times \to \mathbb{C}$ denotes the embedding (see Proposition 4.5.9). Since the canonical morphism $DR_{\mathbb{C}}(N_\lambda) \to Rk_*k^{-1}\,DR_{\mathbb{C}}(N_\lambda)$ is an isomorphism outside of the origin, we have only to show the isomorphism $DR_{\mathbb{C}}(N_\lambda)_0 \simeq (Rk_*k^{-1}\,DR_{\mathbb{C}}(N_\lambda))_0$ for the stalks. Set $\nabla = \partial + \frac{\lambda}{x}$. Then $(DR_{\mathbb{C}}(N_\lambda))_0$ (resp. $(Rk_*k^{-1}\,DR_{\mathbb{C}}(N_\lambda))_0$) is represented by the complex

$$[K \xrightarrow{\nabla} K], \qquad \Big(\text{resp. } [\tilde{K} \xrightarrow{\nabla} \tilde{K}]\Big),$$

where $K = \mathcal{O}_{\mathbb{C}^{\mathrm{an}}}[x^{-1}]$ (resp. $\tilde{K} = k_*\mathcal{O}_{(\mathbb{C}^\times)^{\mathrm{an}}}$). From this we easily see by considering the Laurent series expansions of functions in $K$ and $\tilde{K}$ that $(DR_{\mathbb{C}}(N_\lambda))_0 \to (Rk_*k^{-1}\,DR_{\mathbb{C}}(N_\lambda))_0$ is an isomorphism.

Now we consider the general case where $D$ is an arbitrary divisor on $X$. By Hironaka's theorem there exists a proper surjective morphism $f : X' \to X$ of complex manifolds such that $D' := f^{-1}D$ is a normal crossing divisor on $X'$ and the restriction $X' \setminus D' \to X \setminus D = Y$ of $f$ is an isomorphism. We denote by $j' : X' \setminus D' \to X'$ the embedding. By Theorem 5.2.20 and Lemma 5.2.14 there

exists a regular meromorphic connection $N'$ on $X'$ along $D'$ such that $N \simeq \int_f N'$. Then we have

$$\begin{aligned} DR_X(N) &\simeq DR_X\Big(\int_f N'\Big) \simeq Rf_* DR_{X'}(N') \simeq Rf_* Rj'_* j'^{-1} DR_{X'}(N') \\ &\simeq Rj_* j^{-1} DR_X(N) \end{aligned}$$

by Theorem 4.2.5. The proof is complete. □

**Remark 5.2.25.** In the course of the proof of Theorem 5.2.24 we have used Proposition 4.7.8 and Proposition 4.5.9 in order to reduce the proof of the case of normal crossing divisors to the one-dimensional situation. We can avoid this argument by directly considering $DR_X(N)$ as follows.

Let $D$ be the normal crossing divisor on the complex manifold $X = \mathbb{C}^n$ defined by $x_1 \cdots x_r = 0$, where $(x_1, \ldots, x_n)$ is the coordinate of $\mathbb{C}^n$. Let $N$ be the simple regular meromorphic connection corresponding to the simple local system $\mathbb{C} x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_r^{\lambda_r}$ $(0 \leq \operatorname{Re} \lambda_i < 1)$ on $Y = X \setminus D$. Then $N$ is identified with $\mathcal{O}_X[D]$ as an $\mathcal{O}_X[D]$-module, and the action of $\frac{\partial}{\partial x_i}$ $(1 \leq i \leq n)$ is given by

$$\begin{cases} \nabla_{\frac{\partial}{\partial x_i}} u = \dfrac{\partial u}{\partial x_i} - \dfrac{\lambda_i}{x_i} u, & i = 1, 2, \ldots, r \\ \nabla_{\frac{\partial}{\partial x_i}} u = \dfrac{\partial u}{\partial x_i}, & i = r+1, r+2, \ldots, n \end{cases}$$

for $u \in \mathcal{O}_X[D]$ $(0 \leq \operatorname{Re} \lambda_i < 1)$. In this situation $DR_X(N)$ is represented by the *Koszul complex* $K(\nabla_1, \nabla_2, \ldots, \nabla_n; \mathcal{O}_X[D])$ (up to some degree shift) defined by the mutually commuting differentials $\nabla_i = \nabla_{\frac{d}{dx_i}} : \mathcal{O}_X[D] \longrightarrow \mathcal{O}_X[D]$ $(i = 1, 2, \ldots, n)$. On the other hands, $Rj_* j^{-1} DR_X(N)$ is represented by the Koszul complex $K(\nabla_1, \nabla_2, \ldots, \nabla_n; j_* \mathcal{O}_{X \setminus D})$ because the hypersurface complement $Y = X \setminus D$ is locally a Stein open subset of $X = \mathbb{C}^n$. Hence it suffices to show that the natural morphism

$$K(\nabla_1, \nabla_2, \ldots, \nabla_n; \mathcal{O}_X[D]) \longrightarrow K(\nabla_1, \nabla_2, \ldots, \nabla_n; j_* \mathcal{O}_{X \setminus D})$$

induced by $\mathcal{O}_X[D] \longrightarrow j_* \mathcal{O}_{X \setminus D}$ is a quasi-isomorphism. The sections of $\mathcal{O}_X[D]$ have poles along $D$, while those of $j_* \mathcal{O}_{X \setminus D}$ may have essential singularities along $D$. We have to compare the cohomology groups of the de Rham complexes in these two different sheaves. First, note that the Koszul complex $K(\nabla_1, \nabla_2, \ldots, \nabla_n; \mathcal{O}_X[D])$ (resp. $K(\nabla_1, \nabla_2, \ldots, \nabla_n; j_* \mathcal{O}_{X \setminus D})$) is the simplified complex of the double complex $K(\nabla_2, \ldots, \nabla_n; \mathcal{O}_X[D] \xrightarrow{\nabla_1} \mathcal{O}_X[D])$ (resp. $K(\nabla_2, \ldots, \nabla_n; j_* \mathcal{O}_{X \setminus D} \xrightarrow{\nabla_1} j_* \mathcal{O}_{X \setminus D})$ ). Let us consider the morphism of complexes

$$\big[\mathcal{O}_X[D] \xrightarrow{\nabla_1} \mathcal{O}_X[D]\big] \longrightarrow \big[j_* \mathcal{O}_{X \setminus D} \xrightarrow{\nabla_1} j_* \mathcal{O}_{X \setminus D}\big]$$

induced by $\mathcal{O}_X[D] \longrightarrow j_* \mathcal{O}_{X \setminus D}$. We set $X' = \{x_1 = 0\} \subset X$ and $D' = \{x_2 \cdots x_r = 0\} \subset X'$. Then we can easily see that the cohomology groups of the left complex

(resp. right complex) are $\mathcal{O}_{X'}[D']$ (resp. $j'\mathcal{O}_{X'\setminus D'}$, $j' : X' \setminus D' \hookrightarrow X'$) or zero (it depends on the condition $\lambda_1 = 0$ or not) by expanding the functions in $\mathcal{O}_X[D]$ (resp. in $j_*\mathcal{O}_{X\setminus D}$) into the Laurent series in $x_1$. So repeating this argument also for $x_2, x_3, \ldots, x_n$ we can finally obtain the quasi-isomorphism

$$K(\nabla_1, \nabla_2, \ldots, \nabla_n; \mathcal{O}_X[D]) \xrightarrow{\sim} K(\nabla_1, \nabla_2, \ldots, \nabla_n; j_*\mathcal{O}_{X\setminus D}).$$

## 5.3 Regular integrable connections on algebraic varieties

In this subsection $X$ denotes a smooth algebraic variety. The corresponding complex manifold is denoted by $X^{\mathrm{an}}$.

Assume that we are given an open embedding $j : X \hookrightarrow V$ of $X$ into a smooth variety $V$ such that $D := V \setminus X$ is a divisor on $V$. We set $\mathcal{O}_V[D] := j_*\mathcal{O}_X$. As in the analytic situation $\mathcal{O}_V[D]$ is a coherent sheaf of rings. We say that a $D_V$-module is an *algebraic meromorphic connection* along $D$ if it is isomorphic as an $\mathcal{O}_V$-module to a coherent $\mathcal{O}_V[D]$-module. We denote by $\mathrm{Conn}(V; D)$ the category of algebraic meromorphic connections along $D$. It is an abelian category. Unlike the analytic situation an extension of an integrable connection on $X$ to an algebraic meromorphic connection on $V$ is unique as follows.

**Lemma 5.3.1.** *The functor* $j^{-1} : \mathrm{Conn}(V; D) \to \mathrm{Conn}(X)$ *induces an equivalence*

$$\mathrm{Conn}(V; D) \xrightarrow{\sim} \mathrm{Conn}(X)$$

*of categories. Its quasi-inverse is given by* $j_*$.

*Proof.* This follows easily from the fact that the category of coherent $\mathcal{O}_V[D]$-modules is naturally equivalent to that of coherent $\mathcal{O}_X$-modules. □

It follows that $\mathrm{Conn}(V; D)$ is a subcategory of $\mathrm{Mod}_h(D_V)$ by Theorem 3.2.3 (i).

**Definition 5.3.2.** An integrable connection $M$ on $X$ is called *regular* if for any morphism $i_C : C \to X$ from a smooth algebraic curve $C$ the induced integrable connection $i_C^*M$ on $C$ is regular in the sense of Section 5.1.3.

**Notation 5.3.3.** We denote by $\mathrm{Conn}^{\mathrm{reg}}(X)$ the full subcategory $\mathrm{Conn}(X)$ consisting of regular integrable connections.

By Lemma 5.1.17 we have the following.

**Proposition 5.3.4.** *Let*

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$$

*be an exact sequence of integrable connections on* $X$. *Then* $M_2$ *is regular if and only if* $M_1$ *and* $M_3$ *are regular.*

By Lemma 5.1.18 we easily see the following.

**Proposition 5.3.5.** *Let $M$ and $N$ be regular integrable connections on $X$. Then the integrable connections $M \otimes_{\mathcal{O}_X} N$ and $\mathcal{H}om_{\mathcal{O}_X}(M, N)$ are also regular.*

We give below some criteria for the regularity of integrable connections. Let us take a smooth completion $j : X \hookrightarrow \overline{X}$ of $X$ such that $D = \overline{X} \setminus X$ is a divisor on $\overline{X}$. Such a completion always exists thanks to Hironaka's theorem (in fact, we can take a completion $\overline{X}$ so that $D = \overline{X} \setminus X$ is a normal crossing divisor). We call such a completion $\overline{X}$ of $X$ a *divisor completion*. For a divisor completion $j : X \hookrightarrow \overline{X}$ of $X$ we can consider the analytic meromorphic connection

$$(j_*M)^{\mathrm{an}} = \mathcal{O}_{\overline{X}^{\mathrm{an}}} \otimes_{\mathcal{O}_{\overline{X}}} j_*M \in \mathrm{Conn}(\overline{X}^{\mathrm{an}}; D^{\mathrm{an}})$$

on $\overline{X}^{\mathrm{an}}$ along $D^{\mathrm{an}}$.

**Proposition 5.3.6.** *The following three conditions on an integrable connection $M$ on $X$ are equivalent:*

(i) *$M$ is a regular integrable connection.*
(ii) *For some divisor completion $j : X \hookrightarrow \overline{X}$ of $X$ the analytic meromorphic connection $(j_*M)^{\mathrm{an}}$ is regular.*
(iii) *For any divisor completion $j : X \hookrightarrow \overline{X}$ of $X$ the analytic meromorphic connection $(j_*M)^{\mathrm{an}}$ is regular.*

*Proof.* We first prove the part (ii) $\Longrightarrow$ (i). Assume that for a divisor completion $j : X \hookrightarrow \overline{X}$ the meromorphic connection $(j_*M)^{\mathrm{an}}$ is regular. We need to show that for any morphism $i_C : C \to X$ from an algebraic curve $C$ the induced integrable connection $i_C^*M$ is regular. We may assume that the image of $C$ is not a single point. We take a smooth completion $j_C : C \hookrightarrow \overline{C}$ of $C$ and a morphism $i_{\overline{C}} : \overline{C} \to \overline{X}$ so that the following diagram:

$$\begin{array}{ccc} C & \xrightarrow{j_C} & \overline{C} \\ {\scriptstyle i_C}\downarrow & & \downarrow{\scriptstyle i_{\overline{C}}} \\ X & \xrightarrow{j} & \overline{X} \end{array}$$

is commutative. We may also assume that this diagram is cartesian by replacing $C$ with $(i_{\overline{C}})^{-1}(X)$ (see Lemma 5.1.16). In this situation we have a natural isomorphism

$$\left[(j_C)_*i_C^*M\right]^{\mathrm{an}} \simeq \left[i_{\overline{C}}^*j_*M\right]^{\mathrm{an}} \simeq (i_{\overline{C}}^{\mathrm{an}})^*(j_*M)^{\mathrm{an}}.$$

Since $(j_*M)^{\mathrm{an}}$ is regular, $(i_{\overline{C}}^{\mathrm{an}})^*(j_*M)^{\mathrm{an}}$ (and hence $\left[(j_C)_*i_C^*M\right]^{\mathrm{an}}$) is regular by the definition of (analytic) regular meromorphic connections. It follows that $i_C^*M$ is regular by Lemma 5.1.14. It remains to show (i) $\Longrightarrow$ (iii). By Corollary 5.2.22 (i) it is sufficient to verify the condition (R) for $(j_*M)^{\mathrm{an}}$. We can take $\varphi$ in the condition (R) so that $\varphi_x = \varphi|_{B\times\{x\}}$ comes from an algebraic morphism, for which the condition (R) can be easily checked by the argument used in the proof of (ii) $\Longrightarrow$ (i). □

Let $j : X \hookrightarrow \overline{X}$ be a divisor completion of $X$ such that $D = \overline{X} \setminus X$ is normal crossing. In this situation we give another criterion of the regularity of an integrable connection $M$ on $X$. Let $\mathcal{I}$ be the defining ideal of $D$ ($\mathcal{I} := \{\varphi \in \mathcal{O}_{\overline{X}} \mid \varphi|_D = 0\}$) and consider the sheaf

$$\Theta_{\overline{X}}\langle D\rangle := \{\,\theta \in \Theta_{\overline{X}} \mid \theta\mathcal{I} \subset \mathcal{I}\,\}$$

as in the analytic case. We denote by $D_{\overline{X}}\langle D\rangle$ the subalgebra of $D_{\overline{X}}$ generated by $\Theta_{\overline{X}}\langle D\rangle$ and $\mathcal{O}_{\overline{X}}$. In terms of a local coordinate $\{x_i,\ \partial_i\}$ of $\overline{X}$ for which $D$ is defined by $x_1x_2\cdots x_r = 0$, $\Theta_{\overline{X}}\langle D\rangle$ is generated by $x_i\partial_i$ ($1 \le i \le r$) and $\partial_j$ ($j > r$) over $\mathcal{O}_{\overline{X}}$.

**Theorem 5.3.7 (Deligne [De1]).** *Under the above notation the following three conditions on an integrable connection $M$ on $X$ are equivalent to each other:*

(i) *$M$ is regular.*
(ii) *The $D_{\overline{X}}$-module $j_*M$ is a union of $\mathcal{O}_{\overline{X}}$-coherent $D_{\overline{X}}\langle D\rangle$-submodules.*
(iii) *For any irreducible component $D_1$ of $D$ there exists an open dense subset $D_1' \subset D_1$ satisfying the condition:*

$$\begin{cases} \text{For each point } p \in D_1', \text{ there exists an algebraic curve } \overline{C} \hookrightarrow \overline{X} \text{ which} \\ \text{intersects with } D_1' \text{ transversally at } p \text{ such that the integrable connection} \\ i_C^*M \text{ on } C = \overline{C} \setminus \{p\}\ (i_C : C = \overline{C} \setminus \{p\} \hookrightarrow X) \text{ has a regular singular} \\ \text{point at } p \in \overline{C}. \end{cases}$$

*Proof.* (i) $\Longrightarrow$ (ii): Assume that $M$ is regular. Then $N = (j_*M)^{\mathrm{an}}$ is an analytic regular meromorphic connection along $D^{\mathrm{an}}$. By Corollary 5.2.23 $N$ is a union of $\Theta_{\overline{X}^{\mathrm{an}}}\langle D^{\mathrm{an}}\rangle$-stable coherent $\mathcal{O}_{\overline{X}^{\mathrm{an}}}$-submodules $\widetilde{L}_i$ ($i \in I$). Since $\overline{X}$ is projective, there exists a coherent $\mathcal{O}_{\overline{X}}$-submodule $L_i$ of $j_*M$ such that $L_i^{\mathrm{an}} = \widetilde{L}_i$ by GAGA [Ser1]. Denote by $N_i$ the image of $\Theta_{\overline{X}}\langle D\rangle \otimes_{\mathbb{C}} L_i \to j_*M$. Then by $N_i^{\mathrm{an}} \subset L_i^{\mathrm{an}}$ we obtain $N_i \subset L_i$ by GAGA. Namely, each $L_i$ is $\Theta_{\overline{X}}\langle D\rangle$-stable. We also have $\bigcup_i L_i = j_*M$, and hence (ii) holds.

(ii) $\Longrightarrow$ (i): If (ii) holds, then the $D_{\overline{X}^{\mathrm{an}}}$-module $(j_*M)^{\mathrm{an}}$ is a union of $\Theta_{\overline{X}^{\mathrm{an}}}\langle D^{\mathrm{an}}\rangle$-stable coherent $\mathcal{O}_{\overline{X}^{\mathrm{an}}}$-submodules. Hence it is a regular meromorphic connection along $D^{\mathrm{an}}$ by Corollary 5.2.23. Therefore, $M$ is regular by Proposition 5.3.6.

(i) $\Longrightarrow$ (iii): this is trivial.

(iii) $\Longrightarrow$ (i). Under the condition (iii), the corresponding analytic meromorphic connection $N = (j_*M)^{\mathrm{an}}$ satisfies the condition (R). Hence $(j_*M)^{\mathrm{an}}$ is regular by Corollary 5.2.22 (i). □

The following version of the Riemann–Hilbert correspondence in the algebraic situation will play fundamental roles in establishing more general correspondence for algebraic regular holonomic $D_X$-modules (Theorem 7.2.2 below).

**Theorem 5.3.8 (Deligne [De1]).** *Let $X$ be a smooth algebraic variety. Then the functor $M \mapsto M^{\mathrm{an}}$ induces an equivalence*

$$\mathrm{Conn}^{\mathrm{reg}}(X) \xrightarrow{\sim} \mathrm{Conn}(X^{\mathrm{an}})$$

*of categories.*

Since any integrable connection on a smooth projective algebraic variety is regular by Proposition 5.3.6, we have the following.

**Corollary 5.3.9.** *For a smooth projective variety $X$ we have an equivalence*

$$\mathrm{Conn}(X) \xrightarrow{\sim} \mathrm{Conn}(X^{\mathrm{an}})$$

*of categories.*

By Theorem 5.3.8 and Theorem 4.2.4 we also have the following.

**Corollary 5.3.10.** *For a smooth algebraic variety $X$ we have an equivalence*

$$\mathrm{Conn}^{\mathrm{reg}}(X) \simeq \mathrm{Loc}(X^{\mathrm{an}})$$

*of categories.*

This gives a totally algebraic description of the category of local systems on $X^{\mathrm{an}}$.

The rest of this section is devoted to the proof of Theorem 5.3.8. We fix a divisor completion $j : X \hookrightarrow \overline{X}$ of $X$ and set $D = \overline{X} \setminus X$. We denote by $\mathrm{Conn}^{\mathrm{reg}}(\overline{X}; D)$ the full subcategory of $\mathrm{Conn}(\overline{X}; D)$ consisting of $M \in \mathrm{Conn}(\overline{X}; D)$ such that $M|_X$ is regular. Then we have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Conn}^{\mathrm{reg}}(X; D) & \longrightarrow & \mathrm{Conn}^{\mathrm{reg}}(\overline{X}^{\mathrm{an}}; D^{\mathrm{an}}) \\ \downarrow & & \downarrow \\ \mathrm{Conn}^{\mathrm{reg}}(X) & \longrightarrow & \mathrm{Conn}(X^{\mathrm{an}}), \end{array}$$

where vertical arrows are given by restrictions and horizontal arrows are given by $M \mapsto M^{\mathrm{an}}$. Since the vertical arrows are equivalences by Lemma 5.3.1 and Theorem 5.2.20, our assertion is equivalent to the equivalence of

$$\mathrm{Conn}^{\mathrm{reg}}(X; D) \longrightarrow \mathrm{Conn}^{\mathrm{reg}}(\overline{X}^{\mathrm{an}}; D^{\mathrm{an}}). \tag{5.3.1}$$

We denote by $\mathrm{Mod}_c(\mathcal{O}_{\overline{X}}[D])$ (resp. $\mathrm{Mod}^e_c(\mathcal{O}_{\overline{X}^{\mathrm{an}}}[D^{\mathrm{an}}])$) the category of coherent $\mathcal{O}_{\overline{X}}[D]$-modules (resp. the category of coherent $\mathcal{O}_{\overline{X}^{\mathrm{an}}}[D^{\mathrm{an}}]$-modules generated by a coherent $\mathcal{O}_{\overline{X}^{\mathrm{an}}}$-submodule). Note that any coherent $\mathcal{O}_{\overline{X}}[D]$-module is generated by its coherent $\mathcal{O}_X$-submodule by Proposition 1.4.16.

By Corollary 5.2.23 any regular meromorphic connection on $\overline{X}^{\mathrm{an}}$ along $D^{\mathrm{an}}$ is effective. Namely, the underlying $\mathcal{O}_{\overline{X}^{\mathrm{an}}}[D^{\mathrm{an}}]$-module of an object of $\mathrm{Conn}^{\mathrm{reg}}(\overline{X}^{\mathrm{an}}; D^{\mathrm{an}})$ belongs to $\mathrm{Mod}^e_c(\mathcal{O}_{\overline{X}^{\mathrm{an}}}[D^{\mathrm{an}}])$. We denote by $\mathrm{Conn}^e(\overline{X}^{\mathrm{an}}; D^{\mathrm{an}})$ the full subcategory of effective meromorphic connections on $\overline{X}^{\mathrm{an}}$ along $D^{\mathrm{an}}$. By Proposition 5.3.6 the equivalence of (5.3.1) follows from the equivalence of

$$\mathrm{Conn}(\overline{X}; D) \longrightarrow \mathrm{Conn}^e(\overline{X}^{\mathrm{an}}; D^{\mathrm{an}}). \tag{5.3.2}$$

Let us show the equivalence of (5.3.2). We first show the following.

**Lemma 5.3.11.** *The functor*

$$\mathrm{Mod}_c(\mathcal{O}_{\overline{X}}[D]) \longrightarrow \mathrm{Mod}_c^e(\mathcal{O}_{\overline{X}^{\mathrm{an}}}[D^{\mathrm{an}}])$$

*given by* $M \mapsto M^{\mathrm{an}}$ *gives an equivalence of categories.*

*Proof.* We first show that the functor $M \mapsto M^{\mathrm{an}}$ is essentially surjective. Let $\widetilde{M} \in \mathrm{Mod}_c^e(\mathcal{O}_{\overline{X}^{\mathrm{an}}}[D^{\mathrm{an}}])$. Take a coherent $\mathcal{O}_{\overline{X}^{\mathrm{an}}}$-submodule $\widetilde{L}$ of $\widetilde{M}$ generating $\widetilde{M}$. Then we have $\widetilde{M} \simeq \mathcal{O}_{\overline{X}^{\mathrm{an}}}[D^{\mathrm{an}}] \otimes_{\mathcal{O}_{\overline{X}^{\mathrm{an}}}} \widetilde{L}$. By the GAGA principle [Ser1] there exists a coherent $\mathcal{O}_{\overline{X}}$-module $L$ such that $\mathcal{O}_{\overline{X}^{\mathrm{an}}} \otimes_{\mathcal{O}_{\overline{X}}} L \simeq \widetilde{L}$. Set $M := \mathcal{O}_{\overline{X}}[D] \otimes_{\mathcal{O}_{\overline{X}}} L \in \mathrm{Mod}_c(\mathcal{O}_{\overline{X}}[D])$. Then we have

$$M^{\mathrm{an}} = \mathcal{O}_{\overline{X}^{\mathrm{an}}} \otimes_{\mathcal{O}_{\overline{X}}} \mathcal{O}_{\overline{X}}[D] \otimes_{\mathcal{O}_{\overline{X}}} L \simeq \mathcal{O}_{\overline{X}^{\mathrm{an}}}[D^{\mathrm{an}}] \otimes_{\mathcal{O}_{\overline{X}^{\mathrm{an}}}} \widetilde{L} \simeq \widetilde{M}.$$

Let us show that the functor $M \mapsto M^{\mathrm{an}}$ is fully faithful. Namely, we prove that the canonical morphism

$$\mathrm{Hom}_{\mathcal{O}_{\overline{X}}[D]}(M, N) \longrightarrow \mathrm{Hom}_{\mathcal{O}_{\overline{X}^{\mathrm{an}}}[D^{\mathrm{an}}]}(M^{\mathrm{an}}, N^{\mathrm{an}}).$$

is an isomorphism for any $M, N \in \mathrm{Mod}_c(\mathcal{O}_{\overline{X}}[D])$. Let us take a coherent $\mathcal{O}_{\overline{X}}$-submodule $M_0 \subset M$ such that $\mathcal{O}_{\overline{X}}[D] \otimes_{\mathcal{O}_{\overline{X}}} M_0 \simeq M$. Then we obtain

$$\begin{cases} \mathrm{Hom}_{\mathcal{O}_{\overline{X}}[D]}(M, N) \simeq \mathrm{Hom}_{\mathcal{O}_{\overline{X}}}(M_0, N) \\ \mathrm{Hom}_{\mathcal{O}_{\overline{X}^{\mathrm{an}}}[D^{\mathrm{an}}]}(M^{\mathrm{an}}, N^{\mathrm{an}}) \simeq \mathrm{Hom}_{\mathcal{O}_{\overline{X}^{\mathrm{an}}}}(M_0^{\mathrm{an}}, N^{\mathrm{an}}). \end{cases}$$

Furthermore, $N$ being a union $\bigcup_{i\in I} N_i$ of coherent $\mathcal{O}_{\overline{X}}$-submodules $N_i \subset N$, we have

$$\begin{aligned} \mathrm{Hom}_{\mathcal{O}_{\overline{X}}}(M_0, N) &\simeq \bigcup_{i\in I} \mathrm{Hom}_{\mathcal{O}_{\overline{X}}}(M_0, N_i) \\ &\simeq \bigcup_{i\in I} \mathrm{Hom}_{\mathcal{O}_{\overline{X}^{\mathrm{an}}}}(M_0^{\mathrm{an}}, N_i^{\mathrm{an}}) \qquad \text{(by GAGA)} \\ &\simeq \mathrm{Hom}_{\mathcal{O}_{\overline{X}^{\mathrm{an}}}}(M_0^{\mathrm{an}}, N^{\mathrm{an}}). \end{aligned}$$

This completes the proof □

Note that $\mathrm{Conn}(\overline{X}; D)$ consists of pairs $(M, \nabla)$ of $M \in \mathrm{Mod}_c(\mathcal{O}_{\overline{X}}[D])$ and $\nabla \in \mathrm{Hom}_{\mathbb{C}}(M, \Omega^1_{\overline{X}} \otimes_{\mathcal{O}_{\overline{X}}} M)$ satisfying

$$\nabla(\varphi s) = d\varphi \otimes s + \varphi \nabla s \qquad (\varphi \in \mathcal{O}_{\overline{X}},\ s \in M), \tag{5.3.3}$$

$$[\nabla_\theta,\ \nabla_{\theta'}] = \nabla_{[\theta,\theta']} \qquad (\theta,\ \theta' \in \Theta_{\overline{X}}). \tag{5.3.4}$$

In view of Lemma 5.3.11 $\mathrm{Conn}^e(\overline{X}^{\mathrm{an}}; D^{\mathrm{an}})$ is equivalent to the category consisting of pairs $(M, \widetilde{\nabla})$ of $M \in \mathrm{Mod}_c(\mathcal{O}_{\overline{X}}[D])$ and $\widetilde{\nabla} \in \mathrm{Hom}_{\mathbb{C}}(M^{\mathrm{an}}, \Omega^1_{\overline{X}^{\mathrm{an}}} \otimes_{\mathcal{O}_{\overline{X}^{\mathrm{an}}}} M^{\mathrm{an}})$ satisfying

$$\widetilde{\nabla}(\varphi s) = d\varphi \otimes s + \varphi \widetilde{\nabla} s \qquad (\varphi \in \mathcal{O}_{\overline{X}^{\mathrm{an}}},\ s \in M^{\mathrm{an}}), \tag{5.3.5}$$

$$[\widetilde{\nabla}_\theta,\ \widetilde{\nabla}_{\theta'}] = \widetilde{\nabla}_{[\theta,\theta']} \qquad (\theta,\ \theta' \in \Theta_{\overline{X}^{\mathrm{an}}}). \tag{5.3.6}$$

Hence it is sufficient to show that for $M \in \mathrm{Mod}(\mathcal{O}_{\overline{X}}[D])$ the two sets

$$\Lambda = \{\nabla \in \mathrm{Hom}_{\mathbb{C}}(M, \Omega^1_{\overline{X}} \otimes_{\mathcal{O}_{\overline{X}}} M) \mid \nabla \text{ satisfies (5.3.3) and (5.3.4)}\},$$

$$\widetilde{\Lambda} = \{\widetilde{\nabla} \in \mathrm{Hom}_{\mathbb{C}}(M^{\mathrm{an}}, \Omega^1_{\overline{X}^{\mathrm{an}}} \otimes_{\mathcal{O}_{\overline{X}^{\mathrm{an}}}} M^{\mathrm{an}}) \mid \widetilde{\nabla} \text{ satisfies (5.3.5) and (5.3.6)}\}$$

are in bijective correspondence. Since these two sets are defined by $\mathbb{C}$-linear morphisms (not by $\mathcal{O}$-linear morphisms), we cannot directly use GAGA. We will show the correspondence by rewriting the conditions in terms of $\mathcal{O}$-linear morphisms.

We first show that the two sets

$$\Lambda_1 = \{\nabla \in \mathrm{Hom}_{\mathbb{C}}(M, \Omega^1_{\overline{X}} \otimes_{\mathcal{O}_{\overline{X}}} M) \mid \nabla \text{ satisfies (5.3.3)}\},$$

$$\widetilde{\Lambda}_1 = \{\widetilde{\nabla} \in \mathrm{Hom}_{\mathbb{C}}(M^{\mathrm{an}}, \Omega^1_{\overline{X}^{\mathrm{an}}} \otimes_{\mathcal{O}_{\overline{X}^{\mathrm{an}}}} M^{\mathrm{an}}) \mid \widetilde{\nabla} \text{ satisfies (5.3.5)}\}$$

are in bijective correspondence. We need some preliminaries on differential operators.

Let $Y$ be a complex manifold or a smooth algebraic variety. For $\mathcal{O}_Y$-modules $K$ and $L$ we define the subsheaves $F_pD(K, L)$ $(p \in \mathbb{Z})$ of $\mathcal{H}om_{\mathbb{C}}(M, N)$ recursively on $p$ by $F_pD(K, L) = 0$ for $p < 0$ and

$$F_pD(K, L) = \{P \in \mathcal{H}om_{\mathbb{C}}(M, N) \mid Pf - fP \in F_{p-1}D(K, L)\ (\forall f \in \mathcal{O}_Y)\}$$

for $p \geq 0$. Sections of $F_pD(K, L)$ are called differential operators of order $p$. Let us give a different description of $F_pD(K, L)$. Let $\delta : Y \to Y \times Y$ be the diagonal embedding, and let $p_j : Y \times Y \to Y$ $(j = 1, 2)$ be the projections. We denote by $J \subset \mathcal{O}_{Y\times Y}$ the defining ideal of $\delta(Y)$. By taking $\delta^{-1}$ of the canonical morphism $p_j^{-1}\mathcal{O}_Y \to \mathcal{O}_{Y\times Y}$ we obtain two ring homomorphisms

$$\alpha_j : \mathcal{O}_Y = \delta^{-1}p_j^{-1}\mathcal{O}_Y \to \delta^{-1}\mathcal{O}_{Y\times Y}$$

for $j = 1, 2$. In particular, we have two $\mathcal{O}_Y$-module structures on $\delta^{-1}\mathcal{O}_{Y\times Y}$. Since $J$ is an ideal of $\mathcal{O}_{Y\times Y}$, we also have two $\mathcal{O}_Y$-module structures on $\delta^{-1}J^k$ and $\delta^{-1}(J^k/J^l)$ $(k < l)$. We note that the two $\mathcal{O}_Y$-module structures on $\delta^{-1}(J^k/J^{k+1})$ coincide and that $\delta^{-1}(J/J^2)$ is identified with $\Omega^1_Y$. We consider $\mathcal{H}om_{\mathcal{O}_Y}(\delta^{-1}\mathcal{O}_{Y\times Y} \otimes_{\mathcal{O}_Y} K, L)$, where the tensor product $\delta^{-1}\mathcal{O}_{Y\times Y} \otimes_{\mathcal{O}_Y} K$ is taken with respect to the $\mathcal{O}_Y$-module structure on $\delta^{-1}\mathcal{O}_{Y\times Y}$ induced by $\alpha_2$, and $\delta^{-1}\mathcal{O}_{Y\times Y} \otimes_{\mathcal{O}_Y} K$ is regarded as an $\mathcal{O}_Y$-module via the $\mathcal{O}_Y$-module structure on $\delta^{-1}\mathcal{O}_{Y\times Y}$ induced by $\alpha_1$. Define

$$\beta : \mathcal{H}om_{\mathcal{O}_Y}(\delta^{-1}\mathcal{O}_{Y\times Y} \otimes_{\mathcal{O}_Y} K, L) \to \mathcal{H}om_{\mathbb{C}}(K, L)$$

by $(\beta(\psi))(s) = \psi(1 \otimes s)$. Then we have the following.

**Lemma 5.3.12.** *The morphism $\beta$ induces an isomorphism*

$$\mathcal{H}om_{\mathcal{O}_Y}(\delta^{-1}(\mathcal{O}_{Y\times Y}/J^{p+1}) \otimes_{\mathcal{O}_Y} K, L) \simeq F_pD(K, L).$$

The proof is left to the readers.

Now we return to the original situation. Note that we have

$$\Lambda_1 \subset F_1 D(M, \Omega^1_{\overline{X}} \otimes_{\mathcal{O}_{\overline{X}}} M), \qquad \widetilde{\Lambda}_1 \subset F_1 D(M^{\mathrm{an}}, \Omega^1_{\overline{X}^{\mathrm{an}}} \otimes_{\mathcal{O}_{\overline{X}^{\mathrm{an}}}} M^{\mathrm{an}}).$$

By examining the condition (5.3.3) we see by Lemma 5.3.12 that $\Lambda_1$ is in bijective correspondence with the set

$$\{\varphi \in \mathrm{Hom}_{\mathbb{C}}(\delta^{-1}(\mathcal{O}_{\overline{X}\times\overline{X}}/J^2) \otimes_{\mathcal{O}_{\overline{X}}} M, \Omega^1_{\overline{X}} \otimes_{\mathcal{O}_{\overline{X}}} M) \mid \varphi|_{\Omega^1_{\overline{X}}\otimes_{\mathcal{O}_{\overline{X}}} M} = \mathrm{id}\},$$

where $\delta^{-1}(J/J^2)$ is identified with $\Omega^1_{\overline{X}}$. We have obtained a description of $\Lambda_1$ in terms of $\mathcal{O}$-modules. The same argument holds true in the analytic category and we have a similar description of $\widetilde{\Lambda}_1$. Now we can apply GAGA to conclude that $\Lambda_1$ is in bijective correspondence with $\widetilde{\Lambda}_1$.

Let us finally give a reformulation of the conditions (5.3.4), (5.3.6). For $\nabla \in \mathrm{Hom}_{\mathbb{C}}(M, \Omega^1_{\overline{X}} \otimes_{\mathcal{O}_{\overline{X}}} M)$ satisfying (5.3.3) we define $\nabla^1 \in \mathrm{Hom}_{\mathbb{C}}(\Omega^1_{\overline{X}} \otimes_{\mathcal{O}_{\overline{X}}} M, \Omega^2_{\overline{X}} \otimes_{\mathcal{O}_{\overline{X}}} M)$ by $\nabla^1(\omega \otimes s) = d\omega \otimes s - \omega \wedge \nabla s$. Then we have $\nabla^1 \circ \nabla \in \mathrm{Hom}_{\mathcal{O}_{\overline{X}}}(M, \Omega^2_{\overline{X}} \otimes_{\mathcal{O}_{\overline{X}}} M)$, and $\nabla$ satisfies the condition (5.3.4) if and only if $\nabla^1 \circ \nabla = 0$. This gives a reformulation of the condition (5.3.4) in terms of an $\mathcal{O}_{\overline{X}}$-linear morphism. In the analytic category we also have a similar reformulation of the condition (5.3.6). Now we can apply GAGA to obtain the desired bijection $\Lambda \simeq \widetilde{\Lambda}$. The proof of Theorem 5.3.8 is now complete.

# 6

# Regular Holonomic $D$-Modules

In this chapter we give the definition of regular holonomic $D$-modules, and prove that the regular holonomicity is preserved by various functorial operations of $D$-modules that we introduced in earlier chapters.

## 6.1 Definition and main theorems

There are several mutually equivalent ways to define the regularity of holonomic $D$-modules. Here, following the notes of Bernstein [Ber3] we adopt a definition based on the classification theorem of simple holonomic $D$-modules (Theorem 3.4.2) and the regularity of integrable connections. One of the advantages of this definition is that it gives a concrete description of regular holonomic $D$-modules.

**Definition 6.1.1.** Let $X$ be a smooth algebraic variety. We say that a holonomic $D_X$-module $M$ is *regular* if any composition factor of $M$ is isomorphic to the minimal extension $L(Y, N)$ of some regular integrable connection $N$ on a locally closed smooth subvariety $Y$ of $X$ such that the inclusion $Y \to X$ is affine (see Theorem 3.4.2).

**Notation 6.1.2.**

(i) We denote by $\mathrm{Mod}_{rh}(D_X)$ the full subcategory of $\mathrm{Mod}_h(D_X)$ consisting of regular holonomic $D_X$-modules.

(ii) We denote by $D^b_{rh}(D_X)$ the full subcategory of $D^b_h(D_X)$ consisting of objects $M^{\cdot} \in D^b_h(D_X)$ such that $H^i(M^{\cdot}) \in \mathrm{Mod}_{rh}(D_X)$ for any $i \in \mathbb{Z}$.

It follows immediately from the above definition that the category $\mathrm{Mod}_{rh}(D_X)$ of regular holonomic $D_X$-modules is closed under the operations of taking submodules, quotient modules, and extensions inside $\mathrm{Mod}_{qc}(D_X)$. Consequently $\mathrm{Mod}_{rh}(D_X)$ is an abelian subcategory of $\mathrm{Mod}_h(D_X)$, and $D^b_{rh}(D_X)$ is a triangulated subcategory of $D^b_h(D_X)$.

**Remark 6.1.3.** By Proposition 5.3.4 an integrable connection is regular in the sense of Section 5.3 if and only if all of its composition factors are as well. Therefore, we have $\mathrm{Conn}^{\mathrm{reg}}(X) = \mathrm{Conn}(X) \cap \mathrm{Mod}_{rh}(D_X)$.

**Remark 6.1.4.** Let $C$ be a smooth curve. Then any simple holonomic $D_C$-module is of the form $L(Y, N)$ where $Y$ consists of a single point or is a (connected) non-empty open subset of $C$, and $N$ is an integrable connection on $Y$. If $Y$ consists of a single point, $L(Y, N)$ is always a regular integrable connection. Hence a holonomic $D_C$-module is regular in the sense of Definition 6.1.1 if and only if it is regular in the sense of Definition 5.1.21.

The main results in this chapter are the following two theorems.

**Theorem 6.1.5.** *Let $X$ be a smooth algebraic variety.*

(i) *The duality functor $\mathbb{D}_X$ preserves $D^b_{rh}(D_X)$.*
(ii) *Let $f : X \to Y$ be a morphism of smooth algebraic varieties. Then the functors $\int_f$, $\int_{f!}$ sends $D^b_{rh}(D_X)$ to $D^b_{rh}(D_Y)$, and the functors $f^\dagger$, $f^\bigstar$ sends $D^b_{rh}(D_Y)$ to $D^b_{rh}(D_X)$.*

**Theorem 6.1.6 (Curve testing criterion).** *Let $X$ be a smooth algebraic variety. The following conditions on $M^\cdot \in D^b_h(D_X)$ are equivalent:*

(i) $M^\cdot \in D^b_{rh}(D_X)$.
(ii) $i_C^\dagger M^\cdot \in D^b_{rh}(D_C)$ *for any locally closed embedding $i_C : C \hookrightarrow X$ of a smooth algebraic curve $C$.*
(iii) $k^\dagger M^\cdot \in D^b_{rh}(D_C)$ *for any morphism $k : C \to X$ from a smooth algebraic curve $C$.*

The proof of Theorem 6.1.5 and Theorem 6.1.6 will be given in the next section.

**Remark 6.1.7.** It is known that a holonomic $D_X$-module $M$ is regular if and only if $i_C^* M (= H^{\dim X - 1}(i_C^\dagger M)) \in \mathrm{Mod}_{rh}(D_C)$ for any $i_C : C \hookrightarrow X$ as in Theorem 6.1.6 (ii) (see Mebkhout [Me5, p. 163, Prop. 5.4.2]). Namely, we do not have to consider all cohomology sheaves of $i_C^\dagger M$.

In this book we do not go into detail about the theory of regular holonomic $D$-modules on complex manifolds. Here, we only give its definition and state some known facts without proofs.

**Definition 6.1.8.** Let $X$ be a complex manifold. We say that a holonomic $D_X$-module $M$ is *regular* if there locally exists a goof filtration $F$ of $M$ such that for any $P \in F^m D_X$ satisfying $\sigma_m(P)|_{\mathrm{Ch}(M)} = 0$ we have $P F^k M \subset F^{k+m-1} M$ for any $k$.

For a complex manifold $X$ we denote by $\mathrm{Mod}_{rh}(D_X)$ the full subcategory of $\mathrm{Mod}_h(D_X)$ consisting of regular holonomic $D_X$-modules. It is known that the category $\mathrm{Mod}_{rh}(D_X)$ is closed under the operations of taking submodules, quotient modules, and extensions inside the category $\mathrm{Mod}_c(D_X)$ of coherent $D_X$-modules. In particular, it is an abelian category. We denote by $D^b_{rh}(D_X)$ the full subcategory of $D^b_h(D_X)$ consisting of objects $M^\cdot \in D^b_h(D_X)$ such that $H^i(M^\cdot) \in \mathrm{Mod}_{rh}(D_X)$ for any $i \in \mathbb{Z}$. This is a triangulated subcategory of $D^b_h(D_X)$. The following results are proved by Kashiwara–Kawai [KK3].

**Theorem 6.1.9.** *Let $X$ be a complex manifold. Then any regular holonomic $D_X$-module admits a global good filtration.*

**Theorem 6.1.10.** *Let $X$ be a complex manifold. Then the duality functor $\mathbb{D}_X$ sends $D^b_{rh}(D_X)$ to $D^b_{rh}(D_X)^{\mathrm{op}}$.*

**Theorem 6.1.11.** *Let $f : X \to Y$ be a morphism of complex manifolds.*

(i) *The inverse image functor $f^\dagger$ sends $D^b_{rh}(D_Y)$ to $D^b_{rh}(D_X)$.*
(ii) *If $f$ is proper, the direct image functor $\int_f$ sends $D^b_{rh}(D_X)$ to $D^b_{rh}(D_Y)$.*

One can show the following using a result in Kashiwara–Kawai [KK3].

**Theorem 6.1.12.** *Let $X$ be a smooth algebraic variety, and let $M$ be a holonomic $D_X$-module. Take a divisor completion $j : X \hookrightarrow \overline{X}$ of $X$. Then $M$ is regular if and only if $(j_*M)^{\mathrm{an}}$ is regular.*

## 6.2 Proof of main theorems

In this section we give a proof of Theorem 6.1.5 and Theorem 6.1.6. It is divided into several steps.

**(Step 1)** We show that the conditions (ii) and (iii) in Theorem 6.1.6 are equivalent.

We only need to show that (ii) implies (iii). Assume (ii). Let $k : C \to X$ be a morphism from a smooth algebraic curve $C$. We may assume that $C$ is connected. If $\mathrm{Im}(k)$ consists of a single point $p$, then we have $k^\dagger M^\cdot \simeq \mathcal{O}_C \otimes_{\mathbb{C}} r^\dagger M^\cdot[1]$, where $r : \{p\} \hookrightarrow X$ is the inclusion. Hence the assertion is obvious. Assume that $\dim \overline{\mathrm{Im}(k)} = 1$. Take a non-empty smooth open subset $C'$ of $\overline{\mathrm{Im}(k)}$ and denote by $k_0 : C_0 := k^{-1}C' \to C'$ the canonical morphism. Then we have $k^\dagger M^\cdot|_{C_0} = k_0^\dagger i_{C'}^\dagger M^\cdot \in D^b_{rh}(D_{C_0})$ by (ii) and Lemma 5.1.23. Hence we obtain $k^\dagger M^\cdot \in D^b_{rh}(D_C)$ by Remark 6.1.4.

**(Step 2)** Let us show Theorem 6.1.5 (i).

We need to show $\mathbb{D}_X M^\cdot \in D^b_{rh}(D_X)$ for $M^\cdot \in D^b_{rh}(D_X)$. By induction on the cohomological length of $M^\cdot$ we may assume that $M^\cdot = M \in \mathrm{Mod}_{rh}(D_X)$. We can also reduce the problem to the case when $M$ is simple by induction on the length of the composition series of $M$. In this case $M$ is the minimal extension $L(Y,\ N)$ of a regular integrable connection $N$ on a locally closed smooth subvariety $Y$ of $X$ such that the inclusion $i : Y \to X$ is affine. By Proposition 3.4.3 and Example 2.6.10 we have

$$\mathbb{D}_X L(Y,\ N) \simeq L(Y,\ \mathbb{D}_Y N), \qquad \mathbb{D}_Y N \simeq \mathcal{H}om_{\mathcal{O}_Y}(N,\ \mathcal{O}_Y).$$

Hence the assertion is a consequence of Proposition 5.3.5. The proof of Theorem 6.1.5 (i) is complete.

It follows from **Step 2** that the proof of Theorem 6.1.5 (ii) is reduced to the following two statements for a morphism $f : X \to Y$ of smooth algebraic varieties:

(a) For any $M^{\cdot} \in D^b_{rh}(D_X)$ we have $\int_f M^{\cdot} \in D^b_{rh}(D_Y)$.
(b) For any $M^{\cdot} \in D^b_{rh}(D_Y)$ we have $f^{\dagger} M^{\cdot} \in D^b_{rh}(D_X)$.

We first verify (a) in some special cases in **Step 3** and **Step 4**.

**(Step 3)** Let $X$ be a smooth algebraic variety and let $\overline{X}$ be a smooth completion of $X$ such that the complementary set $D = \overline{X} \setminus X$ is a normal crossing divisor on $\overline{X}$. We show that (a) holds when $f$ is the embedding $j : X \hookrightarrow \overline{X}$ and $M^{\cdot} = M \in \mathrm{Conn}^{\mathrm{reg}}(X)$.

The functor $j_*$ being exact, we may assume that $M$ is simple by induction on the length of $M$. By Theorem 3.4.2 (ii), each composition factor $L$ of $j_*M$ is of the form $L = L(Y,\ N)$ ($Y \subset \overline{X}$ is affine and $N$ is an integrable connection on $Y$), and we need to show that $N$ is a regular integrable connection. If $j^{\dagger} L \neq 0$, then we have $N \simeq M$ by Theorem 3.4.2, and hence $N$ is regular in this case. Therefore, we assume that $j^{\dagger} L = 0$ in the following.

**Claim.** Let $D = \overline{X} \setminus X = \bigcup_{i=1}^{r} D_i$ be the irreducible decomposition of $D$. For each subset $I \subset \{1, 2, \ldots, r\}$ set $D_I := \bigcap_{i \in I} D_i$ and consider its irreducible decomposition $D_I = \bigcup_{\alpha} D_{I,\alpha}$. Also set $D'_{I,\alpha} := D_{I,\alpha} \setminus \bigcup_{j \notin I} D_j$ (an open subset of $D_{I,\alpha}$). Then for the above composition factor $L$ satisfying $j^{\dagger} L = 0$, there exist a pair $(I,\ \alpha)$ with $I \neq \emptyset$ and an integrable connection $N$ over $D'_{I,\alpha}$ such that $L \simeq L(D'_{I,\alpha},\ N)$.

*Proof of claim.* Theorem 5.3.7 implies that $j_*M$ is generated by a $\Theta_{\overline{X}}\langle D \rangle$-stable coherent $\mathcal{O}_{\overline{X}}$-submodule $K$. Define a good filtration $F$ of $j_*M$ by $F_p(j_*M) = (F_p D_{\overline{X}})K$ for $p \geq 0$. Then $\Theta_{\overline{X}}\langle D \rangle (\subset \Theta_{\overline{X}} = \mathrm{gr}^F_1 D_{\overline{X}})$ acts trivially on $\mathrm{gr}^F M$. From this we easily see that

$$\mathrm{Ch}(j_*M) \subset \bigcup_{I,\alpha} T^*_{D_{I,\alpha}} \overline{X}.$$

In other words, the characteristic variety $\mathrm{Ch}(j_*M)$ is a union of conormal bundles $T^*_{D_{I,\alpha}} \overline{X}$. Since $\mathrm{Ch}(L) \subset \mathrm{Ch}(j_*M)$, $L$ must satisfy the same condition. Therefore, the support of $L$ is a union of $D_{I,\alpha}$'s. Let us take an irreducible component $D_{I,\alpha}$ of the support of $L$. Let $i_{D'_{I,\alpha}} : D'_{I,\alpha} \to \overline{X}$ be the inclusion. By Kashiwara's equivalence and Lemma 2.3.5 we see that the characteristic variety of the pull-back $N := (i_{D'_{I,\alpha}})^{\dagger} L$ to $D'_{I,\alpha}$ coincides with the zero-section $T^*_{D'_{I,\alpha}} D'_{I,\alpha}$, and hence $N$ is a non-zero integrable connection by Proposition 2.2.5. By (the proof of) Theorem 3.4.2 (ii) we easily see from the simplicity of $L$ that $L$ coincides with the minimal extension $L(D'_{I,\alpha},\ N)$ of $N$. □

Now our task is to prove that the integrable connection $N$ in the claim above is regular. By Theorem 5.3.7 we have $j_*M = \bigcup_i W_i$ for coherent $\mathcal{O}_{\overline{X}}$-submodules $W_i$ of $j_*M$ stable under the action of $D_{\overline{X}}[D]$. Hence the composition factor $L$ of $j_*M$ also satisfies the same condition $L = \bigcup_i W'_i$. Note that the pair $(D'_{I,\alpha},\ D_{I,\alpha})$ is geometrically very similar to $(X,\ \overline{X})$, and that $\Delta_{I,\alpha} := D_{I,\alpha} \setminus D'_{I,\alpha}$ is a normal crossing

divisor on $D_{I,\alpha}$. Now recall that $\operatorname{supp} L = D_{I,\alpha}$. Therefore, by Kashiwara's equivalence for the inclusion $i_{D_{I,\alpha}} : D_{I,\alpha} \hookrightarrow \overline{X}$, the complex $(i_{D_{I,\alpha}})^\dagger L$ is concentrated in degree 0.

Let us show that $(i_{D_{I,\alpha}})^\dagger L$ also satisfies the same property as $L$ itself, that is, $(i_{D_{I,\alpha}})^\dagger L$ is a union of coherent $\mathcal{O}_{D_{I,\alpha}}$-submodules stable under the action of $D_{D_{I,\alpha}}[\Delta_{I,\alpha}]$. First consider the case where $\overline{X} \supset D = \{x_1 \cdots x_r = 0\} \supset D_I = \{x_1 = 0\}$. Then we have $H^0(i_{D_I})^\dagger L = \operatorname{Ker}(x_1 : L \to L)$ and the coherent $\mathcal{O}_{\overline{X}}$-submodule $W_i'$ stable under the actions of $\theta_i = x_i\partial_i$ $(1 \le i \le r)$ induces a coherent $\mathcal{O}_{D_I}$-coherent submodule $V_i = W_i' \cap \operatorname{Ker}(x_1 : L \to L)$ of $(i_{D_I})^\dagger L$ stable under the actions of $\theta_i$ $(2 \le i \le r)$. We also have $(i_{D_I})^\dagger L = \bigcup_i V_i$. In the general case where the codimension of $D_{I,\alpha}$ is greater than one we can repeat this procedure.

Now let $h = 0$ be the defining equation of the divisor $\Delta_{I,\alpha}$ on $D_{I,\alpha}$. Then we see that

$$(i_{D'_{I,\alpha} \hookrightarrow D_{I,\alpha}})_* N = \mathcal{O}_{D_{I,\alpha}}[\Delta_{I,\alpha}] \otimes_{\mathcal{O}_{D_{I,\alpha}}} (i_{D_{I,\alpha}})^\dagger L$$

also has the same property $\bigcup_{k \ge 0,\ i} [\mathcal{O}_{D_{I,\alpha}} h^{-k} \otimes V_i] = \mathcal{O}_{D_{I,\alpha}}[\Delta_{I,\alpha}] \otimes_{\mathcal{O}_{D_{I,\alpha}}} (i_{D_{I,\alpha}})^\dagger L$. Therefore, Theorem 5.3.7 implies that $N$ is a regular connection on $D'_{I,\alpha}$.

**(Step 4)** For an algebraic variety $Y$ we denote the singular locus of $Y$ by $\operatorname{Sing} Y$. We show the following.

**Claim.** Let $S$ (resp. $C$) be a smooth projective algebraic variety of dimension 2 (resp. 1), and let $S_0 \subset S$ (resp. $C_0 \subset C$) be a non-empty open subset. Set $\Delta_S := S \setminus S_0$ and $\Delta_C := C \setminus C_0$. We assume that $\Delta_S$ is a normal crossing divisor on $S$. Let $f : S \to C$ be a morphism satisfying the following conditions:

(i) $f(S_0) \subset C_0$ and $f|_{S_0} : S_0 \to C_0$ is smooth.
(ii) $f(\operatorname{Sing} \Delta_S) \subset \Delta_C$.

We denote by $i : S_0 \hookrightarrow S$ the embedding. Then for any regular integrable connection $M$ on $S_0$ we have $\int_f i_* M \in D^b_{rh}(D_C)$.

This result is due to Griffiths [Gri] and it was later generalized to higher dimensions by Deligne. We call this fact the regularity of *Gauss–Manin connections*.

*Proof of claim.* Recall the notation $D_S\langle\Delta_S\rangle$, $D_C\langle\Delta_C\rangle$, etc., in Section 5.3. Similarly to $D_{S\to C}$, we define a $(D_S\langle\Delta_S\rangle, f^{-1}D_C\langle\Delta_C\rangle)$-bimodule $D_{S\to C}\langle\Delta\rangle$ by

$$D_{S\to C}\langle\Delta\rangle := \mathcal{O}_S \otimes_{f^{-1}\mathcal{O}_C} f^{-1}D_C\langle\Delta_C\rangle.$$

For the sake of simplicity we consider the corresponding equivalent problem in the category of "right" $D$-modules. Recall that the direct image under $f$ of a complex $M^\cdot \in D^b(D_S^{\mathrm{op}})$ of right $D_S$-modules is given by

$$\int_f M^\cdot := Rf_*(M^\cdot \otimes^L_{D_S} D_{S\to C}).$$

We define a functor

$$\int_f^{\langle\Delta\rangle} : D^b(\mathrm{Mod}(D_S\langle\Delta_S\rangle^{\mathrm{op}})) \longrightarrow D^b(\mathrm{Mod}(D_C\langle\Delta_C\rangle^{\mathrm{op}}))$$

by

$$\int_f^{\langle\Delta\rangle} M^{\cdot} := Rf_*(M^{\cdot} \otimes^L_{D_S\langle\Delta_S\rangle} D_{S\to C}\langle\Delta\rangle).$$

Then we have

(1) For $M \in \mathrm{Mod}(D_{S_0}^{\mathrm{op}})$ we have an isomorphism

$$\int_f i_*M \simeq \int_f^{\langle\Delta\rangle} i_*M$$

in $D^b(\mathrm{Mod}(D_C\langle\Delta_C\rangle^{\mathrm{op}}))$.

(2) Let $L$ be a right $D_S\langle\Delta_S\rangle$-module which is coherent over $\mathcal{O}_S$. Then all the cohomology sheaves of $\int_f^{\langle\Delta\rangle} L$ are right $D_C\langle\Delta_C\rangle$-modules which are coherent over $\mathcal{O}_C$.

The assertion (1) follows immediately from $D_S\langle\Delta_S\rangle|_{S_0} = D_{S_0}$ and $D_{S\to C}\langle\Delta\rangle|_{S_0} = D_{S_0\to C_0}$. Let us prove (2). By our conditions (i), (ii) on $f$ we see that the canonical morphism

$$\Theta_S\langle\Delta_S\rangle \longrightarrow f^*\Theta_C\langle\Delta_C\rangle = \mathcal{O}_S \otimes_{f^{-1}\mathcal{O}_C} f^{-1}\Theta_C\langle\Delta_C\rangle$$

is an epimorphism. Denote by $\Theta_{S/C}\langle\Delta\rangle$ its kernel. Then we have an exact sequence

$$0 \longrightarrow D_S\langle\Delta_S\rangle \otimes_{\mathcal{O}_S} \Theta_{S/C}\langle\Delta\rangle \longrightarrow D_S\langle\Delta_S\rangle \longrightarrow D_{S\to C}[\Delta] \longrightarrow 0$$

of $D_S\langle\Delta_S\rangle$-modules. We can regard this as a locally free resolution of $D_{S\to C}\langle\Delta\rangle$, and hence for a right $D_S\langle\Delta_S\rangle$-module $L$ we have

$$\int_f^{\langle\Delta\rangle} L = Rf_*\big[L \otimes_{\mathcal{O}_S} \Theta_{S/C}\langle\Delta\rangle \longrightarrow L\big].$$

Since $f$ is proper, the cohomology sheaves $H^*(\int_f^{\langle\Delta\rangle} L)$ are coherent over $\mathcal{O}_C$ if $L$ is coherent over $\mathcal{O}_S$. This completes the proof of (2).

Now we can finish the proof of our claim. Since $M$ is a regular integrable connection on $S_0$, we see by Theorem 5.3.7 that there exist $\mathcal{O}_S$-coherent $D_S\langle\Delta_S\rangle$-submodules $L_\alpha \subset i_*M$ such that $i_*M = \varinjlim_\alpha L_\alpha$. By (1) there exist isomorphisms

$$H^*\int_f i_*M \simeq H^*\int_f^{\langle\Delta\rangle} i_*M \simeq H^*\int_f^{\langle\Delta\rangle} \varinjlim_\alpha L_\alpha \simeq \varinjlim_\alpha H^*\int_f^{\langle\Delta\rangle} L_\alpha$$

of $D_C\langle\Delta_C\rangle$-modules. By (2) $H^*(\int_f^{\langle\Delta\rangle} L_\alpha)$ is coherent over $\mathcal{O}_C$, and hence we have $\int_f i_*M \in D^b_{rh}(D_C)$ by Theorem 5.3.7. The proof of our claim is complete. □

In **Step 5**–**Step 8** we will show the statement (a) and the equivalence of (i) and (ii) in Theorem 6.1.6 simultaneously by induction on $\dim \mathrm{supp}(M^{\cdot})$, where $\mathrm{supp}(M^{\cdot}) := \bigcup_j \mathrm{supp}(H^j(M^{\cdot}))$.

**(Step 5)** We show that the statement (a) holds when $f$ is an affine embedding $i : X \hookrightarrow Y$ and $M = M^{\cdot} \in \mathrm{Conn}^{\mathrm{reg}}(X)$.

Let us consider the distinguished triangle

$$\int_{i!} M \longrightarrow \int_{i} M \longrightarrow C_i(M) \overset{+1}{\longrightarrow}$$

in $D^b_h(D_Y)$ associated to the morphism $\int_{i!} \to \int_i$ (see Section 3.4). Since $i$ is an affine embedding, we have $H^p \int_i M = 0\,(p \neq 0)$ and $H^p \int_{i!} M = 0\,(p \neq 0)$. Therefore, by considering the cohomology long exact sequence associated to the above distinguished triangle we see that any composition factor of $H^0(\int_{i!} M)$ or $H^0(\int_i M)$ is isomorphic to that of $H^* C_i(M)$ or to that of the minimal extension $L(X,\ M) = \mathrm{Im}\big(H^0(\int_{i!} M) \to H^0(\int_i M)\big)$. Since $L(X,\ M)$ is regular by definition, we have only to show that $C_i(M) \in D^b_{rh}(D_Y)$.

By a theorem of Hironaka we can decompose the morphism $i$ into the composite of an open immersion $j : X \to \overline{X}$ and a proper morphism $f : \overline{X} \to Y$, where $\overline{X} \setminus X$ is a normal crossing divisor on $\overline{X}$. By applying the functor $\int_f = \int_{f!}$ to the distinguished triangle

$$\int_{j!} M \longrightarrow \int_{j} M \longrightarrow C_j(M) \overset{+1}{\longrightarrow}$$

associated to the open embedding $j$ we obtain a distinguished triangle

$$\int_{i!} M \longrightarrow \int_{i} M \longrightarrow \int_f C_j(M) \overset{+1}{\longrightarrow}.$$

Hence we have $\int_f C_j(M) \simeq C_i(M)$. On the other hand we know already that $\int_j M,\ \int_{j!} M \in D^b_{rh}(D_{\overline{X}})$ by **Step 2** and **Step 3**. Hence it follows from the above distinguished triangle that $C_j(M) \in D^b_{rh}(D_{\overline{X}})$. Moreover, by the construction in Theorem 3.4.2 we have

$$\dim \mathrm{supp}\, C_j(M) < \dim \mathrm{supp}\, M.$$

Therefore, we obtain $C_i(M) \simeq \int_f C_j(M) \in D^b_{rh}(D_Y)$ by our hypothesis of induction.

**(Step 6)** We give a proof of (i)$\Longrightarrow$(ii) in Theorem 6.1.6, i.e., we show $i_C^{\dagger} M^{\cdot} \in D^b_{rh}(D_C)$ for any $M^{\cdot} \in D^b_{rh}(D_X)$ and any embedding $i_C : C \hookrightarrow X$ of an algebraic curve.

By induction on the cohomological length of $M^{\cdot}$ we may assume $M^{\cdot} = M \in \mathrm{Mod}_{rh}(D_X)$. We may also assume that $M$ is a simple object of $\mathrm{Mod}_{rh}(D_X)$ by induction on the length of $M$. In this case $M$ is a minimal extension $M = L(Y,\ N)$ ($j : Y \hookrightarrow X$: affine) of a simple regular integrable connection $N$ on $Y$. Set $Q = (\int_j N)/M$. By Theorem 3.4.2 we have $\dim \mathrm{supp}\, Q < \dim \mathrm{supp}\, M$. Moreover, we have $Q \in D^b_{rh}(D_X)$ by **Step 5**. Therefore, the hypothesis of induction implies that we have $i_C^{\dagger} Q \in D^b_{rh}(D_C)$. Hence by the distinguished triangle

$$i_C^\dagger M \longrightarrow i_C^\dagger \int_j N \longrightarrow i_C^\dagger Q \xrightarrow{+1}$$

we have only to show $i_C^\dagger \int_j N \in D^b_{rh}(D_C)$. By applying the base change theorem (Theorem 1.7.3) to the cartesian square

$$\begin{array}{ccc} Y \cap C & \xrightarrow{i_0} & Y \\ {\scriptstyle j_0}\downarrow & & \downarrow{\scriptstyle j} \\ C & \xrightarrow{i_C} & X, \end{array}$$

we get $i_C^\dagger \int_j N \simeq \int_{j_0} i_0^\dagger N$ (note that $Y \cap C$ is smooth since $C$ is one-dimensional). Hence it is sufficient to show that $\int_{j_0} i_0^\dagger N \in D^b_{rh}(D_C)$. In the case $\dim Y \cap C = 0$ this is obvious. If $\dim Y \cap C = 1$, then $Y \cap C$ is an open subset of $C$. In this case $i_0^\dagger N$ is a regular integrable connection on $Y \cap C$ up to a shift of degrees by definition. Hence by the definition of regular holonomic $D$-modules on algebraic curves, we have $\int_{j_0} i_0^\dagger N \in D^b_{rh}(D_C)$.

**(Step 7)** We give a proof of (ii)$\Longrightarrow$(i) in Theorem 6.1.6.

Let $M^\cdot \in D^b_h(D_X)$. We assume that $i_C^\dagger M^\cdot \in D^b_{rh}(D_C)$ for any embedding $i_C : C \hookrightarrow X$ of an algebraic curve, and will show that $M^\cdot \in D^b_{rh}(D_X)$. Set $S = \operatorname{supp} M^\cdot$. By Proposition 3.1.6 there exists an open dense subset $Y$ of $S$ such that $H^*(i^\dagger M^\cdot)$ are integrable connections on $Y$ ($i : Y \hookrightarrow X$). We may assume that $i : Y \hookrightarrow X$ is an affine embedding. Take an open dense subset $U$ of $X$ such that $S \cap U = Y$ and set $Z = X \setminus U$. We denote by $k : Y \to U$ and $j : U \to X$ the embeddings. Note that $i = j \circ k$ and that $j$ (resp. $k$) is an open (resp. a closed) embedding. By the distinguished triangle

$$R\Gamma_Z(M^\cdot) \longrightarrow M^\cdot \longrightarrow \int_j j^\dagger M^\cdot \xrightarrow{+1}$$

in $D^b_h(D_X)$ associated to the triplet $U \hookrightarrow X \hookleftarrow Z$ it is sufficient to show that $R\Gamma_Z(M^\cdot)$, $\int_j j^\dagger M^\cdot \in D^b_{rh}(D_X)$.

We first show $R\Gamma_Z(M^\cdot) \in D^b_{rh}(D_X)$. By $\operatorname{supp} R\Gamma_Z(M^\cdot) \subset S \cap Z$ we have $\dim \operatorname{supp} R\Gamma_Z(M^\cdot) < \dim \operatorname{supp} M^\cdot$. Hence we have only to show to $i_C^\dagger R\Gamma_Z(M^\cdot) \in D^b_{rh}(D_C)$ for any curve $C$ by the hypothesis of induction. Applying the functor $i_C^\dagger$ to the above distinguished triangle we obtain a distinguished triangle

$$i_C^\dagger R\Gamma_Z(M^\cdot) \longrightarrow i_C^\dagger M^\cdot \longrightarrow i_C^\dagger \int_j j^\dagger M^\cdot \xrightarrow{+1}$$

Hence it is sufficient to show $i_C^\dagger \int_j j^\dagger M^\cdot \in D^b_{rh}(D_X)$. By applying the base change theorem to the cartesian square

$$\begin{array}{ccc} U & \xrightarrow{j} & X \\ \uparrow & & \uparrow{\scriptstyle i_C} \\ U \cap C & \xrightarrow{\alpha} & C \end{array}$$

we obtain $i_C^\dagger \int_j j^\dagger M^\cdot \simeq \int_\alpha \alpha^\dagger i_C^\dagger M^\cdot$, and hence we have only to show that $\int_\alpha \alpha^\dagger i_C^\dagger M^\cdot \in D^b_{rh}(D_C)$. This is obvious if $\dim U \cap C = 0$. Assume that $\dim U \cap C = 1$. By our assumption we have $i_C^\dagger M^\cdot \in D^b_{rh}(D_C)$, and hence we obtain $\int_\alpha \alpha^\dagger i_C^\dagger M^\cdot \in D^b_{rh}(D_C)$ by the definition of the regularity for holonomic $D$-modules on algebraic curves.

Next let us show $\int_j j^\dagger M^\cdot \in D^b_{rh}(D_X)$. By applying Kashiwara's equivalence (Theorem 1.6.2) to the closed embedding $k : Y \hookrightarrow U$ we easily see that $\int_j j^\dagger M^\cdot \simeq \int_i i^\dagger M^\cdot$. Recall that $H^*(i^\dagger M^\cdot)$ are integrable connections on $Y$. Since $i$ is affine, it is sufficient to show $H^*(i^\dagger M^\cdot) \in \mathrm{Conn}^{\mathrm{reg}}(Y)$ by **Step 5**. For this it is sufficient to show $j_C^\dagger H^*(i^\dagger M^\cdot) \in D^b_{rh}(D_C)$ for any locally closed embedding $j_C : C \to Y$ of a curve $C$ into $Y$. Since $H^*(i^\dagger M^\cdot)$ are integrable connections, we have $j_C^\dagger H^*(i^\dagger M^\cdot) \simeq H^{*-1+d_Y}(j_C^\dagger i^\dagger M^\cdot)[1-d_Y] \simeq H^{*-1+d_Y}((i \circ j_C)^\dagger M^\cdot)[1-d_Y]$. Hence it belongs to $D^b_{rh}(D_C)$ by our assumption on $M^\cdot$.

**(Step 8)** We give a proof of (a) for general $f$.

It is enough to prove the assertion separately for the case of closed embeddings, and for the case of projections.

We first treat the case when $f : X \to Y$ is a closed embedding. As before we may assume that $M^\cdot = M = L(X_1, N)$ for some locally closed smooth subvariety $X_1$ of $X$ such that the embedding $i : X_1 \to X$ is affine and $N \in \mathrm{Conn}^{\mathrm{reg}}(X_1)$. Since the dimension of the support of the cokernel of the morphism $L(X_1, N) \hookrightarrow \int_i N$ is less than that of $M^\cdot = L(X_1, N)$, it is enough to prove $\int_{f\circ i} N \in D^b_{rh}(D_Y)$ by our hypothesis of induction. This assertion has already been proved in **Step 5** ($f \circ i$ is an affine embedding).

Next we deal with the case of a projection $f : X = Z \times Y \to Y$ and $M^\cdot \in D^b_{rh}(D_X)$. By the argument in the proof of Lemma 3.2.5 we may assume that $Z$ is affine from the beginning. Let us choose a closed embedding $j : Z \to \mathbb{A}^N$ and factorize $f$ as the composite of $j \times \mathrm{Id}_Y : X = Z \times Y \to \mathbb{A}^N \times Y$ and a projection $\mathbb{A}^N \times Y \to Y$. Since the morphism $j \times \mathrm{Id}_Y$ is a closed embedding, it is enough to prove our assertion for $p : \mathbb{A}^N \times Y \to Y$. Moreover, decomposing $\mathbb{A}^N \times Y \to Y$ into $\mathbb{A}^N \times Y \to \mathbb{A}^{N-1} \times Y \to \cdots \to \mathbb{A}^1 \times Y \to Y$, we can further reduce the problem to the case of projections $\mathbb{A}^1 \times Y \to Y$ (with fiber $\mathbb{A}^1$). By **Step 6**, **Step 7**, and Theorem 1.7.3 we may also assume that $Y$ is an algebraic curve $C$. Namely, for $p : \mathbb{A}^1 \times C \to C$ and $M^\cdot \in D^b_{rh}(D_{\mathbb{A}^1 \times C})$, we have only to prove $\int_p M^\cdot \in D^b_{rh}(D_C)$. As before, we may assume that $M^\cdot = M = L(X_1, N)$ for some locally closed smooth subvariety $X_1$ of $\mathbb{A}^1 \times C$ such that $i : X_1 \to \mathbb{A}^1 \times C$ is affine and $N \in \mathrm{Conn}^{\mathrm{reg}}(X_1)$. By $\dim \mathrm{supp}\,\mathrm{Coker}(L(X_1, N) \to \int_i N) < \dim \mathrm{supp}(L(X_1, N)$ and the hypothesis of induction it is sufficient to show $\int_{p\circ i} N \in D^b_{rh}(D_C)$. The case $\dim X_1 \leq 1$ is

already known (see Lemma 5.1.23) and hence we may assume that $X_1$ is an open subset of $\mathbb{A}^1 \times C$. Namely, it is sufficient to show following statement.

**Claim.** Let $U$ be a non-empty open subset of $\mathbb{A}^1 \times C$. We denote by $i : U \to \mathbb{A}^1 \times C$ and $p : \mathbb{A}^1 \times C \to C$ the embedding and the projection, respectively. Then for any regular integrable connection $N$ on $U$ we have $\int_{p \circ i} N \in D^b_{rh}(D_C)$.

*Proof of claim.* We first note that we can always replace $C$ and $U$ with a non-empty open dense subset $C'$ of $C$ and $U' = p^{-1}C' \cap U$, respectively, by the definition of regularity for holonomic $D$-modules on algebraic curves. Take a smooth completion $j : C \to \overline{C}$ of $C$, and regard $\mathbb{A}^1 \times C$ as an open subset of $\mathbb{P}^1 \times \overline{C}$. We denote by $\overline{p} : \mathbb{P}^1 \times \overline{C} \to \overline{C}$ the projection. Note that $(\mathbb{P}^1 \times \overline{C}) \setminus U$ is a union of a divisor and finitely many points. Hence by replacing $C$ with its non-empty open subset we may assume from the beginning that $\overline{p}(\mathrm{Sing}((\mathbb{P}^1 \times \overline{C}) \setminus U)) \subset \overline{C} \setminus C$. Then by Hironaka's desingularization theorem we can take a proper surjective morphism $\pi : S \to \mathbb{P}^1 \times \overline{C}$ such that the morphism $\pi_0 : \pi^{-1}(U) \to U$ induced by $\pi$ is an isomorphism, $\Delta_S := \pi^{-1}((\mathbb{P}^1 \times \overline{C}) \setminus U)$ is a normal crossing divisor on $S$, and $\pi(\mathrm{Sing}\, \Delta_S) \subset \mathrm{Sing}((\mathbb{P}^1 \times \overline{C}) \setminus U)$. Set $S_0 = \pi^{-1}U$, $f = \overline{p} \circ \pi : S \to \overline{C}$, and let $\overline{i} : S_0 \to S$ be the embedding. Then we have $f(\mathrm{Sing}\, \Delta = S) \subset \overline{C} \setminus C$, and

$$j_* \int_{p \circ i} N \simeq \int_f \overline{i}_* \pi_0^* N.$$

Hence our claim follows from **Step 4**. □

**(Step 9)** The statement (b) follows easily from Theorem 6.1.6.

The proof of Theorem 6.1.5 and Theorem 6.1.6 is now complete.

# 7

# Riemann–Hilbert Correspondence

This chapter is concerned with one of the most important theorems in $D$-module theory. This fundamental theorem, which is now called the Riemann–Hilbert correspondence, establishes an equivalence between the category of regular holonomic $D$-modules and that of perverse sheaves. It builds a bridge from analysis to topology leading us to a number of applications in various fields in mathematics.

## 7.1 Commutativity with de Rham functors

Recall that for a smooth algebraic variety $X$ we have the duality functors

$$\mathbb{D}_X : D^b_h(D_X) \to D^b_h(D_X)^{\mathrm{op}}, \qquad \mathbf{D}_X : D^b_c(X) \to D^b_c(X)^{\mathrm{op}}.$$

Recall also that for a morphism $f : X \to Y$ of smooth algebraic varieties we have the functors

$$\begin{aligned}
\int_f &: D^b_h(D_X) \longrightarrow D^b_h(D_Y), \quad Rf_* : D^b_c(X) \longrightarrow D^b_c(Y),\\
\int_{f!} &: D^b_h(D_X) \longrightarrow D^b_h(D_Y), \quad Rf_! : D^b_c(X) \longrightarrow D^b_c(Y),\\
f^\dagger &: D^b_h(D_Y) \longrightarrow D^b_h(D_X), \quad f^! : D^b_c(Y) \longrightarrow D^b_c(X),\\
f^\bigstar &: D^b_h(D_Y) \longrightarrow D^b_h(D_X), \quad f^{-1} : D^b_c(Y) \longrightarrow D^b_c(X).
\end{aligned}$$

By Theorem 6.1.5 all of the functors $\mathbb{D}_X$, $\int_f$, $\int_{f!}$, $f^\dagger$, $f^\bigstar$ preserve $D^b_{rh}$. We know already that

$$\mathbf{D}_X\, DR_X(M^{\cdot}) \simeq DR_X(\mathbb{D}_X M^{\cdot}) \qquad (M^{\cdot} \in D^b_h(D_X)) \tag{7.1.1}$$

(see Corollary 4.6.5 and Proposition 4.7.9), i.e., the de Rham functor on $D^b_h$ commutes with the duality functors. In this section we will also prove the commutativity of the de Rham functor on $D^b_{rh}$ with the inverse and direct image functors.

We first note that there exists a canonical morphism

$$DR_Y\Big(\int_f M^{\cdot}\Big) \longrightarrow Rf_*\, DR_X(M^{\cdot}) \qquad (M^{\cdot} \in D^b_h(D_X)) \tag{7.1.2}$$

by Proposition 4.7.5. Hence by (7.1.1) we also have a canonical morphism

$$Rf_!\, DR_X(M^{\cdot}) \longrightarrow DR_Y\Big(\int_{f!} M^{\cdot}\Big) \qquad (M^{\cdot} \in D^b_h(D_X)). \tag{7.1.3}$$

On the other hand we have a canonical morphism

$$DR_X(f^{\dagger}N^{\cdot}) \longrightarrow f^!\, DR_Y(N^{\cdot}) \qquad (N^{\cdot} \in D^b_h(D_Y)) \tag{7.1.4}$$

as the image of $\mathrm{Id}_{f^{\dagger}N^{\cdot}}$ under the composite of the morphisms

$$\begin{aligned}
\mathrm{Hom}_{D^b_h(D_X)}(f^{\dagger}N^{\cdot}, f^{\dagger}N^{\cdot}) &\simeq \mathrm{Hom}_{D^b_h(D_Y)}\Big(\int_{f!} f^{\dagger}N^{\cdot}, N^{\cdot}\Big)\\
&\to \mathrm{Hom}_{D^b_c(Y)}\left(DR_Y\Big(\int_{f!} f^{\dagger}N^{\cdot}\Big), DR_Y(N^{\cdot})\right)\\
&\to \mathrm{Hom}_{D^b_c(Y)}(Rf_!\, DR_X(f^{\dagger}N^{\cdot}), DR_Y(N^{\cdot}))\\
&\simeq \mathrm{Hom}_{D^b_c(X)}(DR_X(f^{\dagger}N^{\cdot}), f^!\, DR_Y(N^{\cdot})).
\end{aligned}$$

Here we have used the fact that $f^{\dagger}$ is right adjoint to $\int_{f!}$ (Corollary 3.2.15) and $f^!$ is right adjoint to $Rf_!$. Hence by (7.1.1) we also have a canonical morphism

$$f^{-1}\, DR_Y(N^{\cdot}) \longrightarrow DR_X(f^{\bigstar}N^{\cdot}) \qquad (N^{\cdot} \in D^b_h(D_Y)). \tag{7.1.5}$$

**Theorem 7.1.1.** *Let $f : X \to Y$ be a morphism of smooth algebraic varieties. Then the canonical morphisms* (7.1.2), (7.1.3), (7.1.4), (7.1.5) *are isomorphisms if $M^{\cdot} \in D^b_{rh}(D_X)$ and $N^{\cdot} \in D^b_{rh}(D_Y)$. Namely, we have the following isomorphisms of functors:*

$$\begin{aligned}
DR_Y \circ \int_f &\simeq Rf_* \circ DR_X : D^b_{rh}(D_X) \longrightarrow D^b_c(Y),\\
DR_Y \circ \int_{f!} &\simeq Rf_! \circ DR_X : D^b_{rh}(D_X) \longrightarrow D^b_c(Y),\\
DR_X \circ f^{\dagger} &\simeq f^! \circ DR_Y : D^b_{rh}(D_Y) \longrightarrow D^b_c(X),\\
DR_X \circ f^{\bigstar} &\simeq f^{-1} \circ DR_Y : D^b_{rh}(D_Y) \longrightarrow D^b_c(X).
\end{aligned}$$

*Proof.* Let us show that (7.1.2) is an isomorphism for $M^{\cdot} \in D^b_{rh}(D_X)$. Note that this is already verified if $f$ is projective (Proposition 4.7.5). We first deal with the case $M^{\cdot} = M \in \mathrm{Conn}^{\mathrm{reg}}(X)$. By a theorem of Hironaka $f$ can be factorized as $f = p \circ j$, where $j : X \hookrightarrow \bar{X}$ is an open embedding such that $\overline{X} \setminus X$ is a normal crossing divisor on $\overline{X}$ and $p : \bar{X} \to Y$ is projective. Hence we may assume that $f = j$. In this case

the assertion follows from Theorem 5.2.24 and Proposition 5.3.6. Now we treat the general case. We may assume $M^{\cdot} = M \in \mathrm{Mod}_{rh}(D_X)$. By induction on $\dim \mathrm{supp}\, M$ we may also assume that $M = \int_i L$, where $i : Z \to X$ is an affine embedding of a smooth locally closed subvariety $Z$ of $X$ and $L$ is a regular integrable connection on $Z$. Then we have

$$\begin{aligned} DR_Y \int_f M = DR_Y \int_f \int_i L &\simeq DR_Y \int_{f\circ i} L \simeq R(f\circ i)_* DR_Z\, L \\ &\simeq Rf_* Ri_* DR_Z\, L \simeq Rf_* DR_X \int_i L = Rf_* DR_X\, M. \end{aligned}$$

We have proved that (7.1.2) is an isomorphism for $M^{\cdot} \in D^b_{rh}(D_X)$.

Next we show that (7.1.4) is an isomorphism for $N^{\cdot} \in D^b_{rh}(D_Y)$. Note that this is already verified if $f$ is smooth (Corollary 4.3.3). By factorizing $f$ into a composite of the graph embedding $X \hookrightarrow X \times Y$ and the projection $p : X \times Y \to Y$ we have only to deal with the case where $f$ is a closed embedding $i : X \hookrightarrow Y$. Let $j : Y \setminus X \hookrightarrow Y$ be the corresponding open embedding. Then for $N^{\cdot} \in D^b_{rh}(D_Y)$ we have the following morphism of distinguished triangles:

$$\begin{array}{ccccccc} DR_Y \int_i i^{\dagger} N^{\cdot} & \longrightarrow & DR_Y N^{\cdot} & \longrightarrow & DR_Y \int_j j^{\dagger} N^{\cdot} & \xrightarrow{+1} & \\ \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \mathrm{Id}} & & \downarrow{\scriptstyle \varphi} & & \\ Ri_* i^{!} DR_Y N^{\cdot} & \longrightarrow & DR_Y N^{\cdot} & \longrightarrow & Rj_* j^{!} DR_Y N^{\cdot} & \xrightarrow{+1} & \end{array}$$

Since $j$ is smooth, we have $DR_{Y\setminus X}\, j^{\dagger} N^{\cdot} \simeq j^{!}\, DR_Y\, N^{\cdot}$. Hence by $j^{\dagger} N^{\cdot} \in D^b_{rh}(D_{Y\setminus X})$ we have

$$DR_Y \int_j j^{\dagger} N^{\cdot} \simeq Rj_*\, DR_{Y\setminus X}\, j^{\dagger} N^{\cdot} \simeq Rj_* j^{!} DR_Y N^{\cdot},$$

i.e., the morphism $\varphi$ is an isomorphism. Therefore, $\psi$ is also an isomorphism. By

$$DR_Y \int_i i^{\dagger} N^{\cdot} \xrightarrow{\sim} Ri_*\, DR_X\, i^{\dagger} N^{\cdot},$$

we obtain an isomorphism

$$Ri_*\, DR_X\, i^{\dagger} N^{\cdot} \simeq Ri_* i^{!}\, DR_Y\, N^{\cdot}.$$

Note that $i^{-1} Ri_* = \mathrm{Id}$ since $i$ is a closed embedding. Hence we obtain the desired result by applying $i^{-1}$ to the above isomorphism. We have shown that (7.1.4) is an isomorphism for $N^{\cdot} \in D^b_{rh}(D_Y)$.

The remaining assertions follow from the ones proved thus far in view of (7.1.1). □

**Remark 7.1.2.** As we saw above, the assumption of the regularity of $D$-modules is essential for the proof of the commutativity $DR_Y \int_j \simeq Rj_*\, DR_X$ (resp. $DR_X\, i^{\dagger} \simeq i^{!}\, DR_Y$) of $DR$ with the direct image (resp. the inverse image) for an open embedding $j$ (resp. for a closed embedding $i$). Using this property of regular holonomic $D$-modules, Mebkhout [Me6] defined certain objects in $D^b_c(\mathbb{C}_X)$ to measure the "irregularity" of holonomic $D$-modules (perverse sheaves called irregularity sheaves).

## 7.2 Riemann–Hilbert correspondence

Now we can state one of the most important results in the theory of $D$-modules, which is now called the *Riemann–Hilbert correspondence*. The following original version of this theorem for regular holonomic $D$-modules on complex manifolds was established by Kashiwara in [Kas6], [Kas10] (a different proof was also given later by Mebkhout [Me4]).

**Theorem 7.2.1.** *For a complex manifold $X$ the de Rham functor*

$$DR_X : D^b_{rh}(D_X) \longrightarrow D^b_c(X)$$

*gives an equivalence of categories.*

We do not give a proof of this result in this book.

After the appearance of the above original analytic version Beilinson–Bernstein developed systematically a theory of regular holonomic $D$-modules on smooth algebraic varieties and obtained an algebraic version of the Riemann–Hilbert correspondence stated below (see also Brylinski [Br], where the algebraic version is deduced from the analytic one).

**Theorem 7.2.2.** *For a smooth algebraic variety $X$ the de Rham functor*

$$DR_X : D^b_{rh}(D_X) \longrightarrow D^b_c(X)$$

*gives an equivalence of categories.*

*Sketch of proof.* We first prove that the functor $DR_X$ is fully faithful, i.e., there exists an isomorphism

$$\mathrm{Hom}_{D^b_{rh}(D_X)}(M^{\cdot},\ N^{\cdot}) \xrightarrow{\sim} \mathrm{Hom}_{D^b_c(X)}(DR_X\, M^{\cdot},\ DR_X\, N^{\cdot}) \tag{7.2.1}$$

for $M^{\cdot},\ N^{\cdot} \in D^b_{rh}(D_X)$. In fact, we will prove a more general result:

$$R\,\mathrm{Hom}_{D_X}(M^{\cdot},\ N^{\cdot}) \simeq R\,\mathrm{Hom}_{\mathbb{C}_{X^{\mathrm{an}}}}(DR_X\, M^{\cdot},\ DR_X\, N^{\cdot}). \tag{7.2.2}$$

Let $\Delta : X \hookrightarrow X \times X$ be the diagonal embedding and let $p : X \to \mathrm{pt}$ be the unique morphism from $X$ to the variety pt consisting of a single point. By Corollary 2.6.15 we have

$$R\,\mathrm{Hom}_{D_X}(M^{\cdot},\ N^{\cdot}) \simeq \int_p \Delta^{\dagger}(\mathbb{D}_X M^{\cdot} \boxtimes N^{\cdot}). \tag{7.2.3}$$

To calculate the right-hand side of (7.2.2) we need

$$R\,\mathrm{Hom}_{\mathbb{C}_{X^{\mathrm{an}}}}(F^{\cdot},\ G^{\cdot}) \simeq Rp_*\Delta^!(\mathbf{D}_X F^{\cdot} \boxtimes G^{\cdot}) \qquad (F^{\cdot}, G^{\cdot} \in D^b_c(X)). \tag{7.2.4}$$

This follows by applying $Rp_*(\bullet) = R\Gamma(X, \bullet)$ to

$$\Delta^!(\mathbf{D}_X F^{\cdot} \boxtimes G^{\cdot}) \simeq \Delta^! \mathbf{D}_{X\times X}(F^{\cdot} \boxtimes \mathbf{D}_X G^{\cdot})$$

$$
\begin{aligned}
&\simeq \mathbf{D}_X \Delta^{-1}(F^{\cdot} \boxtimes \mathbf{D}_X G^{\cdot}) \\
&\simeq \mathbf{D}_X (F^{\cdot} \otimes_{\mathbb{C}} \mathbf{D}_X G^{\cdot}) \\
&\simeq R\mathcal{H}om_{\mathbb{C}}(F^{\cdot} \otimes_{\mathbb{C}} \mathbf{D}_X G^{\cdot}, \omega_X) \\
&\simeq R\mathcal{H}om_{\mathbb{C}}(F^{\cdot}, R\mathcal{H}om_{\mathbb{C}}(\mathbf{D}_X G^{\cdot}, \omega_X)) \\
&\simeq R\mathcal{H}om_{\mathbb{C}}(F^{\cdot}, \mathbf{D}_X^2 G^{\cdot}) \\
&\simeq R\mathcal{H}om_{\mathbb{C}}(F^{\cdot}, G^{\cdot}).
\end{aligned}
$$

Therefore, we obtain the desired result by

$$
\begin{aligned}
&R\operatorname{Hom}_{\mathbb{C}_{X^{\mathrm{an}}}}(DR_X M^{\cdot}, DR_X N^{\cdot}) \\
&\quad \simeq Rp_* \Delta^{!}((\mathbf{D}_X DR_X M^{\cdot}) \boxtimes DR_X N^{\cdot}) && (7.2.4) \\
&\quad \simeq Rp_* \Delta^{!}(DR_X(\mathbb{D}_X M^{\cdot}) \boxtimes DR_X N^{\cdot}) && (\text{Proposition } 4.7.9) \\
&\quad \simeq Rp_* \Delta^{!}(DR_{X\times X}((\mathbb{D}_X M^{\cdot}) \boxtimes N^{\cdot})) && (\text{Proposition } 4.7.8) \\
&\quad \simeq Rp_* DR_X(\Delta^{\dagger}(\mathbb{D}_X M^{\cdot} \boxtimes N^{\cdot})) && (\text{Theorem } 7.1.1) \\
&\quad \simeq DR_{\mathrm{pt}} \int_p \Delta^{\dagger}(\mathbb{D}_X M^{\cdot} \boxtimes N^{\cdot}) && (\text{Theorem } 7.1.1) \\
&\quad \simeq \int_p \Delta^{\dagger}(\mathbb{D}_X M^{\cdot} \boxtimes N^{\cdot}) && (DR_{\mathrm{pt}} = \mathrm{Id}) \\
&\quad \simeq R\operatorname{Hom}_{D_X}(M^{\cdot}, N^{\cdot}) && (7.2.3)
\end{aligned}
$$

(note that the regular holonomicity is necessary whenever we used Theorem 7.1.1).

It remains to prove that the functor $DR_X$ is essentially surjective, i.e., for any $F^{\cdot} \in D_c^b(X)$ there exists an object $M^{\cdot} \in D_{rh}^b(D_X)$ satisfying $DR_X(M^{\cdot}) \simeq F^{\cdot}$. It is enough to check it for generators of the triangulated category $D_c^b(X)$. Hence we may assume that $F^{\cdot} = Ri_* L \in D_c^b(\mathbb{C}_X)$ for an affine embedding $i : Z \hookrightarrow X$ of a locally closed smooth subvariety $Z$ of $X$ and a local system $L$ on $Z^{\mathrm{an}}$. By Theorem 5.3.8 there exists a (unique) regular integrable connection $N$ on $Z$ such that $DR_Z N \simeq L[\dim Z]$. Set $M^{\cdot} = \int_i N[-\dim Z] \in D_{rh}^b(D_X)$. Then we have

$$
DR_X(M^{\cdot}) = DR_X \int_i N[-\dim Z] \simeq Ri_* DR_Z N[-\dim Z] \simeq Ri_* L = F^{\cdot}.
$$

The proof is complete. □

**Remark 7.2.3.** It is not totally trivial whether the isomorphism (7.2.1) constructed in the sketch of proof coincides with the one induced by $DR_X$. To make it more precise, we need to check many relations among various functors (we refer to Morihiko Saito [Sa2, §4] for details about this problem). That is why we add the terminology "Sketch of."

By Proposition 4.2.1 we obtain the following.

**Corollary 7.2.4.** *The solution functor*

$$\mathrm{Sol}_X : D^b_{rh}(D_X) \xrightarrow{\sim} D^b_c(X)^{\mathrm{op}}$$

*gives an equivalence of categories.*

The image of the full subcategory $\mathrm{Mod}_{rh}(D_X)$ of $D^b_{rh}(D_X)$ under the de Rham functor $DR_X : D^b_{rh}(D_X) \xrightarrow{\sim} D^b_c(X)$ is described by the following.

**Theorem 7.2.5.** *The de Rham functor induces an equivalence*

$$DR_X : \mathrm{Mod}_{rh}(D_X) \xrightarrow{\sim} \mathrm{Perv}(\mathbb{C}_X)$$

*of categories. In particular,* $\mathrm{Perv}(\mathbb{C}_X)$ *is an abelian category.*

The rest of this section is devoted to the proof of Theorem 7.2.5.

For $n \in \mathbb{Z}$ we denote by $D^{\geq n}_h(D_X)$ (resp. $D^{\leq n}_h(D_X)$) the full subcategory of $D^b_h(D_X)$ consisting of $M^\cdot$ satisfying $H^j(M^\cdot) = 0$ for $j < n$ (resp. $j > n$). We also define full subcategories $D^{\geq n}_c(X)$ and $D^{\leq n}_c(X)$ of $D^b_c(X)$ similarly.

**Lemma 7.2.6.** *For* $M^\cdot \in D^b_h(D_X)$ *we have*

$$M^\cdot \in D^{\geq 0}_h(D_X) \Longleftrightarrow \mathbb{D}_X M^\cdot \in D^{\leq 0}_h(D_X).$$

*Proof.* Let us show ($\Longrightarrow$). We may assume that $M^\cdot \neq 0$. Let $j$ be the smallest (non-negative) integer such that $H^j(M^\cdot) \neq 0$. Then we have a distinguished triangle

$$H^j(M^\cdot)[-j] \longrightarrow M^\cdot \longrightarrow \tau^{\geqslant j+1} M^\cdot \xrightarrow{+1} .$$

Hence by induction on the cohomological length of $M^\cdot$ it is sufficient to show $\mathbb{D}_X(N[-j]) \in D^b_h(D_X)$ for any $N \in \mathrm{Mod}_h(D_X)$ and any non-negative integer $j$. This follows from $\mathbb{D}_X(N[-j]) = \mathbb{D}_X(N)[j]$ and Corollary 2.6.8 (iii). The proof of ($\Longleftarrow$) is similar. □

**Corollary 7.2.7.** *An object* $M^\cdot$ *of* $D^b_h(D_X)$ *belongs to* $\mathrm{Mod}_h(D_X)$ *if and only if both* $M^\cdot$ *and* $\mathbb{D}_X M^\cdot$ *belong to* $D^{\geq 0}_h(D_X)$.

**Lemma 7.2.8.** *The following conditions on* $M^\cdot \in D^b_h(D_X)$ *are equivalent:*

(i) $M^\cdot \in D^{\geq 0}_h(D_X)$.

(ii) *For any locally closed subvariety* $Y$ *of* $X$ *there exists a smooth open dense subset* $Y_0$ *of* $Y$ *such that* $i^\dagger_{Y_0} M^\cdot \in D^{\geq 0}_h(D_{Y_0})$ *and* $H^j(i^\dagger_{Y_0} M^\cdot)$ *is coherent over* $\mathcal{O}_{Y_0}$ *for any* $j$, *where* $i_{Y_0} : Y_0 \to X$ *denotes the embedding.*

*Proof.* (i) $\Rightarrow$ (ii): In view of Proposition 3.1.6 it is sufficient to show that for any smooth open dense subset $Y_0$ of $Y$ we have $i^\dagger_{Y_0} M^\cdot \in D^{\geq 0}_h(D_{Y_0})$. This follows from the fact that the right $i^{-1}_{Y_0} D_X$-module $D_{Y_0 \to X}$ locally admits a free resolution of length $d_X - d_{Y_0}$ (see Section 1.5).

(ii) $\Rightarrow$ (i): Suppose that we have $H^j(M^\cdot) \neq 0$ for a negative integer $j < 0$. Let $j_0$ be the smallest integer satisfying this property, and set $Y = \operatorname{supp} H^{j_0}(M^\cdot)$. It is sufficient to show $H^{j_0}(i_{Y_0}^! M^\cdot) \neq 0$ for any smooth open dense subset $Y_0$ of $Y$. By applying $i_{Y_0}^\dagger$ to the distinguished triangle

$$H^{j_0}(M^\cdot)[-j_0] \longrightarrow M^\cdot \longrightarrow \tau^{>j_0} M^\cdot \xrightarrow{+1}$$

we obtain a distinguished triangle

$$i_{Y_0}^\dagger H^{j_0}(M^\cdot)[-j_0] \longrightarrow i_{Y_0}^\dagger M^\cdot \longrightarrow i_{Y_0}^\dagger \tau^{>j_0} M^\cdot \xrightarrow{+1}.$$

By $H^j(i_{Y_0}^\dagger \tau^{>j_0} M^\cdot) = 0$ for $j \geq j_0$ we have $H^{j_0}(i_{Y_0}^! M^\cdot) \simeq H^0(i_{Y_0}^! H^{j_0}(M^\cdot))$. Hence the assertion follows from Kashiwara's equivalence. □

By Lemma 7.2.6 we obtain the following.

**Lemma 7.2.9.** *The following conditions on $M^\cdot \in D_h^b(D_X)$ are equivalent:*

(i) $M^\cdot \in D_h^{\leq 0}(D_X)$.
(ii) *For any locally closed subvariety $Y$ of $X$ there exists a smooth open dense subset $Y_0$ of $Y$ such that $i_{Y_0}^\bigstar M^\cdot \in D_h^{\leq 0}(D_{Y_0})$ and $H^j(i_{Y_0}^\bigstar M^\cdot)$ is coherent over $\mathcal{O}_{Y_0}$ for any $j$, where $i_{Y_0} : Y_0 \to X$ denotes the embedding.*

*Proof of Theorem* 7.2.5. By Lemma 7.2.6 and the commutativity of the de Rham functor with the duality functors it is sufficient to show that the following conditions on $M^\cdot \in D_{rh}^b(D_X)$ are equivalent:

(a) $M^\cdot \in D_{rh}^{\leq 0}(D_X)$,
(b) $\dim \operatorname{supp} H^j(DR_X M^\cdot) \leq -j$ for any $j \in \mathbb{Z}$.

Set $F^\cdot = DR_X M^\cdot$. By Lemma 7.2.9 and the commutativity of the de Rham functor with the inverse image (on $D_{rh}^b$) the condition (a) is equivalent to the following;

(c) for any locally closed subvariety $Y$ of $X$ there exists a smooth open dense subset $Y_0$ of $Y$ such that $i_{Y_0}^{-1} F^\cdot \in D_c^{\leq -d_{Y_0}}(Y_0)$ and $H^j(i_{Y_0}^{-1} F^\cdot)$ is a local system for any $j$, where $i_{Y_0} : Y_0 \to X$ denotes the embedding.

(c) $\Rightarrow$ (b): Set $Y = \operatorname{supp}(H^j(F^\cdot))$, and take a smooth open dense subset $Y_0$ of $Y$ as in (c). Then by $H^j(i_{Y_0}^{-1} F^\cdot) = i_{Y_0}^{-1} H^j(F^\cdot) \neq 0$ we have $j \leq -d_{Y_0}$. Hence $\dim Y \leq -j$.

(b) $\Rightarrow$ (c): We may assume that $Y$ is connected. Take a stratification $X = \sqcup_\alpha X_\alpha$ of $X$ such that $H^j(F^\cdot)|_{X_\alpha}$ is a local system for any $\alpha$ and $j$. Then there exists some $\alpha$ such that $Y \cap X_\alpha$ is open dense in $Y$. Let $Y_0$ be a smooth open dense subset of $Y$ contained in $Y \cap X_\alpha$. Since $H^j(i_{Y_0}^{-1} F^\cdot)(\simeq i_{Y_0}^{-1} H^j(F^\cdot))$ is a local system, it is non-zero only if $d_{Y_0} \leq -j$ by the assumption (b). Hence we have $i_{Y_0}^{-1} F^\cdot \in D_c^{\leq -d_{Y_0}}(Y_0)$. □

**Remark 7.2.10.** Let $Y$ be an algebraic subvariety of $X$ and consider a local system $L$ on $U^{\mathrm{an}}$ for an open dense subset $U$ of the regular part of $Y$. Then we can associate to it an intersection cohomology complex $\mathrm{IC}_X(L)^{\cdot} \in \mathrm{Perv}(\mathbb{C}_X)$ on $X$ (see Section 8.2). This is an irreducible object in the abelian category $\mathrm{Perv}(\mathbb{C}_X)$ whose support is contained in $Y$. Let us consider also the regular integrable connection $M$ on $U$ which corresponds to $L$ by the equivalence of categories in Theorem 5.3.8 ($DR_U\, M = L[\dim U]$). Then it follows easily from the construction of $\mathrm{IC}_X(L)^{\cdot}$ that the minimal extension $L(U,\, M) \in \mathrm{Mod}_{rh}(D_X)$ of $M$ (see Theorem 3.4.2) corresponds to $\mathrm{IC}_X(L)^{\cdot}$ through the Riemann–Hilbert correspondence in Theorem 7.2.5:

$$DR_X\, L(U,\, M) \simeq \mathrm{IC}(L).$$

**Remark 7.2.11.** In [Kas10] Kashiwara constructed also an explicit inverse of the solution functor $\mathrm{Sol}_X : D^b_{rh}(D_X) \to D^b_c(\mathbb{C}_X)^{\mathrm{op}}$ for a complex manifold $X$ in order to establish the Riemann–Hilbert correspondence. He used Schwartz distributions to construct this inverse functor, which is denoted by $RH_X(\bullet)$ in [Kas10]. Motivated by this construction, Andronikof [An2] and Kashiwara–Schapira [KS3] (see also Colin [Co]) developed new theories which enable us to study Schwartz distributions and $C^\infty$-functions by purely algebraic methods. In particular, microlocalizations of those important function spaces are constructed. We also mention here recent results on the microlocal Riemann–Hilbert correspondence due to Andronikof [An1] and Waschkies [Was] (see also Gelfand–MacPherson–Vilonen [GMV2]).

## 7.3 Comparison theorem

As an application of the Riemann–Hilbert correspondence we give a proof of the *comparison theorem*. This theorem is concerned with historical motivation of the theory of regular holonomic $D$-modules. It is also important from the viewpoint of applications.

Let $\widehat{\mathcal{O}}_{X,x}$ be the formal completion of the local ring $\mathcal{O}_{X,x}$ at $x \in X$, i.e.,

$$\widehat{\mathcal{O}}_{X,x} := \varprojlim_l \mathcal{O}_{X,x}/\mathfrak{m}_x^l,$$

where $\mathfrak{m}_x$ denotes the maximal ideal of $\mathcal{O}_{X,x}$ at $x$. It is identified with the formal power series ring $\mathbb{C}[[x_1, x_2, \ldots, x_n]]$ in the local coordinate $\{x_i\}$ of $X$, and the analytic local ring $\mathcal{O}_{X^{\mathrm{an}},x}$ (the ring of convergent power series at $x$) is naturally a subring of $\widehat{\mathcal{O}}_{X,x}$. For each point $x \in X$ and $M^{\cdot} \in D^b(D_X)$, the inclusion $\mathcal{O}_{X^{\mathrm{an}},x} \hookrightarrow \widehat{\mathcal{O}}_{X,x}$ induces a morphism

$$\nu_x : R\,\mathrm{Hom}_{D_{X,x}}(M_x^{\cdot},\, \mathcal{O}_{X^{\mathrm{an}},x}) \longrightarrow R\,\mathrm{Hom}_{D_{X,x}}(M_x^{\cdot},\, \widehat{\mathcal{O}}_{X,x})$$

of complexes, where $M_x^{\cdot}$ is the stalk of $M^{\cdot}$ at $x$. Note that the left-hand side of $\nu_x$ is isomorphic to the stalk $(\mathrm{Sol}_X\, M^{\cdot})_x$ of $\mathrm{Sol}_X\, M^{\cdot} \in D^b(\mathbb{C}_{X^{\mathrm{an}}})$ at $x$. On the other hand the right-hand side $R\,\mathrm{Hom}_{D_{X,x}}(M_x^{\cdot},\, \widehat{\mathcal{O}}_{X,x})$ can be considered as a higher-degree generalization of the *formal solutions* $\mathrm{Hom}_{D_{X,x}}(M_x^{\cdot},\, \widehat{\mathcal{O}}_{X,x})$ of $M^{\cdot}$ at $x$.

**Proposition 7.3.1.** *If $M^{\cdot} \in D^b_{rh}(D_X)$, then $\nu_x$ is an isomorphism for each $x \in X$.*

*Proof.* By Proposition 4.2.1

$$\begin{aligned}(\mathrm{Sol}_X M^{\cdot})_x &\simeq (DR_X \mathbb{D}_X M^{\cdot}[-n])_x && (n = \dim X)\\ &\simeq i_x^{-1} DR_X \mathbb{D}_X M^{\cdot}[-n] && (i_x : \{x\} \hookrightarrow X)\\ &\simeq DR_{\{x\}} \mathbb{D}_{\{x\}} i_x^{\dagger} M^{\cdot}[-n] && \text{(Theorem 7.1.1)}\\ &\simeq \mathbb{D}_{\{x\}} i_x^{\dagger} M^{\cdot}[-n].\end{aligned}$$

For a $D_{X,x}$-free resolution $P^{\cdot} \xrightarrow{\sim} M^{\cdot}_x$ of $M^{\cdot}_x$ we have by definition

$$i_x^{\dagger} M^{\cdot} \simeq i_x^{\dagger} P^{\cdot} = \mathbb{C} \otimes_{\mathcal{O}_{X,x}} P^{\cdot}[-n]$$

and hence

$$(\mathrm{Sol}_X M^{\cdot})_x \simeq (\mathbb{C} \otimes_{\mathcal{O}_{X,x}} P^{\cdot})^*.$$

On the other hand, since we have

$$\begin{aligned}\mathbb{C} \otimes_{\mathcal{O}_{X,x}} D_{X,x} &= D_{X,x}/\mathfrak{m}_x D_{X,x}\\ &\simeq \mathbb{C}[\partial_1, \partial_2, \ldots, \partial_n]\end{aligned}$$

($\{x_i, \partial_i\}$ is a local coordinate system at $x$), there exists a natural isomorphism

$$\begin{aligned}\widehat{\mathcal{O}}_{X,x} &\xrightarrow{\sim} \mathrm{Hom}_{\mathbb{C}}(D_{X,x}/\mathfrak{m}_x D_{X,x}, \mathbb{C}) = (\mathbb{C} \otimes_{\mathcal{O}_{X,x}} D_{X,x})^*\\ f &\longmapsto [p(\partial) \longmapsto (p(\partial) f)(x)].\end{aligned}$$

Therefore, we get

$$\mathrm{Hom}_{D_{X,x}}(D_{X,x}, \widehat{\mathcal{O}}_{X,x}) \simeq \widehat{\mathcal{O}}_{X,x} \simeq (\mathbb{C} \otimes_{\mathcal{O}_{X,x}} D_{X,x})^*,$$

from which we obtain

$$\mathrm{Hom}_{D_{X,x}}(P^{\cdot}, \widehat{\mathcal{O}}_{X,x}) \simeq (\mathbb{C} \otimes_{\mathcal{O}_{X,x}} P^{\cdot})^*.$$

Since $P^{\cdot}$ is a free resolution of $M^{\cdot}_x$, the left-hand side is isomorphic to

$$R\,\mathrm{Hom}_{D_{X,x}}(M^{\cdot}_x, \widehat{\mathcal{O}}_{X,x}).$$

Therefore, we obtain the desired isomorphism

$$R\,\mathrm{Hom}_{D_{X,x}}(M^{\cdot}_x, \widehat{\mathcal{O}}_{X,x}) \simeq (\mathrm{Sol}_X M^{\cdot})_x.$$

The proof is complete. □

**Remark 7.3.2.** Let $X$ be a complex manifold. It is known that a holonomic $D_X$-module is regular if and only if the canonical morphism

$$\nu_x : R\,\mathrm{Hom}_{D_{X,x}}(M^{\cdot}_x, \mathcal{O}_{X,x}) \to R\,\mathrm{Hom}_{D_{X,x}}(M^{\cdot}_x, \widehat{\mathcal{O}}_{X,x})$$

is an isomorphism at each $x \in X$. This fact was proved by Malgrange [Ma3] for ordinary differential equations (the one-dimensional case), and by Kashiwara–Kawai [KK3] in the general case.

# 8

# Perverse Sheaves

In this chapter we will give a self-contained account of the theory of perverse sheaves and intersection cohomology groups assuming the basic notions concerning constructible sheaves presented in Section 4.5. We also include a survey on the theory of Hodge modules.

## 8.1 Theory of perverse sheaves

An obvious origin of the theory of perverse sheaves is the Riemann–Hilbert correspondence. Indeed, as we have seen in Section 7.2 one naturally encounters the category of perverse sheaves as the image under the de Rham functor of the category of regular holonomic $D$-modules. Another origin is the intersection cohomology groups due to Goresky–MacPherson. Perverse sheaves provide the theory of intersection cohomology groups with a sheaf-theoretical foundation.

In this section we present a systematic treatment of the theory of perverse sheaves on analytic spaces or (not necessarily smooth) algebraic varieties based on the language of $t$-structures.

### 8.1.1 $t$-structures

The derived category of an abelian category $\mathcal{C}$ contains $\mathcal{C}$ as a full abelian subcategory; however, it sometimes happens that it also contains another natural full abelian subcategory besides the standard one $\mathcal{C}$. For example, for a smooth algebraic variety (or a complex manifold) $X$ the derived category $D^b_c(X)$ contains the subcategory $\mathrm{Perv}(\mathbb{C}_X)$ of perverse sheaves as a non-standard full abelian subcategory. It is the image of the standard one $\mathrm{Mod}_{rh}(D_X)$ of $D^b_{rh}(D_X)$ by the de Rham functor

$$DR_X : D^b_{rh}(D_X) \xrightarrow{\sim} D^b_c(X).$$

More generally one can consider the following problem: in what situation does a triangulated category contain a natural abelian subcategory? An answer is given by the theory of $t$-structures due to Beilinson–Bernstein–Deligne [BBD].

In this subsection we give an account of the theory of $t$-structures. Besides the basic reference [BBD] we are also indebted to Kashiwara–Schapira [KS2, Chapter X].

**Definition 8.1.1.** Let $\mathbf{D}$ be a triangulated category, and let $\mathbf{D}^{\leqslant 0}$, $\mathbf{D}^{\geqslant 0}$ be its full subcategories. Set $\mathbf{D}^{\leqslant n} = \mathbf{D}^{\leqslant 0}[-n]$ and $\mathbf{D}^{\geqslant n} = \mathbf{D}^{\geqslant 0}[-n]$ for $n \in \mathbb{Z}$. We say that the pair $(\mathbf{D}^{\leqslant 0}, \mathbf{D}^{\geqslant 0})$ defines a *$t$-structure* on $\mathbf{D}$ if the following three conditions are satisfied:

(T1) $\mathbf{D}^{\leqslant -1} \subset \mathbf{D}^{\leqslant 0}$, $\mathbf{D}^{\geqslant 1} \subset \mathbf{D}^{\geqslant 0}$.
(T2) For any $X \in \mathbf{D}^{\leqslant 0}$ and any $Y \in \mathbf{D}^{\geqslant 1}$ we have $\mathrm{Hom}_{\mathbf{D}}(X, Y) = 0$.
(T3) For any $X \in \mathbf{D}$ there exists a distinguished triangle

$$X_0 \longrightarrow X \longrightarrow X_1 \xrightarrow{+1}$$

such that $X_0 \in \mathbf{D}^{\leqslant 0}$ and $X_1 \in \mathbf{D}^{\geqslant 1}$.

**Example 8.1.2.** Let $\mathcal{C}$ be an abelian category and $\mathcal{C}'$ a thick abelian subcategory of $\mathcal{C}$ in the sense of Definition B.4.6. Then by Proposition B.4.7 the full subcategory $D^{\sharp}_{\mathcal{C}'}(\mathcal{C})$ of $D^{\sharp}(\mathcal{C})$ ($\sharp = \emptyset, +, -, b$) consisting of objects $F^{\cdot} \in D^{\sharp}(\mathcal{C})$ satisfying $H^j(F^{\cdot}) \in \mathcal{C}'$ for any $j$ is a triangulated category. We show that $\mathbf{D} = D^{\sharp}_{\mathcal{C}'}(\mathcal{C})$ admits a standard $t$-structure $(\mathbf{D}^{\leqslant 0}, \mathbf{D}^{\geqslant 0})$ given by

$$\mathbf{D}^{\leqslant 0} = \{F^{\cdot} \in D^{\sharp}_{\mathcal{C}'}(\mathcal{C}) \mid H^j(F^{\cdot}) = 0 \text{ for } {}^{\forall} j > 0\},$$
$$\mathbf{D}^{\geqslant 0} = \{F^{\cdot} \in D^{\sharp}_{\mathcal{C}'}(\mathcal{C}) \mid H^j(F^{\cdot}) = 0 \text{ for } {}^{\forall} j < 0\}.$$

Since the condition (T1) is trivially satisfied, we will check (T2) and (T3). For $F^{\cdot} \in \mathbf{D}^{\leqslant 0}$, $G^{\cdot} \in \mathbf{D}^{\geqslant 1}$ and $f \in \mathrm{Hom}_{\mathbf{D}}(F^{\cdot}, G^{\cdot})$ we have a natural commutative diagram

$$\begin{array}{ccc} F^{\cdot} & \xrightarrow{f} & G^{\cdot} \\ \uparrow \wr & & \uparrow \\ \tau^{\leqslant 0}F^{\cdot} & \xrightarrow{\tau^{\leqslant 0}(f)} & \tau^{\leqslant 0}G^{\cdot}, \end{array}$$

where $\tau^{\leqslant 0} : \mathbf{D} \longrightarrow \mathbf{D}^{\leqslant 0}$ denotes the usual truncation functor. By $G^{\cdot} \in \mathbf{D}^{\geqslant 1}$ we easily see that $\tau^{\leqslant 0}G^{\cdot}$ is isomorphic in $\mathbf{D}$ to the zero object 0, and hence $f$ is zero. The condition (T2) is thus verified. The remaining condition (T3) follows from the distinguished triangle

$$\tau^{\leqslant 0}F^{\cdot} \longrightarrow F^{\cdot} \longrightarrow \tau^{\geqslant 1}F^{\cdot} \xrightarrow{+1}$$

for $F^{\cdot} \in \mathbf{D}$.

Now let $(\mathbf{D}^{\leqslant 0}, \mathbf{D}^{\geqslant 0})$ be a $t$-structure of a triangulated category $\mathbf{D}$.

**Definition 8.1.3.** We call the full subcategory $\mathcal{C} = \mathbf{D}^{\leqslant 0} \cap \mathbf{D}^{\geqslant 0}$ of $\mathbf{D}$ the *heart* (or core) of the $t$-structure $(\mathbf{D}^{\leqslant 0}, \mathbf{D}^{\geqslant 0})$.

We will see later that hearts of $t$-structures are abelian categories (Theorem 8.1.9).

**Proposition 8.1.4.** *Denote by* $\iota : \mathbf{D}^{\leqslant n} \longrightarrow \mathbf{D}$ (*resp.* $\iota' : \mathbf{D}^{\geqslant n} \longrightarrow \mathbf{D}$) *the inclusion. Then there exists a functor* $\tau^{\leqslant n} : \mathbf{D} \longrightarrow \mathbf{D}^{\leqslant n}$ (*resp.* $\tau^{\geqslant n} : \mathbf{D} \longrightarrow \mathbf{D}^{\geqslant n}$) *such that for any* $Y \in \mathbf{D}^{\leqslant n}$ *and any* $X \in \mathbf{D}$ (*resp. for any* $X \in \mathbf{D}$ *and any* $Y \in \mathbf{D}^{\geqslant n}$) *we have an isomorphism*

$$\mathrm{Hom}_{\mathbf{D}^{\leqslant n}}(Y, \tau^{\leqslant n} X) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{D}}(\iota(Y), X)$$

$$(\textit{resp.}\ \mathrm{Hom}_{\mathbf{D}^{\geqslant n}}(\tau^{\geqslant n} X, Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{D}}(X, \iota'(Y))),$$

*i.e.,* $\tau^{\leqslant n}$ *is right adjoint to* $\iota$*, and* $\tau^{\geqslant n}$ *is left adjoint to* $\iota'$*.*

*Proof.* It is sufficient to show that for any $X \in \mathbf{D}$ there exist $Z \in \mathbf{D}^{\leqslant n}$ and $Z' \in \mathbf{D}^{\geqslant m}$ such that

$$\mathrm{Hom}_{\mathbf{D}}(Y, Z) \simeq \mathrm{Hom}_{\mathbf{D}}(Y, X) \qquad (Y \in \mathbf{D}^{\leqslant n}),$$
$$\mathrm{Hom}_{\mathbf{D}}(Z', Y') \simeq \mathrm{Hom}_{\mathbf{D}}(X, Y') \qquad (Y' \in \mathbf{D}^{\geqslant m}).$$

We may assume that $n = 0$ and $m = 1$. Let $X_0$ and $X_1$ be as in (T3). We will show that $Z = X_0$ and $Z' = X_1$ satisfy the desired property. We will only show the statement for $X_0$ (the one for $X_1$ is proved similarly). Let $Y \in \mathbf{D}^{\leqslant 0}$. Applying the cohomological functor $\mathrm{Hom}_{\mathbf{D}}(Y, \bullet)$ to the distinguished triangle

$$X_0 \longrightarrow X \longrightarrow X_1 \xrightarrow{+1}$$

we obtain an exact sequence

$$\mathrm{Hom}_{\mathbf{D}}(Y, X_1[-1]) \longrightarrow \mathrm{Hom}_{\mathbf{D}}(Y, X_0) \longrightarrow \mathrm{Hom}_{\mathbf{D}}(Y, X) \longrightarrow \mathrm{Hom}_{\mathbf{D}}(Y, X_1).$$

By (T2) we have $\mathrm{Hom}_{\mathbf{D}}(Y, X_1[-1]) = \mathrm{Hom}_{\mathbf{D}}(Y, X_1) = 0$, and hence we obtain

$$\mathrm{Hom}_{\mathbf{D}}(Y, X_0) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{D}}(Y, X).$$

The proof is complete. □

Note that if a right (resp. left) adjoint functor exists, then it is unique up to isomorphisms. We call the functors

$$\tau^{\leqslant n} : \mathbf{D} \longrightarrow \mathbf{D}^{\leqslant n}, \qquad \tau^{\geqslant n} : \mathbf{D} \longrightarrow \mathbf{D}^{\geqslant n}$$

the truncation functors associated to the $t$-structure $(\mathbf{D}^{\leqslant 0}, \mathbf{D}^{\geqslant 0})$. We use the convention $\tau^{>n} = \tau^{\geqslant n+1}$ and $\tau^{<n} = \tau^{\leqslant n-1}$.

By definition we have canonical morphisms

$$\tau^{\leqslant n} X \longrightarrow X, \quad X \longrightarrow \tau^{\geqslant n} X \qquad (X \in \mathbf{D}). \tag{8.1.1}$$

By the proof of Proposition 8.1.4 we easily see the following.

**Proposition 8.1.5.**
(i) *The canonical morphisms* (8.1.1) *is embedded into a distinguished triangle*

$$\tau^{\leqslant n}(X) \longrightarrow X \longrightarrow \tau^{\geqslant n+1}(X) \xrightarrow{+1} \tag{8.1.2}$$

*in* $\mathbf{D}$.
(ii) *For* $X \in \mathbf{D}$ *let*

$$X_0 \longrightarrow X \longrightarrow X_1 \xrightarrow{+1}$$

*be as in* (T3). *Then there exist identifications* $X_0 \simeq \tau^{\leqslant 0}X$ *and* $X_1 \simeq \tau^{\geqslant 1}X$ *by which the morphisms* $X_0 \to X$ *and* $X \to X_1$ *are identified with the canonical ones* (8.1.1). *In particular,* $X_0$ *and* $X_1$ *are uniquely determined from* $X \in \mathbf{D}$.

**Proposition 8.1.6.** *The following conditions on* $X \in \mathbf{D}$ *are equivalent:*

(i) *We have* $X \in \mathbf{D}^{\leqslant n}$ (*resp.* $X \in \mathbf{D}^{\geqslant n}$).
(ii) *The canonical morphism* $\tau^{\leqslant n}X \to X$ (*resp.* $X \to \tau^{\geqslant n}X$) *is an isomorphism.*
(iii) *We have* $\tau^{>n}X = 0$ (*resp.* $\tau^{<n}X = 0$).

*Proof.* The equivalence of (ii) and (iii) is obvious in view of (8.1.2). The implication (ii)$\Longrightarrow$(i) is obvious by $\tau^{\leqslant n}X \in \mathbf{D}^{\leqslant n}$ and $\tau^{\geqslant n}X \in \mathbf{D}^{\geqslant n}$. It remains to show (i)$\Longrightarrow$(ii). Let us show that the canonical morphism $\tau^{\leqslant n}X \to X$ is an isomorphism for $X \in \mathbf{D}^{\leqslant n}$. We may assume that $n = 0$. By applying Proposition 8.1.5 (ii) to the obvious distinguished triangle

$$X \xrightarrow{\mathrm{id}} X \longrightarrow 0 \xrightarrow{+1}$$

we see that the canonical morphism $\tau^{\leqslant 0}X \to X$ is an isomorphism. The remaining assertion is proved similarly. □

**Lemma 8.1.7.** *Let*

$$X' \longrightarrow X \longrightarrow X'' \xrightarrow{+1}$$

*be a distinguished triangle in* $\mathbf{D}$. *If* $X'$, $X'' \in \mathbf{D}^{\leqslant 0}$ (*resp.* $\mathbf{D}^{\geqslant 0}$), *then* $X \in \mathbf{D}^{\leqslant 0}$ (*resp.* $\mathbf{D}^{\geqslant 0}$). *In particular, if* $X'$, $X'' \in \mathcal{C}$, *then* $X \in \mathcal{C}$.

*Proof.* We only prove the assertion for $\mathbf{D}^{\leqslant 0}$. Assume that $X'$, $X'' \in \mathbf{D}^{\leqslant 0}$. By Proposition 8.1.6 it is enough to show $\tau^{>0}(X) = 0$. In the exact sequence

$$\mathrm{Hom}_{\mathbf{D}}(X'', \tau^{>0}(X)) \longrightarrow \mathrm{Hom}_{\mathbf{D}}(X, \tau^{>0}(X)) \longrightarrow \mathrm{Hom}_{\mathbf{D}}(X', \tau^{>0}(X))$$

we have $\mathrm{Hom}_{\mathbf{D}}(X'', \tau^{>0}(X)) = \mathrm{Hom}_{\mathbf{D}}(X', \tau^{>0}(X)) = 0$ by (T2), and hence $\mathrm{Hom}_{\mathbf{D}}(\tau^{>0}(X), \tau^{>0}(X)) = \mathrm{Hom}_{\mathbf{D}}(X, \tau^{>0}(X)) = 0$ by Proposition 8.1.4. This implies $\tau^{>0}(X) = 0$. □

**Proposition 8.1.8.** *Let* $a$, $b$ *be two integers.*

(i) *If* $b \geq a$, *then we have* $\tau^{\leqslant b} \circ \tau^{\leqslant a} \simeq \tau^{\leqslant a} \circ \tau^{\leqslant b} \simeq \tau^{\leqslant a}$ *and* $\tau^{\geqslant b} \circ \tau^{\geqslant a} \simeq \tau^{\geqslant a} \circ \tau^{\geqslant b} \simeq \tau^{\geqslant b}$.

(ii) *If $a > b$, then* $\tau^{\leqslant b} \circ \tau^{\geqslant a} \simeq \tau^{\geqslant a} \circ \tau^{\leqslant b} \simeq 0$.
(iii) $\tau^{\geqslant a} \circ \tau^{\leqslant b} \simeq \tau^{\leqslant b} \circ \tau^{\geqslant a}$.

*Proof.* (i) By Proposition 8.1.6 we obtain $\tau^{\leqslant b} \circ \tau^{\leqslant a} \simeq \tau^{\leqslant a}$. We see from Proposition 8.1.4 that for any $X \in \mathbf{D}$ and $Y \in \mathbf{D}^{\leqslant a}$ we have

$$\begin{aligned}\mathrm{Hom}_{\mathbf{D}^{\leqslant a}}(Y, \tau^{\leqslant a}\tau^{\leqslant b}X) &\simeq \mathrm{Hom}_{\mathbf{D}}(Y, \tau^{\leqslant b}X) \simeq \mathrm{Hom}_{\mathbf{D}^{\leqslant b}}(Y, \tau^{\leqslant b}X) \\ &\simeq \mathrm{Hom}_{\mathbf{D}}(Y, X) \simeq \mathrm{Hom}_{\mathbf{D}^{\leqslant a}}(Y, \tau^{\leqslant a}X),\end{aligned}$$

and hence $\tau^{\leqslant a} \circ \tau^{\leqslant b} \simeq \tau^{\leqslant a}$. The remaining assertion can be proved similarly.

(ii) This is an immediate consequence of Proposition 8.1.6.

(iii) By (ii) we may assume $b \geq a$. Let $X \in \mathbf{D}$. We first construct a morphism $\phi : \tau^{\geqslant a}\tau^{\leqslant b}X \longrightarrow \tau^{\leqslant b}\tau^{\geqslant a}X$. By (i) there exists a distinguished triangle

$$\tau^{\leqslant b}\tau^{\geqslant a}X \longrightarrow \tau^{\geqslant a}X \longrightarrow \tau^{>b}X \xrightarrow{+1},$$

from which we conclude $\tau^{\leqslant b}\tau^{\geqslant a}X \in \mathbf{D}^{\geqslant a}$ by Lemma 8.1.7. Hence we obtain a chain of isomorphisms

$$\begin{aligned}\mathrm{Hom}_{\mathbf{D}}(\tau^{\leqslant b}X, \tau^{\geqslant a}X) &\simeq \mathrm{Hom}_{\mathbf{D}}(\tau^{\leqslant b}X, \tau^{\leqslant b}\tau^{\geqslant a}X) \\ &\simeq \mathrm{Hom}_{\mathbf{D}^{\geqslant a}}(\tau^{\geqslant a}\tau^{\leqslant b}X, \tau^{\leqslant b}\tau^{\geqslant a}X).\end{aligned}$$

Then $\phi \in \mathrm{Hom}_{\mathbf{D}}(\tau^{\geqslant a}\tau^{\leqslant b}X, \tau^{\leqslant b}\tau^{\geqslant a}X)$ is obtained as the image of the composite of natural morphisms $\tau^{\leqslant b}X \longrightarrow X \longrightarrow \tau^{\geqslant a}X$ through these isomorphisms. Let us show that $\phi$ is an isomorphism. By (i) there exists a distinguished triangle

$$\tau^{<a}X \longrightarrow \tau^{\leqslant b}X \longrightarrow \tau^{\geqslant a}\tau^{\leqslant b}X \xrightarrow{+1},$$

from which we obtain $\tau^{\geqslant a}\tau^{\leqslant b}X \in \mathbf{D}^{\leqslant b}$ by Lemma 8.1.7. On the other hand, applying the *octahedral axiom* to the three distinguished triangles

$$\begin{cases} \tau^{<a}X \xrightarrow{p} \tau^{\leqslant b}X \longrightarrow \tau^{\geqslant a}\tau^{\leqslant b}X \xrightarrow{+1} \\ \tau^{<a}X \xrightarrow{q\circ p} X \longrightarrow \tau^{\geqslant a}X \xrightarrow{+1} \\ \tau^{\leqslant b}X \xrightarrow{q} X \longrightarrow \tau^{>b}X \xrightarrow{+1}, \end{cases}$$

we get a new one

$$\tau^{\geqslant a}\tau^{\leqslant b}X \longrightarrow \tau^{\geqslant a}X \longrightarrow \tau^{>b}X \xrightarrow{+1}.$$

Then it follows from $\tau^{\geqslant a}\tau^{\leqslant b}X \in \mathbf{D}^{\leqslant b}$ and Proposition 8.1.5 (ii) that $\tau^{\geqslant a}\tau^{\leqslant b}X \simeq \tau^{\leqslant b}(\tau^{\geqslant a}X)$. □

**Theorem 8.1.9.**

(i) *The heart $\mathcal{C} = \mathbf{D}^{\leqslant 0} \cap \mathbf{D}^{\geqslant 0}$ is an abelian category.*

(ii) *An exact sequence*

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

*in* $\mathcal{C}$ *gives rise to a distinguished triangle*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{+1}$$

*in* $\mathbf{D}$.

*Proof.* (i) Let $X, Y \in \mathcal{C}$. By applying Lemma 8.1.7 to the distinguished triangle

$$X \longrightarrow X \oplus Y \longrightarrow Y \xrightarrow{+1}$$

in $\mathbf{D}$ we see that $X \oplus Y \in \mathcal{C}$.

It remains to show that any morphism $f : X \to Y$ in $\mathcal{C}$ admits a kernel and a cokernel and that the canonical morphism $\mathrm{Coim}\, f \longrightarrow \mathrm{Im}\, f$ is an isomorphism. Embed $f$ into a distinguished triangle

$$X \xrightarrow{f} Y \longrightarrow Z \xrightarrow{+1}.$$

Then we have $Z \in \mathbf{D}^{\leqslant 0} \cap \mathbf{D}^{\geqslant -1}$ by Lemma 8.1.7. We will show that the kernel and the cokernel of $f$ are given by

$$\begin{aligned} \mathrm{Coker}\, f &\simeq H^0(Z) = \tau^{\geqslant 0} Z, \\ \mathrm{Ker}\, f &\simeq H^{-1}(Z) = \tau^{\leqslant 0}(Z[-1]). \end{aligned}$$

Consider the exact sequences

$$\begin{aligned} &\mathrm{Hom}_{\mathbf{D}}(X[1], W) \to \mathrm{Hom}_{\mathbf{D}}(Z, W) \to \mathrm{Hom}_{\mathbf{D}}(Y, W) \to \mathrm{Hom}_{\mathbf{D}}(X, W), \\ &\mathrm{Hom}_{\mathbf{D}}(W, Y[-1]) \to \mathrm{Hom}_{\mathbf{D}}(W, Z[-1]) \to \mathrm{Hom}_{\mathbf{D}}(W, X) \to \mathrm{Hom}_{\mathbf{D}}(W, Y) \end{aligned}$$

for $W \in \mathcal{C}$. By (T2) and Proposition 8.1.4 they are rewritten as

$$\begin{aligned} &0 \longrightarrow \mathrm{Hom}_{\mathbf{D}}(\tau^{\geqslant 0} Z, W) \longrightarrow \mathrm{Hom}_{\mathbf{D}}(Y, W) \longrightarrow \mathrm{Hom}_{\mathbf{D}}(X, W), \\ &0 \longrightarrow \mathrm{Hom}_{\mathbf{D}}(W, \tau^{\leqslant 0}(Z[-1])) \longrightarrow \mathrm{Hom}_{\mathbf{D}}(W, X) \longrightarrow \mathrm{Hom}_{\mathbf{D}}(W, Y). \end{aligned}$$

This implies that $\mathrm{Coker}\, f \simeq \tau^{\geqslant 0} Z$ and $\mathrm{Ker}\, f \simeq \tau^{\leqslant 0}(Z[-1])$. Let us show that the canonical morphism $\mathrm{Coim}\, f \longrightarrow \mathrm{Im}\, f$ is an isomorphism. Let us embed $Y \longrightarrow \mathrm{Coker}\, f$ into a distinguished triangle

$$I \longrightarrow Y \longrightarrow \mathrm{Coker}\, f \xrightarrow{+1}.$$

Then $I \in \mathbf{D}^{\geqslant 0}$ by Lemma 8.1.7. Applying the octahedral axiom to the three distinguished triangles

$$\begin{cases} Y \xrightarrow{p} Z \longrightarrow X[1] \xrightarrow{+1} \\ Y \xrightarrow{q\circ p} \mathrm{Coker}\, f \longrightarrow I[1] \xrightarrow{+1} \\ Z \xrightarrow{q} \mathrm{Coker}\, f \longrightarrow \mathrm{Ker}\, f[2] \xrightarrow{+1}, \end{cases}$$

we get new distinguished triangles

$$X[1] \longrightarrow I[1] \longrightarrow \mathrm{Ker}\, f[2] \xrightarrow{+1},$$
$$\mathrm{Ker}\, f \longrightarrow X \longrightarrow I \xrightarrow{+1}.$$

This implies $I \in \mathbf{D}^{\leqslant 0}$ by Lemma 8.1.7 and hence we have $I \in \mathcal{C}$. Then by the argument used in the proof of the existence of a kernel and a cokernel we conclude that

$$\mathrm{Im}\, f = \mathrm{Ker}(Y \to \mathrm{Coker}\, f) \simeq I \simeq \mathrm{Coker}(\mathrm{Ker}\, f \to X) = \mathrm{Coim}\, f.$$

(ii) Embed $X \xrightarrow{f} Y$ into a distinguished triangle

$$X \xrightarrow{f} Y \longrightarrow W \xrightarrow{+1}.$$

Then by $\mathrm{Ker}\, f = 0$ and $\mathrm{Coker}\, f \simeq Z$ we obtain $W \simeq Z$ by the proof of (i). □

**Definition 8.1.10.** We define a functor

$$H^0 : \mathbf{D} \longrightarrow \mathcal{C} = \mathbf{D}^{\leqslant 0} \cap \mathbf{D}^{\geqslant 0}$$

by $H^0(X) = \tau^{\geqslant 0}\tau^{\leqslant 0}X = \tau^{\leqslant 0}\tau^{\geqslant 0}X \in \mathcal{C}$. For $n \in \mathbb{Z}$ we set $H^n(X) = H^0(X[n]) = (\tau^{\geqslant n}\tau^{\leqslant n}X)[n] \in \mathcal{C}$.

**Proposition 8.1.11.** *The functor* $H^0 : \mathbf{D} \longrightarrow \mathcal{C} = \mathbf{D}^{\leqslant 0} \cap \mathbf{D}^{\geqslant 0}$ *is a* **cohomological functor** *in the sense of Definition* B.3.8.

*Proof.* We need to show for a distinguished triangle

$$X \longrightarrow Y \longrightarrow Z \xrightarrow{+1}$$

in $\mathbf{D}$ that

$$H^0(X) \longrightarrow H^0(Y) \longrightarrow H^0(Z)$$

is an exact sequence in $\mathcal{C}$. The proof is divided into several steps.

(a) We prove that

$$0 \longrightarrow H^0(X) \longrightarrow H^0(Y) \longrightarrow H^0(Z) \tag{8.1.3}$$

is exact under the condition $X, Y, Z \in \mathbf{D}^{\geqslant 0}$. For $W \in \mathcal{C}$ consider the exact sequence

$$\mathrm{Hom}_{\mathbf{D}}(W, Z[-1]) \to \mathrm{Hom}_{\mathbf{D}}(W, X) \to \mathrm{Hom}_{\mathbf{D}}(W, Y) \to \mathrm{Hom}_{\mathbf{D}}(W, Z).$$

By (T2) we have $\mathrm{Hom}_{\mathbf{D}}(W, Z[-1]) = 0$. Moreover, for $V \in \mathbf{D}^{\geqslant 0}$ we have $\tau^{\leqslant 0}V \simeq \tau^{\leqslant 0}\tau^{\geqslant 0}V = H^0(V)$, and hence $\mathrm{Hom}_{\mathbf{D}}(W, V) \simeq \mathrm{Hom}_{\mathbf{D}}(W, \tau^{\leqslant 0}V) \simeq \mathrm{Hom}_{\mathcal{C}}(W, H^0(V))$ by Proposition 8.1.4. Hence the above exact sequence is rewritten as

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{C}}(W, H^0(X)) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(W, H^0(Y)) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(W, H^0(Z)),$$

from which we obtain our assertion.

(b) We prove that (8.1.3) is exact assuming only $Z \in \mathbf{D}^{\geqslant 0}$. Let $W \in \mathbf{D}^{<0}$. Then we have $\mathrm{Hom}_{\mathbf{D}}(W, Z) = \mathrm{Hom}_{\mathbf{D}}(W, Z[-1]) = 0$ and hence $\mathrm{Hom}_{\mathbf{D}}(W, X) \simeq \mathrm{Hom}_{\mathbf{D}}(W, Y)$. By Proposition 8.1.4 this implies that the canonical morphism $\tau^{<0}X \longrightarrow \tau^{<0}Y$ is an isomorphism. Therefore, applying the octahedral axiom to the distinguished triangles

$$\begin{cases} \tau^{<0}X \xrightarrow{p} X \longrightarrow \tau^{\geqslant 0}X \xrightarrow{+1} \\ \tau^{<0}X \xrightarrow{q\circ p} Y \longrightarrow \tau^{\geqslant 0}Y \xrightarrow{+1} \\ X \xrightarrow{q} Y \longrightarrow Z \xrightarrow{+1}, \end{cases}$$

we get a new one

$$\tau^{\geqslant 0}X \longrightarrow \tau^{\geqslant 0}Y \longrightarrow Z \xrightarrow{+1}.$$

Hence our assertion is a consequence of (a).

(c) Similarly to (b) we can prove that

$$H^0(X) \longrightarrow H^0(Y) \longrightarrow H^0(Z) \longrightarrow 0$$

is exact under the condition $X \in \mathbf{D}^{\leqslant 0}$.

(d) Finally, let us consider the general case. Embed the composite of the morphisms $\tau^{\leqslant 0}X \longrightarrow X \longrightarrow Y$ into a distinguished triangle

$$\tau^{\leqslant 0}X \longrightarrow Y \longrightarrow W \xrightarrow{+1}.$$

By applying (c) we have an exact sequence

$$H^0(X) \longrightarrow H^0(Y) \longrightarrow H^0(W).$$

Now applying the octahedral axiom to the distinguished triangles

$$\begin{cases} \tau^{\leqslant 0}X \xrightarrow{r} X \longrightarrow \tau^{>0}X \xrightarrow{+1} \\ \tau^{\leqslant 0}X \xrightarrow{s\circ r} Y \longrightarrow W \xrightarrow{+1} \\ X \xrightarrow{s} Y \longrightarrow Z \xrightarrow{+1}, \end{cases}$$

we get a distinguished triangle

$$W \longrightarrow Z \longrightarrow \tau^{>0}X[1] \xrightarrow{+1}.$$

Hence by (b) we have an exact sequence

$$0 \longrightarrow H^0(W) \longrightarrow H^0(Z).$$

This completes the proof. □

For a distinguished triangle

$$X \longrightarrow Y \longrightarrow Z \xrightarrow{+1}$$

in $\mathbf{D}$ we thus obtain a long exact sequence

$$\cdots \longrightarrow H^{-1}(Z) \longrightarrow H^0(X) \longrightarrow H^0(Y) \longrightarrow H^0(Z) \longrightarrow H^1(X) \longrightarrow \cdots$$

in $\mathcal{C}$.

Now let $\mathbf{D}_i$ be triangulated categories endowed with $t$-structures $(\mathbf{D}_i^{\leqslant 0}, \mathbf{D}_i^{\geqslant 0})$ $(i = 1, 2)$, and let $F : \mathbf{D}_1 \longrightarrow \mathbf{D}_2$ be a functor of triangulated categories. We denote by $\mathcal{C}_i$ the heart of $(\mathbf{D}_i^{\leqslant 0}, \mathbf{D}_i^{\geqslant 0})$.

**Definition 8.1.12.** We define an additive functor

$$^pF : \mathcal{C}_1 \longrightarrow \mathcal{C}_2$$

by $^pF = H^0 \circ F \circ \varepsilon_1$, where $\varepsilon_1 : \mathcal{C}_1 \to \mathbf{D}_1$ denotes the inclusion functor.

**Definition 8.1.13.** We say that $F$ is *left $t$-exact* (resp. *right $t$-exact*) if $F(\mathbf{D}_1^{\geqslant 0}) \subset \mathbf{D}_2^{\geqslant 0}$ (resp. $F(\mathbf{D}_1^{\leqslant 0}) \subset \mathbf{D}_2^{\leqslant 0}$). We also say that $F$ is *$t$-exact* if it is both left and right $t$-exact.

**Example 8.1.14.** Let $\mathcal{C}_1, \mathcal{C}_2$ be abelian categories and let $G : \mathcal{C}_1 \to \mathcal{C}_2$ be a left exact functor. Assume that $\mathcal{C}_1$ has enough injectives. Then the right derived functor $RG : D^+(\mathcal{C}_1) \longrightarrow D^+(\mathcal{C}_2)$ is left $t$-exact with respect to the standard $t$-structures of $D^+(\mathcal{C}_i)$ (see Example 8.1.2).

**Proposition 8.1.15.** *Let $\mathbf{D}_i$, $(\mathbf{D}_i^{\leqslant 0}, \mathbf{D}_i^{\geqslant 0})$, $(i = 1, 2)$ and $F : \mathbf{D}_1 \longrightarrow \mathbf{D}_2$ be as above. Assume that $F$ is left $t$-exact.*

(i) *For any $X \in \mathbf{D}_1$ we have $\tau^{\leqslant 0}(F(\tau^{\leqslant 0}X)) \simeq \tau^{\leqslant 0}F(X)$. In particular, for $X \in \mathbf{D}_1^{\geqslant 0}$ there exists an isomorphism $^pF(H^0(X)) \simeq H^0(F(X))$ in $\mathcal{C}_2$.*

(ii) *$^pF : \mathcal{C}_1 \longrightarrow \mathcal{C}_2$ is a left exact functor between abelian categories.*

*Proof.* (i) It is sufficient to show that the canonical morphism

$$\mathrm{Hom}_{\mathbf{D}_2^{\leqslant 0}}\big(W,\ \tau^{\leqslant 0}(F(\tau^{\leqslant 0}(X)))\big) \to \mathrm{Hom}_{\mathbf{D}_2^{\leqslant 0}}\big(W,\ \tau^{\leqslant 0}(F(X))\big)$$

is an isomorphism for any $W \in \mathbf{D}_2^{\leqslant 0}$. By Proposition 8.1.4 we have

$$\mathrm{Hom}_{\mathbf{D}_2^{\leqslant 0}}\big(W,\ \tau^{\leqslant 0}(F(\tau^{\leqslant 0}(X)))\big) \simeq \mathrm{Hom}_{\mathbf{D}_2}\big(W,\ F(\tau^{\leqslant 0}(X))\big),$$

$$\mathrm{Hom}_{\mathbf{D}_2^{\leqslant 0}}\big(W,\ \tau^{\leqslant 0}(F(X))\big) \simeq \mathrm{Hom}_{\mathbf{D}_2}\big(W,\ F(X)\big),$$

and hence we have only to show that the canonical morphism

$$\mathrm{Hom}_{\mathbf{D}_2}\big(W,\ F(\tau^{\leqslant 0}(X))\big) \longrightarrow \mathrm{Hom}_{\mathbf{D}_2}\big(W,\ F(X)\big)$$

is an isomorphism for any $W \in \mathbf{D}_2^{\leqslant 0}$. By the distinguished triangle

$$F(\tau^{\leqslant 0}(X)) \longrightarrow F(X) \longrightarrow F(\tau^{\geqslant 1}(X)) \xrightarrow{+1}$$

we obtain an exact sequence

$$\begin{aligned}\mathrm{Hom}_{\mathbf{D}_2}(W, F(\tau^{\geqslant 1}(X))[-1]) &\longrightarrow \mathrm{Hom}_{\mathbf{D}_2}(W, F(\tau^{\leqslant 0}(X)))\\ &\longrightarrow \mathrm{Hom}_{\mathbf{D}_2}(W, F(X))\\ &\longrightarrow \mathrm{Hom}_{\mathbf{D}_2}(W, F(\tau^{\geqslant 1}(X))).\end{aligned}$$

Since $F$ is left $t$-exact, we have $F(\tau^{\geqslant 1}(X)) \in \mathbf{D}_2^{\geqslant 1}$, and hence

$$\mathrm{Hom}_{\mathbf{D}_2}(W, F(\tau^{\geqslant 1}(X))[n]) = 0 \qquad (n \leq 0)$$

by (T2). Therefore, the assertion follows from the above exact sequence.

(ii) For an exact sequence

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

in $\mathcal{C}_1$ we have a distinguished triangle

$$X \longrightarrow Y \longrightarrow Z \xrightarrow{+1}$$

in $\mathbf{D}_1$ by Theorem 8.1.9 (ii). Hence we obtain a distinguished triangle

$$F(X) \longrightarrow F(Y) \longrightarrow F(Z) \xrightarrow{+1}$$

in $\mathbf{D}_2$. By considering the cohomology long exact sequence associated to it we obtain an exact sequence

$$H^{-1}(F(Z)) \longrightarrow H^0(F(X)) \longrightarrow H^0(F(Y)) \longrightarrow H^0(F(Z)).$$

It remains to show $H^{-1}(F(Z)) = 0$. Since $F$ is left $t$-exact, we have $F(Z) \in \mathbf{D}_2^{\geqslant 0}$. Hence we have $\tau^{\leqslant -1}F(Z) = 0$ by Proposition 8.1.6. It follows that we have $H^{-1}(F(Z)) = \tau^{\geqslant -1}\tau^{\leqslant -1}F(Z)[-1] = 0$. □

**Lemma 8.1.16.** *Let $\mathbf{D}'_i$ be a triangulated category and $\mathbf{D}_i \subset \mathbf{D}'_i$ its full triangulated subcategory with a $t$-structure $(\mathbf{D}_i^{\leqslant 0}, \mathbf{D}_i^{\geqslant 0})$ $(i = 1, 2)$. Assume that $F : \mathbf{D}'_1 \longrightarrow \mathbf{D}'_2$ and $G : \mathbf{D}'_2 \longrightarrow \mathbf{D}'_1$ are functors of triangulated categories and $F$ is the left adjoint functor of $G$.*

(i) *If* $F(\mathbf{D}_1) \subset \mathbf{D}_2$ *and* $F(\mathbf{D}_1^{\leqslant 0}) \subset \mathbf{D}_2^{\leqslant d}$ *for* $d \in \mathbb{Z}$, *then for any* $Y \in \mathbf{D}_2^{\geqslant 0}$ *satisfying* $G(Y) \in \mathbf{D}_1$ *we have* $G(Y) \in \mathbf{D}_1^{\geqslant -d}$.

(ii) *If* $G(\mathbf{D}_2) \subset \mathbf{D}_1$ *and* $G(\mathbf{D}_2^{\geqslant 0}) \subset \mathbf{D}_1^{\geqslant -d}$ *for* $d \in \mathbb{Z}$, *then for any* $X \in \mathbf{D}_1^{\leqslant 0}$ *satisfying* $F(X) \in \mathbf{D}_2$ *we have* $F(X) \in \mathbf{D}_2^{\leqslant d}$.

*Proof.* We prove only the assertion (i). By Proposition 8.1.6 it is enough to show $\tau^{<-d}G(Y) = 0$. According to Proposition 8.1.4, for any $X \in \mathbf{D}_1^{<-d}$ we have an isomorphism

$$\begin{aligned}\mathrm{Hom}_{\mathbf{D}_1^{<-d}}(X,\ \tau^{<-d}G(Y)) &\simeq \mathrm{Hom}_{\mathbf{D}_1}(X,\ G(Y))\\ &\simeq \mathrm{Hom}_{\mathbf{D}_2}(F(X),\ Y) = 0\end{aligned}$$

(note $F(X) \in \mathbf{D}_2^{\leq 0}$ and $Y \in \mathbf{D}_2^{\geqslant 0}$). Therefore, we have $\tau^{<-d}G(Y) = 0$. □

**Corollary 8.1.17.** *Let* $\mathbf{D}_i$ *be a triangulated category with a t-structure* $(i = 1, 2)$. *Assume that* $F : \mathbf{D}_1 \longrightarrow \mathbf{D}_2$ *and* $G : \mathbf{D}_2 \longrightarrow \mathbf{D}_1$ *are functors of triangulated categories and* $F$ *is the left adjoint functor of* $G$. *Then* $F$ *is right t-exact if and only if* $G$ *is left t-exact.*

### 8.1.2 Perverse sheaves

From now on, let $X$ be a (not necessarily smooth) algebraic variety or an analytic space and denote by $D_c^b(X)$ the full subcategory of $D^b(X) = D^b(\mathrm{Mod}(\mathbb{C}_X))$ consisting of objects $F^{\cdot} \in D^b(X)$ such that $H^j(F^{\cdot})$ is a constructible sheaf on $X$ for any $j$. For the definition of constructible sheaves and basic properties of $D_c^b(X)$ see Section 4.5. The aim of this subsection is to introduce the *perverse t-structure* $({}^pD_c^{\leqslant 0}(X),\ {}^pD_c^{\geqslant 0}(X))$ on $\mathbf{D} = D_c^b(X)$ and define the category of perverse sheaves on $X$ to be its heart ${}^pD_c^{\leqslant 0}(X) \cap {}^pD_c^{\geqslant 0}(X)$. We follow the basic reference [BBD]. We are also indebted to [GM1], [G1], and [KS2, Chapter X].

**Remark 8.1.18.**

(i) Although we restrict ourselves to the case of complex coefficients, all of the results that we present in Sections 8.1.2 and 8.2 remain valid even after replacing $\mathrm{Mod}(\mathbb{C}_X)$ with $\mathrm{Mod}(\mathbb{Q}_X)$. In particular, we have the notion perverse sheaves and intersection cohomology groups with coefficients in $\mathbb{Q}$. They are essential for the theory of Hodge modules to be explained in Section 8.3.

(ii) The $t$-structure that we treat here is the one with respect to the "*middle perversity*" in the terminology of [GM1].

(iii) There exists a more general theory of perverse sheaves on subanalytic spaces as explained in [KS2, Chapter X].

**Notation 8.1.19.** For a locally closed analytic subspace $S$ of $X$ we denote its dimension by $d_S$. The inclusion map $S \hookrightarrow X$ is usually denoted by $i_S$.

Recall that we denote by $\mathbf{D}_X : D_c^b(X)^{\mathrm{op}} \xrightarrow{\sim} D_c^b(X)$ the Verdier duality functor.

**Definition 8.1.20.** We define full subcategories ${}^pD_c^{\leqslant 0}(X)$ and ${}^pD_c^{\geqslant 0}(X)$ of $D_c^b(X)$ as follows. For $F^\cdot \in D_c^b(X)$ we have $F^\cdot \in {}^pD_c^{\leqslant 0}(X)$ if and only if

(i) $\dim\{\mathrm{supp}\, H^j(F^\cdot)\} \leq -j$ for any $j \in \mathbb{Z}$,

and $F^\cdot \in {}^pD_c^{\geqslant 0}(X)$ if and only if

(ii) $\dim\{\mathrm{supp}\, H^j(\mathbf{D}_X F^\cdot)\} \leq -j$ for any $j \in \mathbb{Z}$,

We define a full subcategory $\mathrm{Perv}(\mathbb{C}_X)$ of $D_c^b(X)$ by

$$\mathrm{Perv}(\mathbb{C}_X) = {}^pD_c^{\leqslant 0}(X) \cap {}^pD_c^{\geqslant 0}(X).$$

We will show later that the pair $({}^pD_c^{\leqslant 0}(X), {}^pD_c^{\geqslant 0}(X))$ defines a $t$-structure on $D_c^b(X)$, and hence $\mathrm{Perv}(\mathbb{C}_X)$ will turn out to be an abelian category. Since we have $\mathbf{D}_X\mathbf{D}_X F^\cdot \simeq F^\cdot$ for any $F^\cdot \in D_c^b(X)$, the Verdier duality functor $\mathbf{D}_X(\bullet)$ exchanges ${}^pD_c^{\leqslant 0}(X)$ with ${}^pD_c^{\geqslant 0}(X)$.

**Lemma 8.1.21.** *Let $F^\cdot \in D_c^b(X)$. Then we have*

$$\mathrm{supp}\, H^j(\mathbf{D}_X F^\cdot) = \{x \in X | \quad H^{-j}(i_{\{x\}}^! F^\cdot) \neq 0\}$$

*for any $j \in \mathbb{Z}$, where $i_{\{x\}} : \{x\} \hookrightarrow X$ are inclusion maps.*

*Proof.* Since for each $x \in X$ we have

$$i_{\{x\}}^! F^\cdot \simeq i_{\{x\}}^! \mathbf{D}_X \mathbf{D}_X F^\cdot \simeq \mathbf{D}_{\{x\}} i_{\{x\}}^{-1}(\mathbf{D}_X F^\cdot),$$

we obtain an isomorphism $H^{-j}(i_{\{x\}}^! F^\cdot) \simeq [H^j(\mathbf{D}_X F^\cdot)_x]^*$ for any $j \in \mathbb{Z}$. □

**Proposition 8.1.22.** *Let $F^\cdot \in D_c^b(X)$ and $X = \bigsqcup_{\alpha \in A} X_\alpha$ be a complex stratification of $X$ consisting of connected strata such that $i_{X_\alpha}^{-1} F^\cdot$ and $i_{X_\alpha}^! F^\cdot$ have locally constant cohomology sheaves for any $\alpha \in A$. Then*

(i) *$F^\cdot \in {}^pD_c^{\leqslant 0}(X)$ if and only if $H^j(i_{X_\alpha}^{-1} F^\cdot) = 0$ for any $\alpha$ and $j > -d_{X_\alpha}$.*

(ii) *$F^\cdot \in {}^pD_c^{\geqslant 0}(X)$ if and only if $H^j(i_{X_\alpha}^! F^\cdot) = 0$ for any $\alpha$ and $j < -d_{X_\alpha}$.*

*Proof.* (i) Trivial.

(ii) By Lemma 8.1.21, $F^\cdot \in {}^pD_c^{\geqslant 0}(X)$ if and only if

$$\dim\{x \in X | \quad H^{-j}(i_{\{x\}}^! F^\cdot) \neq 0\} \leq -j$$

for any $j \in \mathbb{Z}$. For $x \in X_\alpha$ decompose the morphism $i_{\{x\}} : \{x\} \hookrightarrow X$ into $\{x\} \xrightarrow{j_{\{x\}}} X_\alpha \xrightarrow{i_{X_\alpha}} X$. Then we have an isomorphism

$$i_{\{x\}}^! F^\cdot \simeq j_{\{x\}}^! i_{X_\alpha}^! F^\cdot \simeq j_{\{x\}}^{-1} i_{X_\alpha}^! F^\cdot[-2d_{X_\alpha}], \tag{8.1.4}$$

where we used our assumption on $i_{X_\alpha}^! F^\cdot$ in the last isomorphism. Hence for any $j \in \mathbb{Z}$ by the connectedness of $X_\alpha$, $X_\alpha \cap \{x \in X | \quad H^{-j}(i_{\{x\}}^! F^\cdot) \neq 0\} = X_\alpha \cap \mathrm{supp}\, H^j(\mathbf{D}_X F^\cdot)$ is $X_\alpha$ or $\emptyset$. Moreover, from (8.1.4) we easily see that the following conditions are equivalent for any $\alpha \in A$:

(a) $H^j(i^!_{X_\alpha} F^\cdot) = 0$ for any $j < -d_{X_\alpha}$.
(b) $H^{-j}(i^!_{\{x\}} F^\cdot) = 0$ for any $x \in X_\alpha$ and $j > -d_{X_\alpha}$.
(c) $X_\alpha \cap \operatorname{supp} H^j(\mathbf{D}_X F^\cdot) = \emptyset$ for any $j > -d_{X_\alpha}$.

The last condition (c) implies that for any stratum $X_\alpha$ such that $X_\alpha \subset \operatorname{supp} H^j(\mathbf{D}_X F^\cdot)$ we must have $d_{X_\alpha} \leq -j$. This completes the proof. □

**Corollary 8.1.23.** *Assume that $X$ is a connected complex manifold and all the cohomology sheaves of $F^\cdot \in D^b_c(X)$ are locally constant on $X$. Then*

(i) $F^\cdot \in {}^pD^{\leqslant 0}_c(X)$ *if and only if* $H^j(F^\cdot) = 0$ *for any* $j > -d_X$.
(ii) $F^\cdot \in {}^pD^{\geqslant 0}_c(X)$ *if and only if* $H^j(F^\cdot) = 0$ *for any* $j < -d_X$.

**Proposition 8.1.24.** *Let $F^\cdot \in D^b_c(X)$. Then the following four conditions are equivalent:*

(i) $F^\cdot \in {}^pD^{\geqslant 0}_c(X)$.
(ii) *For any locally closed analytic subset $S$ of $X$ we have*

$$H^j(i^!_S(F^\cdot)) = 0 \quad \textit{for any} \quad j < -d_S.$$

(iii) *For any locally closed analytic subset $S$ of $X$ we have*

$$H^j_S(F^\cdot) = H^j \mathrm{R}\Gamma_S(F^\cdot) = 0 \quad \textit{for any} \quad j < -d_S.$$

(iv) *For any locally closed "smooth" analytic subset $S$ of $X$ we have*

$$H^j(i^!_S(F^\cdot)) = 0 \quad \textit{for any} \quad j < -d_S.$$

*Proof.* First, since we have $H^i_S(\bullet) = H^i Ri_{S*} i^!_S(\bullet)$, the conditions (ii) and (iii) are equivalent. Let us prove the equivalence of (ii) and (iv). Assume that the condition (iv) is satisfied for $F^\cdot \in D^b_c(X)$. We will show that

$$H^j(i^!_Z(F^\cdot)) = 0 \quad \text{for any} \quad j < -d_Z. \tag{8.1.5}$$

for any locally closed (possibly singular) analytic subset $Z$ of $X$ by induction on $\dim Z$. Denote by $Z_{\mathrm{reg}}$ the smooth part of $Z$ and set $Z' = Z \setminus Z_{\mathrm{reg}}$. Then $\dim Z' < \dim Z$ and our hypothesis of induction implies that

$$H^j_{Z'}(F^\cdot) = 0 \quad \text{for any} \quad j < -d_{Z'}.$$

In particular, we have

$$H^j_{Z'}(F^\cdot) = 0 \quad \text{for any} \quad j < -d_Z.$$

So the assertion (8.1.5) follows from (iv) and the distinguished triangle

$$\mathrm{R}\Gamma_{Z'}(F^\cdot) \longrightarrow \mathrm{R}\Gamma_Z(F^\cdot) \longrightarrow \mathrm{R}\Gamma_{Z_{\mathrm{reg}}}(F^\cdot) \xrightarrow{+1} .$$

Now let us take a complex stratification $X = \sqcup_{\alpha\in A} X_\alpha$ of $X$ consisting of connected strata such that $i_{X_\alpha}^{-1}F^\cdot$ and $i_{X_\alpha}^!F^\cdot$ have locally constant cohomology sheaves for any $\alpha \in A$. Then by Proposition 8.1.22 the condition (i) is equivalent to the one $H^j(i_{X_\alpha}^!F^\cdot) = 0$ for any $\alpha$ and $j < -d_{X_\alpha}$. Therefore, if we take $S = X_\alpha$ in (iv) we see that (iv) implies (i). It remains to prove that (i) implies (iv). Assume that $H^j(i_{X_\alpha}^!F^\cdot) = 0$ for any $\alpha$ and $j < -d_{X_\alpha}$. For any locally closed (smooth) analytic subset $S$ in $X$, we need to show that

$$H^j(i_S^!(F^\cdot)) = 0 \quad \text{for any} \quad j < -d_S.$$

Set $X_k = \bigsqcup_{\dim X_\alpha \le k} X_\alpha$ in $X$ ($k = -1, 0, 1, \ldots, d_X$). Since

$$X_{-1} = \emptyset \subset X_0 \subset \cdots \subset X_{d_X} = X,$$

it is enough to prove the following assertions $(\mathrm{P})_k$ by induction on $k$:

$$(\mathrm{P})_k: \quad H^j_{S\cap X_k}(F^\cdot) = 0 \quad \text{for any} \quad j < -d_S. \tag{8.1.6}$$

Moreover, by the distinguished triangles

$$\mathrm{R}\Gamma_{S\cap X_{k-1}}(F^\cdot) \longrightarrow \mathrm{R}\Gamma_{S\cap X_k}(F^\cdot) \longrightarrow \mathrm{R}\Gamma_{S\cap(X_k\setminus X_{k-1})}(F^\cdot) \xrightarrow{+1}$$

for $k = 0, 1, \ldots, d_X$, we can reduce the problem to the proof of the assertions

$$(\mathrm{Q})_k: \quad H^j_{S\cap(X_k\setminus X_{k-1})}(F^\cdot) = 0 \quad \text{for any} \quad j < -d_S. \tag{8.1.7}$$

Note that $X_k \setminus X_{k-1}$ is the union of $k$-dimensional strata. Hence we obtain a direct sum decomposition

$$H^j_{S\cap(X_k\setminus X_{k-1})}(F^\cdot) \simeq \bigoplus_{\dim X_\alpha = k} H^j_{S\cap X_\alpha}(F^\cdot)$$

and it remains to show $H^j(i^!_{S\cap X_\alpha}F^\cdot) \simeq 0$ for any $\alpha \in A$ and $j < -d_S$. Decomposing $i_{S\cap X_\alpha} : S \cap X_\alpha \hookrightarrow X$ into $S \cap X_\alpha \xrightarrow{j_{X_\alpha}} X_\alpha \xrightarrow{i_{X_\alpha}} X$ we obtain an isomorphism $i^!_{S\cap X_\alpha}F^\cdot \simeq j^!_{X_\alpha}(i^!_{X_\alpha}(F^\cdot))$. Therefore, by applying Lemma 8.1.25 below to $Y = X_\alpha$ and $G^\cdot = i^!_{X_\alpha}(F^\cdot) \in D^b_c(Y)$ we obtain

$$H^j j^!_{X_\alpha}(G^\cdot) \simeq H^j i^!_{S\cap X_\alpha}(F^\cdot) \simeq 0 \quad \text{for any} \quad j < -d_{S\cap X_\alpha}.$$

This completes the proof. □

**Lemma 8.1.25.** *Let $Y$ be a complex manifold and $G^\cdot \in D^b_c(Y)$. Assume that all the cohomology sheaves of $G^\cdot$ are locally constant on $Y$ and for an integer $d \in \mathbb{Z}$ we have $H^jG^\cdot = 0$ for $j < d$. Then for any locally closed analytic subset $Z$ of $Y$ we have*

$$H^j_Z(G^\cdot) = 0 \quad \textit{for any} \quad j < d + 2\operatorname{codim}_Y Z.$$

*Proof.* By induction on the cohomological length of $G^\cdot$ we may assume that $G^\cdot$ is a local system $L$ on $Y$. Since the question is local on $Y$, we may further assume that $L$ is the constant sheaf $\mathbb{C}_Y$. Hence it is sufficient to show

$$H^j_Z(\mathbb{C}_Y) = 0 \quad \text{for any} \quad j < 2\operatorname{codim}_Y Z.$$

This well-known result can be proved by induction on the dimension of $Z$ with the help of the distinguished triangle

$$\mathrm{R}\Gamma_{Z\setminus Z_{\mathrm{reg}}}(\mathbb{C}_Y) \longrightarrow \mathrm{R}\Gamma_Z(\mathbb{C}_Y) \longrightarrow \mathrm{R}\Gamma_{Z_{\mathrm{reg}}}(\mathbb{C}_Y) \xrightarrow{+1} . \qquad \square$$

It is easily seen that $F^\cdot \in D^b_c(X)$ belongs to ${}^pD^{\leqslant 0}_c(X)$ if and only if the condition

(i)* for any Zariski locally closed irreducible subvariety $S$ of $X$ there exists a Zariski open dense smooth subset $S_0$ of $S$ such that $H^j(i^{-1}_{S_0}F^\cdot)$ is a local system for any $j$ and $H^j(i^{-1}_{S_0}F^\cdot) = 0$ for any $j > -d_S$

is satisfied (see the proof of Lemma 7.2.9). Also by the proof of Proposition 8.1.24, we easily see that $F^\cdot \in D^b_c(X)$ belongs to ${}^pD^{\geqslant 0}_c(X)$ if and only if the condition

(ii)* for any Zariski locally closed irreducible subvariety $S$ of $X$ there exists a Zariski open dense smooth subset $S_0$ of $S$ such that $H^j(i^!_{S_0}F^\cdot)$ is a local system for any $j$ and $H^j(i^!_{S_0}F^\cdot) = 0$ for any $j < -d_S$

is satisfied.

**Proposition 8.1.26.** *Let $F^\bullet \in {}^pD^{\leqslant 0}_c(X)$ and $G^\bullet \in {}^pD^{\geqslant 0}_c(X)$.*

(i) *We have*

$$H^j(R\mathcal{H}om_{\mathbb{C}_X}(F^\bullet, G^\bullet)) = 0$$

*for any $j < 0$.*

(ii) *The correspondence*

$$\{\textit{open subsets of } X\} \ni U \longmapsto \operatorname{Hom}_{D^b(U)}(F^\bullet|_U,\ G^\bullet|_U)$$

*defines a sheaf on $X$.*

*Proof.* (i) Set $S = \bigcup_{j<0} \operatorname{supp}(H^j(R\mathcal{H}om_{\mathbb{C}_X}(F^\bullet, G^\bullet))) \subset X$. Assume that $S \neq \emptyset$. Let $i_S : S \to X$ be the embedding. For $j < 0$ we have

$$\operatorname{supp}(H^j R\mathcal{H}om_{\mathbb{C}_X}(F^\bullet, G^\bullet)) \subset S,$$

and hence

$$\begin{aligned} H^j R\mathcal{H}om_{\mathbb{C}_X}(F^\bullet, G^\bullet) &\simeq H^j(\mathrm{R}\Gamma_S R\mathcal{H}om_{\mathbb{C}_X}(F^\bullet, G^\bullet)) \\ &\simeq H^j(i_{S*}i^!_S R\mathcal{H}om_{\mathbb{C}_X}(F^\bullet, G^\bullet)) \\ &\simeq i_{S*}H^j(R\mathcal{H}om_{\mathbb{C}_S}(i^{-1}_S F^\bullet, i^!_S G^\bullet)). \end{aligned}$$

Our assumption $F^\cdot \in {}^pD_c^{\leqslant 0}(X)$ implies that

$$\dim \operatorname{supp}\bigl\{H^k(i_S^{-1}F^\cdot)\bigr\} \leq -k$$

for any $k \in \mathbb{Z}$, and the dimension of

$$Z := \bigcup_{k>-d_S} \operatorname{supp}\bigl\{H^k(i_S^{-1}F^\cdot)\bigr\} \subset S$$

is less than $d_S$. Therefore, we obtain $S_0 = S \setminus Z \neq \emptyset$ and $H^j i_{S_0}^{-1} F^\cdot = 0$ for any $j > -d_S$. On the other hand, we have $H^j i_S^! G^\cdot = 0$ for any $j < -d_S$. Hence we obtain $H^j R\mathcal{H}om_{\mathbb{C}_S}(i_S^{-1}F^\cdot, i_S^! G^\cdot)|_{S_0} = 0$ for any $j < 0$. But this contradicts our definition $S = \bigcup_{j<0} \operatorname{supp}\bigl\{H^j R\mathcal{H}om_{\mathbb{C}_X}(F^\cdot, G^\cdot)\bigr\}$.

(ii) By (i) we have

$$\begin{aligned}\operatorname{Hom}_{D^b(U)}(F^\cdot|_U, G^\cdot|_U) &= H^0\bigl(U, R\mathcal{H}om_{\mathbb{C}_X}(F^\cdot, G^\cdot)\bigr)\\ &= \Gamma\bigl(U, H^0(R\mathcal{H}om_{\mathbb{C}_X}(F^\cdot, G^\cdot))\bigr).\end{aligned}$$

Hence the correspondence $U \longmapsto \operatorname{Hom}_{D^b(U)}(F^\cdot|_U,\ G^\cdot|_U)$ gives a sheaf isomorphic to $H^0(R\mathcal{H}om_{\mathbb{C}_X}(F^\cdot, G^\cdot))$. □

Now we are ready to prove the following.

**Theorem 8.1.27.** *The pair* $({}^pD_c^{\leqslant 0}(X), {}^pD_c^{\geqslant 0}(X))$ *defines a t-structure on* $D_c^b(X)$.

*Proof.* Among the conditions of $t$-structures in Definition 8.1.1, (T1) is trivially satisfied and (T2) follows from Proposition 8.1.26 above. Let us show (T3). For $F^\cdot \in D_c^b(X)$, take a stratification $X = \bigsqcup_{\alpha\in A} X_\alpha$ of $X$ such that $i_{X_\alpha}^{-1}F^\cdot$ and $i_{X_\alpha}^! F^\cdot$ have locally constant cohomology sheaves for any $\alpha \in A$. Set $X_k = \bigsqcup_{\dim X_\alpha \leq k} X_\alpha \subset X$ $(k = -1, 0, 1, 2, \dots)$ and consider the following assertions:

$(\mathrm{S})_k$ {

There exists $F_0^\cdot \in {}^pD_c^{\leqslant 0}(X \setminus X_k)$, $F_1^\cdot \in {}^pD_c^{\geqslant 1}(X \setminus X_k)$, and a distinguished triangle

$$F_0^\cdot \longrightarrow F^\cdot|_{X\setminus X_k} \longrightarrow F_1^\cdot \xrightarrow{+1}$$

in $D_c^b(X \setminus X_k)$ such that $F_0^\cdot|_{X_\alpha}$ and $F_1^\cdot|_{X_\alpha}$ have locally constant cohomology sheaves for any $\alpha \in A$ satisfying $X_\alpha \subset X \setminus X_k$.

}

Note that what we want to prove is $(\mathrm{S})_{-1}$. We will show $(\mathrm{S})_k$'s by descending induction on $k \in \mathbb{Z}$. It is trivial for $k \gg 0$. Assume that $(\mathrm{S})_k$ holds. Let us prove $(\mathrm{S})_{k-1}$. Take a distinguished triangle

$$F_0^\cdot \longrightarrow F^\cdot|_{X\setminus X_k} \longrightarrow F_1^\cdot \xrightarrow{+1} \tag{8.1.8}$$

in $D_c^b(X \setminus X_k)$ as in $(\mathrm{S})_k$. Let $j : X \setminus X_k \hookrightarrow X \setminus X_{k-1}$ be the open embedding and $i : X_k \setminus X_{k-1} \hookrightarrow X \setminus X_{k-1}$ be the closed embedding. Since $j_!$ is left adjoint

to $j^!$, the morphism $F_0^\bullet \longrightarrow j^!(F^\bullet|_{X\setminus X_{k-1}}) \simeq F^\bullet|_{X\setminus X_k}$ gives rise to a morphism $j_!F_0^\bullet \longrightarrow F^\bullet|_{X\setminus X_{k-1}}$. Let us embed this morphism into a distinguished triangle

$$j_!F_0^\bullet \longrightarrow F^\bullet|_{X\setminus X_{k-1}} \longrightarrow G^\bullet \xrightarrow{+1} . \tag{8.1.9}$$

We also embed the composite of the morphisms $\tau^{\leqslant -k}i_!i^!G^\bullet \longrightarrow i_!i^!G^\bullet \longrightarrow G^\bullet$ into a distinguished triangle

$$\tau^{\leqslant -k}i_!i^!G^\bullet \longrightarrow G^\bullet \longrightarrow \tilde{F}_1^\bullet \xrightarrow{+1} . \tag{8.1.10}$$

We finally embed the composite of $F^\bullet|_{X\setminus X_{k-1}} \longrightarrow G^\bullet \longrightarrow \tilde{F}_1^\bullet$ into a distinguished triangle

$$\tilde{F}_0^\bullet \longrightarrow F^\bullet|_{X\setminus X_{k-1}} \longrightarrow \tilde{F}_1^\bullet \xrightarrow{+1} . \tag{8.1.11}$$

By our construction $\tilde{F}_0^\bullet|_{X_\alpha}$ and $\tilde{F}_1^\bullet|_{X_\alpha}$ have locally constant cohomology sheaves for any $\alpha \in A$ satisfying $X_\alpha \subset X \setminus X_{k-1}$. It remains to show $\tilde{F}_0^\bullet \in {}^pD_c^{\leqslant 0}(X \setminus X_{k-1})$ and $\tilde{F}_1^\bullet \in {}^pD_c^{\geqslant 1}(X \setminus X_{k-1})$. Applying the functor $j^{-1}(\bullet)$ to (8.1.10) and (8.1.9), we get an isomorphism $j^{-1}\tilde{F}_1^\bullet \simeq j^{-1}G^\bullet$ and a distinguished triangle

$$F_0^\bullet \longrightarrow F^\bullet|_{X\setminus X_k} \longrightarrow j^{-1}\tilde{F}_1^\bullet \xrightarrow{+1} .$$

Hence we have $j^{-1}\tilde{F}_1^\bullet \simeq F_1^\bullet$, and $j^{-1}\tilde{F}_0^\bullet \simeq F_0^\bullet$ by (8.1.8) and (8.1.11). Therefore, we have only to show that

(i) $H^j(i^{-1}\tilde{F}_0^\bullet) = 0$ for ${}^\forall j > -k$
(ii) $H^j(i^!\tilde{F}_1^\bullet) = 0$ for ${}^\forall j < -k+1$

in view of Proposition 8.1.22 (note that $X_k \setminus X_{k-1}$ is a union of $k$-dimensional strata). By applying the octahedral axiom to the three distinguished triangles

$$\begin{cases} j_!F_0^\bullet \longrightarrow F^\bullet|_{X\setminus X_{k-1}} \xrightarrow{f} G^\bullet \xrightarrow{+1} \\ \tilde{F}_0^\bullet \longrightarrow F^\bullet|_{X\setminus X_{k-1}} \xrightarrow{g\circ f} \tilde{F}_1^\bullet \xrightarrow{+1} \\ \tau^{\leqslant -k}i_!i^!G^\bullet \longrightarrow G^\bullet \xrightarrow{g} \tilde{F}_1^\bullet \xrightarrow{+1} \end{cases}$$

we obtain a distinguished triangle

$$j_!F_0^\bullet \longrightarrow \tilde{F}_0^\bullet \longrightarrow \tau^{\leqslant -k}i_!i^!G^\bullet \xrightarrow{+1} .$$

Hence we have $i^{-1}\tilde{F}_0^\bullet \simeq i^{-1}\tau^{\leqslant -k}i_!i^!G^\bullet \simeq i^{-1}i_!\tau^{\leqslant -k}i^!G^\bullet \simeq \tau^{\leqslant -k}i^!G^\bullet$. The assertion (i) is proved. By applying the functor $i^!$ to (8.1.10) we obtain a distinguished triangle

$$i^!\tau^{\leqslant -k}i_!i^!G^\bullet \longrightarrow i^!G^\bullet \longrightarrow i^!\tilde{F}_1^\bullet \xrightarrow{+1} .$$

We have $i^!\tau^{\leqslant -k}i_!i^!G \simeq i^!i_!\tau^{\leqslant -k}i^!G \simeq \tau^{\leqslant -k}i^!G$, and hence we obtain $i^!\tilde{F}_1^\bullet \simeq \tau^{\geqslant -k+1}(i^!G^\bullet)$ by this distinguished triangle. The assertion (ii) is also proved. □

**Definition 8.1.28.** The $t$-structure $({}^pD_c^{\leqslant 0}(X), {}^pD_c^{\geqslant 0}(X))$ of the triangulated category $D_c^b(X)$ is called the perverse $t$-structure. An object of its heart $\mathrm{Perv}(\mathbb{C}_X) = {}^pD_c^{\leqslant 0}(X) \cap {}^pD_c^{\geqslant 0}(X)$ is called a *perverse sheaf* on $X$. We denote by

$$ {}^p\tau^{\leqslant 0} : D_c^b(X) \longrightarrow {}^pD_c^{\leqslant 0}(X), \qquad {}^p\tau^{\geqslant 0} : D_c^b(X) \longrightarrow {}^pD_c^{\geqslant 0}(X) $$

the truncation functors with respect to the perverse $t$-structure. For $n \in \mathbb{Z}$ we define a functor

$$ {}^pH^n : D_c^b(X) \longrightarrow \mathrm{Perv}(\mathbb{C}_X) $$

by ${}^pH^n(F^{\cdot}) = {}^p\tau^{\leqslant 0}{}^p\tau^{\geqslant 0}(F^{\cdot}[n])$. For $F^{\cdot} \in D_c^b(X)$ its image ${}^pH^n(F^{\cdot})$ in $\mathrm{Perv}(\mathbb{C}_X)$ is called the $n$th *perverse cohomology* (or the $n$th perverse part) of $F^{\cdot}$.

Note that for any perverse sheaf $F^{\cdot} \in \mathrm{Perv}(\mathbb{C}_X)$ on $X$ we have $H^i(F^{\cdot}) = 0$ for $i \notin [-d_X, 0]$. By Proposition 8.1.11 a distinguished triangle

$$ F^{\cdot} \longrightarrow G^{\cdot} \longrightarrow H^{\cdot} \xrightarrow{+1} $$

in $D_c^b(X)$ gives rise to a long exact sequence

$$ \cdots \to {}^pH^{n-1}(H^{\cdot}) \to {}^pH^n(F^{\cdot}) \to {}^pH^n(G^{\cdot}) \to {}^pH^n(H^{\cdot}) \to {}^pH^{n+1}(F^{\cdot}) \to \cdots $$

in the abelian category $\mathrm{Perv}(\mathbb{C}_X)$.

By definition, being a perverse sheaf is a local property in the following sense. Let $X = \bigcup_{i \in I} U_i$ be an open covering of $X$. Then $F^{\cdot} \in D_c^b(X)$ is a perverse sheaf if and only if $F^{\cdot}|_{U_i}$ is so for any $i \in I$.

**Remark 8.1.29.** It can be shown that the correspondence

$$ \{\text{open subsets of } X\} \ni U \longmapsto \mathrm{Perv}(\mathbb{C}_U) $$

defines a *stack*, i.e., a kind of sheaf with values in categories. More precisely, let $X = \bigcup_{i \in I} U_i$ be an open covering and assume that we are given a family $F_i^{\cdot} \in \mathrm{Perv}(\mathbb{C}_{U_i})$ equipped with isomorphism $F_i|_{U_i \cap U_j} \simeq F_j|_{U_i \cap U_j}$ satisfying obvious compatibility conditions. Then we can glue it uniquely to get $F^{\cdot} \in \mathrm{Perv}(\mathbb{C}_X)$. This is the reason why we call a complex of sheaves $F^{\cdot} \in \mathrm{Perv}(\mathbb{C}_X)$ a perverse "sheaf."

**Definition 8.1.30.** For a perverse sheaf $F^{\cdot} \in \mathrm{Perv}(\mathbb{C}_X)$, we define the *support* $\mathrm{supp}\, F^{\cdot}$ of $F^{\cdot}$ to be the complement of the largest open subset $U \hookrightarrow X$ such that $F^{\cdot}|_U = 0$.

**Proposition 8.1.31.** *Assume that $X$ is a smooth algebraic variety or a complex manifold. Then for any local system $L$ on $X^{\mathrm{an}}$ we have $L[d_X] \in \mathrm{Perv}(\mathbb{C}_X)$.*

*Proof.* Assume that $X$ is a complex manifold. By $\omega_X{}^{\cdot} \simeq \mathbb{C}_X[2d_X]$ we have

$$ \mathbf{D}_X(L[d_X]) = R\mathcal{H}om_{\mathbb{C}_X}(L[d_X], \mathbb{C}_X[2d_X]) \simeq L^*[d_X], $$

where $L^*$ denotes the dual local system $\mathcal{H}om_{\mathbb{C}_X}(L, \mathbb{C}_X)$. Hence the assertion is clear. The proof for the case where $X$ is a smooth algebraic variety is the same. □

**Remark 8.1.32.** More generally, it is known that if $X$ is a pure-dimensional algebraic variety (resp. analytic space) which is locally a complete intersection, then we have $L[d_X] \in \mathrm{Perv}(\mathbb{C}_X)$ for any local system $L$ on $X^{\mathrm{an}}$ (see for instance [Di, Theorem 5.1.20]).

The following proposition is obvious in view of the definition of ${}^pD_c^{\leqslant 0}(X)$ and ${}^pD_c^{\geqslant 0}(X)$.

**Proposition 8.1.33.** *The Verdier duality functor* $\mathbf{D}_X : D_c^b(X) \to D_c^b(X)^{\mathrm{op}}$ *is t-exact and induces an exact functor*

$$\mathbf{D}_X : \mathrm{Perv}(\mathbb{C}_X) \longrightarrow \mathrm{Perv}(\mathbb{C}_X)^{\mathrm{op}}.$$

**Proposition 8.1.34.** *Let* $i : Z \hookrightarrow X$ *be an embedding of a closed subvariety* $Z$ *of* $X$. *Then the functor* $i_*$ *sends* $\mathrm{Perv}(\mathbb{C}_Z)$ *to* $\mathrm{Perv}(\mathbb{C}_X)$.

*Proof.* It is easily seen that $i_*$ sends ${}^pD_c^{\leqslant 0}(Z)$ to ${}^pD_c^{\leqslant 0}(X)$. Since $Z$ is closed, we have $i_* = i_! = \mathbf{D}_X \circ i_* \circ \mathbf{D}_Z$. Hence $i_*$ sends ${}^pD_c^{\geqslant 0}(Z)$ to ${}^pD_c^{\geqslant 0}(X)$. □

By Propositions 8.1.31 and 8.1.34 we obtain the following.

**Example 8.1.35.**

(i) Let $X$ be a (possibly singular) analytic space and $Y \subset X$ a smooth complex manifold contained in $X$ as a closed subset. Then for any local system $L$ on $Y$, the complex $i_{Y*}(L[d_Y]) \in D_c^b(X)$ is a perverse sheaf on $X$.

(ii) Let $X = \mathbb{C}$ and $U = \mathbb{C} \setminus \{0\} \overset{j}{\hookrightarrow} X$. Then for any local system $L$ on $U$, $Rj_*(L[1])$, $Rj_!(L[1]) = j_!(L[1]) \in D_c^b(X)$ ($j_!$ is an exact functor) are perverse sheaves on $X = \mathbb{C}$.

**Definition 8.1.36.** Let $X, Y$ be algebraic varieties (or analytic spaces). For a functor $F : D_c^b(X) \longrightarrow D_c^b(Y)$ of triangulated categories we define a functor ${}^pF : \mathrm{Perv}(\mathbb{C}_X) \longrightarrow \mathrm{Perv}(\mathbb{C}_Y)$ to be the composite of the functors

$$\mathrm{Perv}(\mathbb{C}_X) \hookrightarrow D_c^b(X) \xrightarrow{F} D_c^b(Y) \xrightarrow{{}^pH^0} \mathrm{Perv}(\mathbb{C}_Y).$$

Let $f : X \longrightarrow Y$ be a morphism of algebraic varieties or analytic spaces. Then we have functors

$${}^pf^{-1}, {}^pf^! : \mathrm{Perv}(\mathbb{C}_Y) \longrightarrow \mathrm{Perv}(\mathbb{C}_X).$$

Assume that $f : X \longrightarrow Y$ is a proper morphism. Then we also have functors

$${}^pRf_*, {}^pRf_! : \mathrm{Perv}(\mathbb{C}_X) \longrightarrow \mathrm{Perv}(\mathbb{C}_Y).$$

**Notation 8.1.37.** We sometimes denote the functors ${}^pRf_*$, ${}^pRf_!$ by ${}^pf_*$, ${}^pf_!$, respectively, to simplify our notation.

**Lemma 8.1.38.**

(i) *For an object $F^\cdot$ of $D^b_c(X)$ we have $F^\cdot = 0$ if and only if ${}^pH^j(F^\cdot) = 0$ for any $j \in \mathbb{Z}$.*

(ii) *A morphism $f : F^\cdot \longrightarrow G^\cdot$ in $D^b_c(X)$ is an isomorphism if and only if the morphism ${}^pH^j(f) : {}^pH^j(F^\cdot) \to {}^pH^j(G^\cdot)$ is an isomorphism for any $j \in \mathbb{Z}$.*

*Proof.* (i) Assume that ${}^pH^j(F^\cdot) = 0$ for any $j \in \mathbb{Z}$. Since $F^\cdot$ is represented by a bounded complex of sheaves, there exist integers $a \leq b$ such that $F^\cdot \in {}^pD_c^{\leqslant b}(X) \cap {}^pD_c^{\geqslant a}(X)$. In the distinguished triangle

$$ {}^p\tau^{\leqslant b-1}(F^\cdot) \longrightarrow F^\cdot \longrightarrow {}^p\tau^{\geqslant b}(F^\cdot) \xrightarrow{+1} $$

we have ${}^p\tau^{\geqslant b}(F^\cdot) \simeq {}^p\tau^{\geqslant b}{}^p\tau^{\leqslant b}(F^\cdot) \simeq {}^pH^b(F^\cdot)[-b] = 0$, and hence we have $F^\cdot \simeq {}^p\tau^{\leqslant b-1}(F^\cdot) \in {}^pD_c^{\leqslant b-1}(X) \cap {}^pD_c^{\geqslant a}(X)$. By repeating this procedure we finally obtain $F^\cdot \in {}^pD_c^{\leqslant a-1}(X) \cap {}^pD_c^{\geqslant a}(X)$, and hence $F^\cdot = 0$.

(ii) Assume that the morphism ${}^pH^j(f) : {}^pH^j(F^\cdot) \to {}^pH^j(G^\cdot)$ is an isomorphism for any $j \in \mathbb{Z}$. Embed the morphism $f : F^\cdot \longrightarrow G^\cdot$ into a distinguished triangle

$$ F^\cdot \xrightarrow{f} G^\cdot \longrightarrow H^\cdot \xrightarrow{+1} . $$

By considering the long exact sequence of perverse cohomologies associated to it we obtain ${}^pH^j(H^\cdot) \simeq 0$ for ${}^\forall j \in \mathbb{Z}$. Hence we have $H^\cdot = 0$ by (i). It follows that $F^\cdot \longrightarrow G^\cdot$ is an isomorphism. □

The following result is an obvious consequence of Proposition 8.1.5 (ii).

**Lemma 8.1.39.** *Let*

$$ F^\cdot \xrightarrow{f} G^\cdot \xrightarrow{g} H^\cdot \xrightarrow{+1} $$

*be a distinguished triangle in $D^b_c(X)$ and assume $F^\cdot \in {}^pD_c^{\leqslant 0}(X)$ and $H^\cdot \in {}^pD_c^{\geqslant 1}(X)$. Then we have $F^\cdot \simeq {}^p\tau^{\leqslant 0}(G^\cdot)$ and $H^\cdot \simeq {}^p\tau^{\geqslant 1}(G^\cdot)$.*

**Proposition 8.1.40.** *Let $f : Y \longrightarrow X$ be a morphism of algebraic varieties or analytic spaces such that* $\dim f^{-1}(x) \leq d$ *for any $x \in X$.*

(i) *For any $F^\cdot \in {}^pD_c^{\leqslant 0}(X)$ we have $f^{-1}(F^\cdot) \in {}^pD_c^{\leqslant d}(Y)$.*

(ii) *For any $F^\cdot \in {}^pD_c^{\geqslant 0}(X)$ we have $f^!(F^\cdot) \in {}^pD_c^{\geqslant -d}(Y)$.*

*Proof.* (i) For $F^\cdot \in {}^pD_c^{\leqslant 0}(X)$ we have

$$ \begin{aligned} \dim\big(\operatorname{supp} H^j(f^{-1}F^\cdot[d])\big) &= \dim\big(f^{-1}(\operatorname{supp} H^{j+d}(F^\cdot))\big) \\ &\leq \dim\big(\operatorname{supp} H^{j+d}(F^\cdot)\big) + d \leq -j - d + d = -j, \end{aligned} $$

and hence $f^{-1}F^\cdot[d] \in {}^pD_c^{\leqslant 0}(Y)$, Therefore, we have $f^{-1}F^\cdot \in {}^pD_c^{\leqslant d}(Y)$.

(ii) This follows from (i) in view of $f^! = \mathbf{D}_Y \circ f^{-1} \circ \mathbf{D}_X$ and Proposition 8.1.33. □

**Corollary 8.1.41.** *Let $Z$ be a locally closed subvariety of $X$ and let $i : Z \to X$ be the embedding.*

(i) *The functor $i^{-1} : D_c^b(X) \longrightarrow D_c^b(Z)$ is right $t$-exact with respect to the perverse $t$-structures.*
(ii) *The functor $i^! : D_c^b(X) \longrightarrow D_c^b(Z)$ is left $t$-exact with respect to the perverse $t$-structures.*

The following propositions are immediate consequences of Proposition 8.1.40 and Corollary 8.1.41 in view of Lemma 8.1.16.

**Proposition 8.1.42.** *Let $f : X \to Y$ be as in Proposition* 8.1.40.

(i) *For any $G^\cdot \in {}^pD_c^{\geqslant 0}(Y)$ such that $Rf_*(G^\cdot) \in D_c^b(X)$ we have $Rf_*(G^\cdot) \in {}^pD_c^{\geqslant -d}(X)$.*
(ii) *For any $G^\cdot \in {}^pD_c^{\leqslant 0}(Y)$ such that $Rf_!(G^\cdot) \in D_c^b(X)$ we have $Rf_!(G^\cdot) \in {}^pD_c^{\leqslant d}(X)$.*

**Proposition 8.1.43.** *Let $i : Z \to X$ be as in Corollary* 8.1.41.

(i) *For any $G^\cdot \in {}^pD_c^{\geqslant 0}(Z)$ such that $Ri_*(G^\cdot) \in D_c^b(X)$ we have $Ri_*(G^\cdot) \in {}^pD_c^{\geqslant 0}(X)$.*
(ii) *For any $G^\cdot \in {}^pD_c^{\leqslant 0}(Z)$ such that $i_!(G^\cdot) \in D_c^b(X)$ we have $i_!(G^\cdot) \in {}^pD_c^{\leqslant 0}(X)$.*

**Corollary 8.1.44.**

(i) *Let $j : U \hookrightarrow X$ be an inclusion of an open subset $U$ of $X$. Then $j^{-1} = j^!$ is $t$-exact with respect to the perverse $t$-structures and induces an exact functor*

$$ {}^pj^{-1} = {}^pj^! : \mathrm{Perv}(\mathbb{C}_X) \longrightarrow \mathrm{Perv}(\mathbb{C}_U). $$

(ii) *Let $i : Z \hookrightarrow X$ be an inclusion of a closed subvariety $Z$. Then $i_* = i_!$ is $t$-exact with respect to the perverse $t$-structures and induces an exact functor*

$$ {}^pi_* = {}^pi_! : \mathrm{Perv}(\mathbb{C}_Z) \longrightarrow \mathrm{Perv}(\mathbb{C}_X). $$

*Moreover, if we denote by $\mathrm{Perv}_Z(\mathbb{C}_X)$ the category of perverse sheaves on $X$ whose supports are contained in $Z$, then the functor ${}^pi^{-1} = {}^pi^! : \mathrm{Perv}_Z(\mathbb{C}_X) \longrightarrow \mathrm{Perv}(\mathbb{C}_Z)$ is well defined and induces an equivalence*

$$ \mathrm{Perv}_Z(\mathbb{C}_X) \underset{{}^pi_*={}^pi_!}{\overset{{}^pi^{-1}={}^pi^!}{\rightleftarrows}} \mathrm{Perv}(\mathbb{C}_Z) $$

*of categories. The functor ${}^pi^{-1}$ is the quasi-inverse of ${}^pi_*$.*

**Lemma 8.1.45.** *For an exact sequence $0 \longrightarrow F^\cdot \longrightarrow G^\cdot \longrightarrow H^\cdot \longrightarrow 0$ of perverse sheaves on $X$, we have* $\mathrm{supp}\, G^\cdot = \mathrm{supp}\, F^\cdot \cup \mathrm{supp}\, H^\cdot$.

*Proof.* We first show $\operatorname{supp} F^{\cdot} \subset \operatorname{supp} G^{\cdot}$. Set $U = X \setminus \operatorname{supp} G^{\cdot}$. It is enough to show $F^{\cdot}|_U = 0$. By Corollary 8.1.44 (i) we have an exact sequence

$$0 \longrightarrow F^{\cdot}|_U \xrightarrow{\phi} G^{\cdot}|_U$$

in the abelian category $\mathrm{Perv}(\mathbb{C}_U)$. Hence we obtain $F^{\cdot}|_U = 0$ from $G^{\cdot}|_U = 0$. The inclusion $\operatorname{supp} H^{\cdot} \subset \operatorname{supp} G^{\cdot}$ can be proved similarly. Let us show $\operatorname{supp} G^{\cdot} \subset \operatorname{supp} F^{\cdot} \cup \operatorname{supp} H^{\cdot}$. It is sufficient to show that if we have $F^{\cdot}|_U = H^{\cdot}|_U = 0$ for an open subset $U$ of $X$, then $G^{\cdot}|_U = 0$. This follows easily from Theorem 8.1.9 (ii). □

**Remark 8.1.46.** The formal properties of perverse sheaves on $X$ that we listed above carry rich information on the singularities of the base space $X$. Indeed, using perverse sheaves, we can easily recover and even extend various classical results in singularity theory. For example, see [Di], [Mas1], [Mas2], [NT], [Schu], [Tk2]. Note also that Kashiwara [Kas19] recently introduced an interesting $t$-structure on the category $D^b_{rh}(D_X)$ whose heart corresponds to the category of constructible sheaves on $X$ (here $X$ is a smooth algebraic variety) through the Riemann–Hilbert correspondence.

## 8.2 Intersection cohomology theory

### 8.2.1 Introduction

Let $X$ be an irreducible projective algebraic variety (or an irreducible compact analytic space) of dimension $d_X$. If $X$ is non-singular, then we have the Poincaré duality $H^i(X, \mathbb{C}_X) \simeq \left[H^{2d_X - i}(X, \mathbb{C}_X)\right]^*$ for any $0 \leq i \leq 2d_X$. However, for a general (singular) variety $X$, we cannot expect such a nice symmetry in its usual cohomology groups. The intersection cohomology theory of Goresky–MacPherson [GM1] is a new theory which enables us to overcome this problem. The basic idea in their theory is to replace the constant sheaf $\mathbb{C}_X$ with a new complex $\mathrm{IC}_X^{\cdot}[-d_X] \in D^b_c(X)$ of sheaves on $X$ and introduce the *intersection cohomology groups*

$$IH^i(X) = H^i\big(X, \mathrm{IC}_X^{\cdot}[-d_X]\big) \quad (0 \leq i \leq 2d_X)$$

by taking the hypercohomology groups of $\mathrm{IC}_X^{\cdot}$. Then we obtain a generalized Poincaré duality

$$IH^i(X) = \left[IH^{2d_X - i}(X)\right]^* \quad (0 \leq i \leq 2d_X)$$

for any projective variety $X$. Moreover, it turns out that intersection cohomology groups admit the Hodge decomposition. Indeed, Morihiko Saito constructed his theory of Hodge modules to obtain this remarkable generalization of the Hodge–Kodaira theory to singular varieties (see Section 8.3). To define the *intersection cohomology complex* $\mathrm{IC}_X^{\cdot} \in D^b_c(X)$ of $X$, first we take a constant perverse sheaf $\mathbb{C}_U[d_X]$ on a Zariski open dense subset $U$ of the smooth part $X_{\mathrm{reg}}$ of $X$. Then $\mathrm{IC}_X^{\cdot}$ is a "minimal" extension of $\mathbb{C}_U[d_X] \in \mathrm{Perv}(\mathbb{C}_U)$ to a perverse sheaf on the whole $X$. Let

us briefly explain the construction of $\mathrm{IC}_X^{\cdot}$. To begin with, take a Whitney stratification $X = \bigsqcup_{\alpha\in A} X_\alpha$ of $X$ and set $X_k = \bigsqcup_{\dim X_\alpha \le k} X_\alpha \subset X$ ($k = -1, 0, 1, 2, \dots$). Then we get a filtration

$$X = X_{d_X} \supset X_{d_X-1} \supset \cdots \supset X_0 \supset X_{-1} = \emptyset$$

of $X$ by closed analytic subsets $X_k$ of $X$ such that $X_k \setminus X_{k-1}$ is a smooth $k$-dimensional complex manifold for any $k \in \mathbb{Z}$. Set $U_k = X \setminus X_{k-1}$ and $j_k : U_k \hookrightarrow U_{k-1}$. Then we have

$$\emptyset = U_{d_X+1} \xrightarrow{j_{d_X+1}} U_{d_X} \xrightarrow{j_{d_X}} U_{d_X-1} \hookrightarrow \cdots \xrightarrow{j_2} U_1 \xrightarrow{j_1} U_0 = X$$

and the perverse sheaf $\mathrm{IC}_X^{\cdot}$ is, in this case, isomorphic to the complex

$$\left(\tau^{\leqslant -1} Rj_{1*}\right) \circ \left(\tau^{\leqslant -2} Rj_{2*}\right) \circ \cdots \circ \left(\tau^{\leqslant -d_X} Rj_{d_X *}\right)(\mathbb{C}_U[d_X])$$

for $U = U_{d_X} \subset X$. We can prove that the Verdier dual of $\mathrm{IC}_X^{\cdot}$ is isomorphic to $\mathrm{IC}_X^{\cdot}$ itself. Namely, we have $\mathbf{D}_X(\mathrm{IC}_X^{\cdot}) \simeq \mathrm{IC}_X^{\cdot}$. This self-duality of $\mathrm{IC}_X^{\cdot}$ is the main reason why the intersection cohomology groups of $X$ satisfy the generalized Poincaré duality. In order to see that this construction of $\mathrm{IC}_X^{\cdot}$ is canonical, it is, in fact, necessary to check that it does not depend on the choice of a Whitney stratification $X = \bigsqcup_{\alpha\in A} X_\alpha$. For this purpose, in [GM1] Goresky and MacPherson introduced a "maximal" filtration

$$X \supset \bar{X}_{d_X} \supset \bar{X}_{d_X-1} \supset \cdots \supset \bar{X}_0 \supset \bar{X}_{-1} = \emptyset$$

of $X$ (that is, any Whitney stratification $X = \bigsqcup_{\alpha\in A} X_\alpha$ of $X$ is finer than the stratification $X = \bigsqcup_{k\in\mathbb{Z}}(\bar{X}_k \setminus \bar{X}_{k-1})$). But here, we define the intersection cohomology complex $\mathrm{IC}_X^{\cdot}$ by using the perverse $t$-structures and prove that it is isomorphic to the complex

$$\left(\tau^{\leqslant -1} Rj_{1*}\right) \circ \cdots \circ \left(\tau^{\leqslant -d_X} Rj_{d_X *}\right)(\mathbb{C}_U[d_X])$$

whenever we fix a Whitney stratification $X = \bigsqcup_{\alpha\in A} X_\alpha$ of $X$. More generally, for any pair $(X, U)$ of an irreducible complex analytic space $X$ and its Zariski open dense subset $U$ ($j : U \hookrightarrow X$), we can introduce a functor

$${}^p j_{!*} : \mathrm{Perv}(\mathbb{C}_U) \longrightarrow \mathrm{Perv}(\mathbb{C}_X)$$

such that ${}^p j_{!*}(\mathbb{C}_U[d_X]) \simeq \mathrm{IC}_X^{\cdot}$ in the above case. Such an extension of a perverse sheaf on $U$ to the one on $X$ will be called a *minimal extension* or a Deligne–Goresky–MacPherson extension (D-G-M extension for short). In addition to the two fundamental papers [BBD] and [GM1] on this subject, we are also indebted to [Bor2], [CG], [Di], [G1], [Ki], [Na2], [Schu].

### 8.2.2 Minimal extensions of perverse sheaves

Let $X$ be an irreducible algebraic variety or an irreducible analytic space and $U$ a Zariski open dense subset of $X$. In what follows, we set $Z = X \setminus U$ and denote by $i : Z \hookrightarrow X$ and $j : U \hookrightarrow X$ the embeddings.

We say that a stratification $X = \bigsqcup_{\alpha\in A} X_\alpha$ of $X$ is *compatible* with $F^\cdot \in D^b_c(U)$ if $U = \bigsqcup_{\alpha\in B} X_\alpha$ for some $B \subset A$, and both $F^\cdot|_{X_\alpha}$, $\mathbf{D}_U F^\cdot|_{X_\alpha}$ have locally constant cohomology sheaves for any $\alpha \in B$. Such a stratification always exists if $X$ is an algebraic variety; however, it does not always exist in the case where $X$ is an analytic space.

**Example 8.2.1.** Regard $X = \mathbb{C}$ and $U = \mathbb{C} \setminus \{0\}$ as an analytic space and its Zariski open subset, respectively. Set $Y_n = \{1/n\}$ for each positive integer $n$. We further set $V = U \setminus (\bigcup_{n=1}^\infty Y_n)$. Then the stratification $U = V \sqcup (\bigsqcup_{n=1}^\infty Y_n)$ of $U$ (and any of its refinement) cannot be extended to that of $X$. In particular, if $F$ is a constructible sheaf on $U$ whose support is exactly $U \setminus V$, then there exists no stratification of $X$ compatible with $F$.

Assume that a stratification $X = \bigsqcup_{\alpha\in A} X_\alpha$ of $X$ is compatible with $F^\cdot \in D^b_c(U)$. By replacing it with its refinement we may assume that the stratification $X = \bigsqcup_{\alpha\in A} X_\alpha$ satisfies the Whitney condition (see Definition E.3.7). In this situation the cohomology sheaves of $Rj_*F^\cdot|_{X_\alpha}$ and $j_!F^\cdot|_{X_\alpha}$ are locally constant for any $\alpha \in A$. In particular, we have $Rj_*F^\cdot$, $j_!F^\cdot \in D^b_c(X)$.

Now let $F^\cdot$ be a perverse sheaf on $U$ and assume that there exists a Whitney stratification of $X$ compatible with $F^\cdot$. We shall consider the problem of extending $F^\cdot$ to a perverse sheaf on $X$.

By taking the 0th perverse cohomology ${}^pH^0$ of the canonical morphism $j_!F^\bullet \longrightarrow Rj_*F^\bullet$ we get a morphism ${}^pj_!F^\bullet \longrightarrow {}^pj_*F^\bullet$ (see Notation 8.1.37) in $\mathrm{Perv}(\mathbb{C}_X)$.

**Definition 8.2.2.** We denote by ${}^pj_{!*}F^\bullet$ the image of the canonical morphism

$$ {}^pj_!F^\bullet \longrightarrow {}^pj_*F^\bullet $$

in $\mathrm{Perv}(\mathbb{C}_X)$, and call it the *minimal extension* of $F^\bullet \in \mathrm{Perv}(\mathbb{C}_U)$.

In other words, the morphism ${}^pj_!F^\bullet \longrightarrow {}^pj_*F^\bullet$ factorizes as ${}^pj_!F^\bullet \twoheadrightarrow {}^pj_{!*}F^\bullet \hookrightarrow {}^pj_*F^\bullet$ ($\twoheadrightarrow$ is an epimorphism and $\hookrightarrow$ is a monomorphism) in $\mathrm{Perv}(\mathbb{C}_X)$. Moreover, by definition, in the algebraic case for any morphism $F^\bullet \to G^\bullet$ in $\mathrm{Perv}(\mathbb{C}_U)$ we obtain a canonical morphism ${}^pj_{!*}F^\bullet \to {}^pj_{!*}G^\bullet$ in $\mathrm{Perv}(\mathbb{C}_X)$.

**Proposition 8.2.3.** *For $F^\bullet \in \mathrm{Perv}(\mathbb{C}_U)$, we have $\mathbf{D}_X({}^pj_{!*}F^\bullet) \simeq {}^pj_{!*}(\mathbf{D}_U F^\bullet)$.*

*Proof.* By applying $\mathbf{D}_X$ to the sequence

$$ {}^pj_!F^\bullet \twoheadrightarrow {}^pj_{!*}F^\bullet \hookrightarrow {}^pj_*F^\bullet $$

of morphisms in $\mathrm{Perv}(\mathbb{C}_X)$ we obtain a sequence

$$ \mathbf{D}_X({}^pj_*F^\bullet) \twoheadrightarrow \mathbf{D}_X({}^pj_{!*}F^\bullet) \hookrightarrow \mathbf{D}_X({}^pj_!F^\bullet) $$

of morphisms in $\mathrm{Perv}(\mathbb{C}_X)$ by Proposition 8.1.33. Furthermore, it follows also from Proposition 8.1.33 that

$$\begin{cases} \mathbf{D}_X({}^pj_*F^\bullet) \simeq {}^pH^0\mathbf{D}_X(Rj_*F^\bullet) \simeq {}^pj_!(\mathbf{D}_U F^\bullet) \\ \mathbf{D}_X({}^pj_!F^\bullet) \simeq {}^pH^0\mathbf{D}_X(Rj_!F^\bullet) \simeq {}^pj_*(\mathbf{D}_U F^\bullet). \end{cases}$$

Therefore, we obtain a sequence

$$ {}^pj_!(\mathbf{D}_U F^\bullet) \longrightarrow\!\!\!\!\!\rightarrow \mathbf{D}_X({}^pj_{!*}F^\bullet) \lhook\joinrel\longrightarrow {}^pj_*(\mathbf{D}_U F^\bullet)$$

of morphisms in $\mathrm{Perv}(\mathbb{C}_X)$, which shows that we should have $\mathbf{D}_X({}^pj_{!*}F^\bullet) \simeq {}^pj_{!*}(\mathbf{D}_U F^\bullet)$. □

**Lemma 8.2.4.** *Let $U'$ be a Zariski open subset of $X$ containing $U$ such that we have $U' = \sqcup_{\alpha\in B'} X_\alpha$ for some $B' \subset A$. We denote by $j_1 : U \to U'$ and $j_2 : U' \to X$ the embeddings.*

(i) *We have ${}^pj_*F^\bullet \simeq {}^pj_{2*}{}^pj_{1*}F^\bullet$ and ${}^pj_!F^\bullet \simeq {}^pj_{2!}{}^pj_{1!}F^\bullet$.*
(ii) ${}^pj_{!*}F^\bullet \simeq {}^pj_{2!*}{}^pj_{1!*}F^\bullet$.

*Proof.* (i) Since the functors $Rj_{1*}$ and $Rj_{2*}$ are left $t$-exact, we have

$$ {}^pj_*F^\bullet = {}^pH^0(Rj_{2*}Rj_{1*}F^\bullet) \simeq {}^pH^0\big(Rj_{2*}{}^pH^0(Rj_{1*}F^\bullet)\big) = {}^pj_{2*}{}^pj_{1*}F^\bullet$$

by Proposition 8.1.15 (i). The proof of the assertion ${}^pj_!F^\bullet \simeq {}^pj_{2!}{}^pj_{1!}F^\bullet$ is similar.

(ii) Recall that ${}^pj_{1!*}F^\bullet$ is a subobject of ${}^pj_{1*}F^\bullet$ in $\mathrm{Perv}(\mathbb{C}_{U'})$ such that the morphism ${}^pj_{1!}F^\bullet \longrightarrow {}^pj_{1*}F^\bullet$ factorizes as

$$ {}^pj_{1!}F^\bullet \longrightarrow\!\!\!\!\!\rightarrow {}^pj_{1!*}F^\bullet \lhook\joinrel\longrightarrow {}^pj_{1*}F^\bullet.$$

By using the right $t$-exactness of ${}^pj_{2!}$ (Propositions 8.1.43 (ii) and 8.1.15 (ii)) and the left $t$-exactness of ${}^pj_{2*}$ (Proposition 8.1.43 (i) and Proposition 8.1.15 (ii)) we obtain

$$\begin{aligned} {}^pj_!F^\bullet = {}^pj_{2!} \circ {}^pj_{1!}F^\bullet &\longrightarrow\!\!\!\!\!\rightarrow {}^pj_{2!} \circ {}^pj_{1!*}F^\bullet \longrightarrow\!\!\!\!\!\rightarrow {}^pj_{2!*} \circ {}^pj_{1!*}F^\bullet \\ &\lhook\joinrel\longrightarrow {}^pj_{2*} \circ {}^pj_{1!*}F^\bullet \lhook\joinrel\longrightarrow {}^pj_{2*} \circ {}^pj_{1*}F^\bullet = {}^pj_*F^\bullet. \end{aligned}$$

It follows that we have ${}^pj_{2!*} \circ {}^pj_{1!*}F^\bullet \simeq {}^pj_{!*}F^\bullet$. □

**Proposition 8.2.5.** *The minimal extension $G^\bullet = {}^pj_{!*}F^\bullet$ of $F^\bullet \in \mathrm{Perv}(\mathbb{C}_U)$ is characterized as the unique perverse sheaf on $X$ satisfying the conditions*

(i) $G^\bullet|_U \simeq F^\bullet$,
(ii) $i^{-1}G^\bullet \in {}^pD_c^{\leqslant -1}(Z)$,
(iii) $i^!G^\bullet \in {}^pD_c^{\geqslant 1}(Z)$.

*Proof.* We first show that the minimal extension $G^\bullet = {}^pj_{!*}F^\bullet$ satisfies the conditions (i), (ii), (iii). Since the functor $j^{-1} = j^!$ is $t$-exact by Corollary 8.1.44 (i), we have

$$\begin{aligned} {}^pj_{!*}F^\bullet|_U &= j^{-1}\,\mathrm{Im}\big[{}^pj_!F^\bullet \longrightarrow {}^pj_*F^\bullet\big] \\ &= \mathrm{Im}\big[j^{-1}{}^pj_!F^\bullet \longrightarrow j^{-1}{}^pj_*F^\bullet\big] \\ &= \mathrm{Im}\big[{}^pH^0(j^{-1}Rj_!F^\bullet) \longrightarrow {}^pH^0(j^{-1}Rj_*F^\bullet)\big] \end{aligned}$$

$$\begin{aligned} &= \mathrm{Im}\big[F^{\cdot} \longrightarrow F^{\cdot}\big] \\ &= F^{\cdot}. \end{aligned}$$

Hence the condition (i) is satisfied. By the distinguished triangle

$$j_! j^{-1} G^{\cdot} \longrightarrow G^{\cdot} \longrightarrow i_* i^{-1} G^{\cdot} \xrightarrow{+1}$$

in $D_c^b(X)$ we obtain an exact sequence

$${}^pH^0(j_! j^{-1} G^{\cdot}) \longrightarrow {}^pH^0(G^{\cdot}) \longrightarrow {}^pH^0(i_* i^{-1} G^{\cdot}) \longrightarrow {}^pH^1(j_! j^{-1} G^{\cdot}).$$

By the definition of $G^{\cdot}$ we have ${}^pH^0(G^{\cdot}) = {}^pj_{!*}F^{\cdot}$. By (i) we have $j^{-1}G^{\cdot} = F^{\cdot}$, and hence ${}^pH^0(j_! j^{-1} G^{\cdot}) = {}^pj_! F^{\cdot}$. Moreover, we have ${}^pH^1(j_! j^{-1} G^{\cdot}) = 0$ by Proposition 8.1.43 (ii). Finally, the canonical morphism ${}^pj_! F^{\cdot} \longrightarrow {}^pj_{!*}F^{\cdot}$ is an epimorphism by the definition of ${}^pj_{!*}$. Therefore, we obtain ${}^pH^0(i_* i^{-1} G^{\cdot}) = 0$ by the above exact sequence. Since $i_*$ is $t$-exact, we have ${}^pH^0(i^{-1}G^{\cdot}) = 0$. Since $i^{-1}$ is right $t$-exact, we have $i^{-1}G^{\cdot} \in {}^pD_c^{\leqq 0}(Z)$. It follows that $i^{-1}G^{\cdot} \in {}^pD_c^{\leqq -1}(Z)$. Hence the condition (ii) is satisfied. The condition (iii) can be checked similarly to (ii) by using the distinguished triangle

$$i_* i^! G^{\cdot} \longrightarrow G^{\cdot} \longrightarrow R j_* j^{-1} G^{\cdot} \xrightarrow{+1}.$$

Let us show that $G^{\cdot} \in \mathrm{Perv}(\mathbb{C}_X)$ satisfying the conditions (i), (ii), (iii) is canonically isomorphic to ${}^pj_{!*}F^{\cdot}$. Since $j^{-1}(= j^!)$ is left adjoint to $Rj_*$ and right adjoint to $j_!$, we obtain canonical morphisms $j_! F^{\cdot} \longrightarrow G^{\cdot} \longrightarrow Rj_* F^{\cdot}$ in $D_c^b(X)$. Hence we obtain canonical morphisms ${}^pj_! F^{\cdot} \longrightarrow G^{\cdot} \longrightarrow {}^pj_* F^{\cdot}$ in $\mathrm{Perv}(\mathbb{C}_X)$. It is sufficient to show that ${}^pj_! F^{\cdot} \longrightarrow G^{\cdot}$ is an epimorphism and $G^{\cdot} \longrightarrow {}^pj_* F^{\cdot}$ is a monomorphism in $\mathrm{Perv}(\mathbb{C}_X)$. We only show that ${}^pj_! F^{\cdot} \longrightarrow G^{\cdot}$ is an epimorphism (the proof of the remaining assertion is similar). Since the cokernel of ${}^pj_! F^{\cdot} \longrightarrow G^{\cdot}$ is supported by $Z$, there exists an exact sequence

$${}^pj_! F^{\cdot} \longrightarrow G^{\cdot} \longrightarrow i_* H^{\cdot} \longrightarrow 0$$

for some $H^{\cdot} \in \mathrm{Perv}(\mathbb{C}_Z)$ (Corollary 8.1.44 (ii)). Since $i^{-1}$ is right $t$-exact, we have an exact sequence ${}^pi^{-1}G^{\cdot} \longrightarrow {}^pi^{-1} i_* H^{\cdot} \longrightarrow 0$. By Corollary 8.1.44 (ii) we have ${}^pi^{-1} i_* H^{\cdot} = i^{-1} i_* H^{\cdot} = H^{\cdot}$. Moreover, by our assumption (ii) we have ${}^pi^{-1}G^{\cdot} = 0$. It follows that $H^{\cdot} = 0$, and hence ${}^pj_! F^{\cdot} \longrightarrow G^{\cdot}$ is an epimorphism. □

**Corollary 8.2.6.** *Assume that $X$ is smooth. Then for any local system $L \in \mathrm{Loc}(X)$ on $X$ we have $L[d_X] \simeq {}^pj_{!*}(L|_U[d_X])$.*

*Proof.* By Proposition 8.2.5 it is sufficient to show $i^{-1}L[d_X] \in {}^pD_c^{\leqq -1}(Z)$ and $i^! L[d_X] \in {}^pD_c^{\geqq 1}(Z)$. By $d_Z < d_X$ we easily see that $i^{-1}L[d_X] \in {}^pD_c^{\leqq -1}(Z)$. Furthermore, we have

$$\begin{aligned} i^! L[d_X] &\simeq i^! \mathbf{D}_X \mathbf{D}_X (L[d_X]) \\ &\simeq \mathbf{D}_X i^{-1}(L^*[d_X]) \in {}^pD_c^{\geqq 1}(X), \end{aligned}$$

where $L^*$ is the dual local system of $L$. □

**Proposition 8.2.7.** *Let $F^\cdot \in \mathrm{Perv}(\mathbb{C}_U)$ be as above. Then*

(i) ${}^pj_*F^\cdot$ *has no non-trivial subobject whose support is contained in* $Z$.
(ii) ${}^pj_!F^\cdot$ *has no non-trivial quotient object whose support is contained in* $Z$.

*Proof.* (i) Let $G^\cdot \subset {}^pj_*F^\cdot$ be a subobject of ${}^pj_*F^\cdot$ such that $\operatorname{supp} G^\cdot \subset Z$. Then by Corollary 8.1.41 $i^!G^\cdot \simeq i^{-1}G^\cdot$ is a perverse sheaf on $Z$ and we obtain ${}^pi^!G^\cdot \simeq i^!G^\cdot$. Since we have $G^\cdot \simeq i_*i^!G^\cdot$, it suffices to show that ${}^pi^!G^\cdot \simeq 0$. Now let us apply the left $t$-exact functor ${}^pi^!$ to the exact sequence $0 \to G^\cdot \to {}^pj_*F^\cdot$. Then we obtain an exact sequence $0 \to {}^pi^!G^\cdot \to {}^pi^!{}^pj_*F^\cdot$. By Proposition 8.1.15 (i) and $i^!Rj_*F^\cdot \simeq 0$ we obtain ${}^pi^!{}^pj_*F^\cdot \simeq {}^pH^0(i^!Rj_*F^\cdot) \simeq 0$. Hence we get ${}^pi^!G^\cdot \simeq 0$. The proof of (ii) is similar. □

**Corollary 8.2.8.** *The minimal extension ${}^pj_{!*}F^\cdot$ has neither non-trivial subobject nor non-trivial quotient object whose support is contained in* $Z$.

*Proof.* By the definition of the minimal extension ${}^pj_{!*}F^\cdot$ the result follows immediately from Proposition 8.2.7. □

In the algebraic case we also have the following.

**Corollary 8.2.9.**

(i) *Let $0 \to F^\cdot \to G^\cdot$ be an exact sequence in* $\mathrm{Perv}(\mathbb{C}_U)$. *Then the associated sequence $0 \to {}^pj_{!*}F^\cdot \to {}^pj_{!*}G^\cdot$ in* $\mathrm{Perv}(\mathbb{C}_X)$ *is also exact.*
(ii) *Let $F^\cdot \to G^\cdot \to 0$ be an exact sequence in* $\mathrm{Perv}(\mathbb{C}_U)$. *Then the associated sequence ${}^pj_{!*}F^\cdot \to {}^pj_{!*}G^\cdot \to 0$ in* $\mathrm{Perv}(\mathbb{C}_X)$ *is also exact.*

*Proof.* (i) Since the kernel $K^\cdot$ of the morphism ${}^pj_{!*}F^\cdot \to {}^pj_{!*}G^\cdot$ is a subobject of ${}^pj_{!*}F^\cdot$ whose support is contained in $Z$, we obtain $K^\cdot \simeq 0$ by Corollary 8.2.8. The proof of (ii) is similar. □

**Corollary 8.2.10.** *Assume that $F^\cdot$ is a simple object in* $\mathrm{Perv}(\mathbb{C}_U)$. *Then the minimal extension ${}^pj_{!*}F^\cdot$ is also a simple object in* $\mathrm{Perv}(\mathbb{C}_X)$.

*Proof.* Let $G^\cdot \subset {}^pj_{!*}F^\cdot$ be a subobject of ${}^pj_{!*}F^\cdot$ in $\mathrm{Perv}(\mathbb{C}_X)$ and consider the exact sequence $0 \to G^\cdot \to {}^pj_{!*}F^\cdot \to H^\cdot \to 0$ associated to it. If we apply the $t$-exact functor $j^! = j^{-1}$ to it, then we obtain an exact sequence $0 \to j^{-1}G^\cdot \to F^\cdot \to j^{-1}H^\cdot \to 0$. Since $F^\cdot$ is simple, $j^{-1}G^\cdot$ or $j^{-1}H^\cdot$ is zero. In other words, $G^\cdot$ or $H^\cdot$ is supported by $Z$. It follows from Corollary 8.2.8 that $G^\cdot$ or $H^\cdot$ is zero. □

Now we focus our attention on the case where $U$ is smooth and $F^\cdot = L[d_X]$ for a local system $L \in \mathrm{Loc}(U)$ on $U$. Note that we assume that $X$ is irreducible as before. We can take a Whitney stratification $X = \bigsqcup_{\alpha\in A} X_\alpha$ of $X$ such that $U$ is a union of strata in it. In view of Lemma 8.2.4 and Corollary 8.2.6 we may assume that $U$ is the unique open stratum in considering the minimal extension of $L[d_X]$. In other words we fix a Whitney stratification $X = \bigsqcup_{\alpha\in A} X_\alpha$ of $X$ and consider the minimal extension ${}^pj_{!*}(L[d_X])$ of $L \in \mathrm{Loc}(U)$, where $U$ is the open stratum and $j : U \to X$ is the embedding.

Set $X_k = \bigsqcup_{\dim X_\alpha \leq k} X_\alpha$ for each $k \in \mathbb{Z}$. Then we get a filtration

$$X = X_{d_X} \supset X_{d_X-1} \supset \cdots \supset X_0 \supset X_{-1} = \emptyset$$

of $X$ by (closed) analytic subsets. Set $U_k = X \setminus X_{k-1} = \bigsqcup_{\dim X_\alpha \geq k} X_\alpha$ and consider the sequence

$$U = U_{d_X} \xhookrightarrow{j_{d_X}} U_{d_X-1} \xhookrightarrow{j_{d_X-1}} \cdots \xhookrightarrow{j_2} U_1 \xhookrightarrow{j_1} U_0 = X$$

of inclusions of open subsets in $X$.

**Proposition 8.2.11.** *In the situation as above, we have an isomorphism*

$${}^p j_{!*}(L[d_X]) \simeq \left(\tau^{\leqslant -1} R j_{1*}\right) \circ \cdots \circ \left(\tau^{\leqslant -d_X} R j_{d_X *}\right)(L[d_X]).$$

*Proof.* In view of Lemma 8.2.4 it is sufficient to show that for any perverse sheaf $F^\cdot$ on $U_k$ whose restriction to each stratum $X_\alpha \subset U_k$ has locally constant cohomology sheaves we have

$${}^p j_{k!*} F^\cdot \simeq \tau^{\leqslant -k} R j_{k*}(F^\cdot).$$

We will show that the conditions (i), (ii), (iii) of Proposition 8.2.5 is satisfied for $G^\cdot = \tau^{\leqslant -k} R j_{k*}(F^\cdot) \in D^b_c(U_{k-1})$. Since $U_k$ consists of strata with dimension $\geq k$, we have $H^r(F^\cdot) = 0$ for $r > -k$ by Proposition 8.1.22. It follows that $\left[\tau^{\leqslant -k} R j_{k*}(F^\cdot)\right]\big|_{U_k} \simeq F^\cdot$, and hence the condition (i) is satisfied. Set $Z = U_{k-1} \setminus U_k = \bigsqcup_{\dim X_\alpha = k-1} X_\alpha$ and let $i : Z \hookrightarrow U_k$ be the embedding. Then $i^{-1} G^\cdot$ has locally constant cohomology sheaves on each $(k-1)$-dimensional stratum $X_\alpha \subset Z$ and we have $H^r(i^{-1} G^\cdot) = 0$ for $r > -k$. It follows that $i^{-1} G^\cdot \in {}^p D^{\leqslant -1}_c(Z)$ by Proposition 8.1.22 (i), and hence the condition (ii) is satisfied. Consider the distinguished triangle

$$G^\cdot = \tau^{\leqslant -k} R j_{k*} F^\cdot \longrightarrow R j_{k*} F^\cdot \longrightarrow \tau^{\geqslant -k+1} R j_{k*} F^\cdot \xrightarrow{+1}.$$

Applying the functor $i^!$ to it, we get $i^! G^\cdot \simeq i^!(\tau^{\geqslant -k+1} R j_{k*} F^\cdot)[-1]$ because $i^! R j_{k*} F^\cdot \simeq 0$. Hence we have $H^r(i^! G^\cdot) = 0$ for $r \leq -k+1$. Since $i^! G^\cdot$ has locally constant cohomology sheaves on each $(k-1)$-dimensional stratum $X_\alpha \subset Z$, we have $i^! G^\cdot \in {}^p D^{\geqslant 1}_c(Z)$ by Proposition 8.1.22 (ii). The condition (iii) is also satisfied. □

**Corollary 8.2.12.** *There exists a canonical morphism* $(j_* L)[d_X] \to {}^p j_{!*}(L[d_X])$ *in* $D^b_c(X)$.

*Proof.* The result follows from the isomorphism

$$\tau^{\leqslant -d_X} {}^p j_{!*}(L[d_X]) \simeq (j_{1*} \circ j_{2*} \cdots \circ j_{d_X *})(L)[d_X] \simeq (j_* L)[d_X]$$

obtained by Proposition 8.2.11. □

**Definition 8.2.13.** For an irreducible algebraic variety (resp. an irreducible analytic space) $X$ we define its *intersection cohomology complex* $\mathrm{IC}_X^{\cdot} \in \mathrm{Perv}(\mathbb{C}_X)$ by

$$\mathrm{IC}_X^{\cdot} = {}^p j_{!*}(\mathbb{C}_{X_{\mathrm{reg}}^{\mathrm{an}}}[d_X]) \qquad (\text{resp. } \mathrm{IC}_X^{\cdot} = {}^p j_{!*}(\mathbb{C}_{X_{\mathrm{reg}}}[d_X]),$$

where $X_{\mathrm{reg}}$ denotes the regular part of $X$ and $j : X_{\mathrm{reg}} \hookrightarrow X$ is the embedding.

By Proposition 8.2.3 we have the following.

**Theorem 8.2.14.** *We have* $\mathbf{D}_X(\mathrm{IC}_X^{\cdot}) = \mathrm{IC}_X^{\cdot}$.

**Proposition 8.2.15.** *There exist canonical morphisms*

$$\mathbb{C}_X \longrightarrow \mathrm{IC}_X^{\cdot}[-d_X] \longrightarrow \omega_X^{\cdot}[-2d_X]$$

*in* $D_c^b(X)$.

*Proof.* By Corollary 8.2.12 there exists a natural morphism $\mathbb{C}_X \longrightarrow \mathrm{IC}_X^{\cdot}[-d_X]$. By taking the Verdier dual we obtain a morphism $\mathrm{IC}_X^{\cdot}[d_X] \longrightarrow \omega_X^{\cdot}$. □

**Definition 8.2.16.** Let $X$ be an irreducible analytic space. For $i \in \mathbb{Z}$ we set

$$\begin{cases} IH^i(X) := H^i(R\Gamma(X, \mathrm{IC}_X^{\cdot}[-d_X])), \\ IH_c^i(X) := H^i(R\Gamma_c(X, \mathrm{IC}_X^{\cdot}[-d_X])). \end{cases}$$

We call $IH^i(X)$ (resp. $IH_c^i(X)$) the $i$th *intersection cohomology group* (resp. the $i$th intersection cohomology group with compact supports) of $X$.

The following theorem is one of the most important results in intersection cohomology theory.

**Theorem 8.2.17.** *Let $X$ be an irreducible analytic space of dimension $d$. Then we have the* **generalized Poincaré duality**

$$IH^i(X) \simeq \left[IH_c^{2d-i}(X)\right]^*$$

*for any* $0 \leq i \leq 2d$.

*Proof.* Let $a_X : X \longrightarrow\!\!\!\!\rightarrow \{\mathrm{pt}\}$ be the unique morphism from $X$ to the variety $\{\mathrm{pt}\}$ consisting of a single point. Then we have an isomorphism

$$R\mathcal{H}om_{\mathbb{C}}(Ra_{X!}\mathrm{IC}_X^{\cdot}, \mathbb{C}) \simeq Ra_{X*}R\mathcal{H}om_{\mathbb{C}_X}(\mathrm{IC}_X^{\cdot}, \omega_X^{\cdot})$$

in $D^b(\{\mathrm{pt}\}) \simeq D^b(\mathrm{Mod}(\mathbb{C}))$ by the Poincaré–Verdier duality theorem. Since $R\mathcal{H}om_{\mathbb{C}_X}(\mathrm{IC}_X^{\cdot}, \omega_X^{\cdot}) = \mathbf{D}_X(\mathrm{IC}_X^{\cdot}) \simeq \mathrm{IC}_X^{\cdot}$ by Theorem 8.2.14, we get an isomorphism

$$\left[R\Gamma_c(X, \mathrm{IC}_X^{\cdot})\right]^* \simeq R\Gamma(X, \mathrm{IC}_X^{\cdot})$$

By taking the $(i-d)$th cohomology groups of both sides, we obtain the desired isomorphism. □

In what follows, let us set $H^i(X) = H^i(X, \mathbb{C}_X)$ and $H^i_c(X) = H^i_c(X, \mathbb{C}_X)$. By the Verdier duality theorem we have an isomorphism $H^{-i}(X, \omega_X^{\cdot}) \simeq [H^i_c(X)]^*$ for any $i \in \mathbb{Z}$. This hypercohomology group $H^{-i}(X, \omega_X^{\cdot})$ is called the $i$th *Borel–Moore homology group* of $X$ and denoted by $H^{BM}_i(X)$. If $X$ is complete (or compact), then $H^{BM}_i(X)$ is isomorphic to the usual homology group $H_i(X) = H_i(X, \mathbb{C})$ of $X$. By Proposition 8.2.15 we obtain the following.

**Proposition 8.2.18.** *There exist canonical morphisms*

$$H^i(X) \longrightarrow IH^i(X) \longrightarrow H^{BM}_{2d_X - i}(X).$$

*for any* $i \in \mathbb{Z}$.

The morphism $H^i(X) \longrightarrow H^{BM}_{2d_X-i}(X)$ can be obtained more directly as follows. Recall that the top-dimensional Borel–Moore homology group $H^{BM}_{2d_X}(X) = [H^{2d_X}_c(X)]^* \simeq \mathbb{C}$ of $X$ contains a canonical generator $[X]$ called the *fundamental class* of $X$ (see, for example, Fulton [F, Section 19.1]). Then by the cup product

$$H^i(X) \times H^{2d_X - i}_c(X) \longrightarrow H^{2d_X}_c(X)$$

and the morphism $H^{2d_X}_c(X) \longrightarrow \mathbb{C}$ obtained by the fundamental class $[X] \in [H^{2d_X}_c(X)]^*$ we obtain a bilinear map

$$H^i(X) \times H^{2d_X - i}_c(X) \longrightarrow \mathbb{C}.$$

This gives a morphism $H^i(X) \longrightarrow H^{BM}_{2d_X-i}(X) = [H^{2d_X-i}_c(X)]^*$.

**Proposition 8.2.19.** *Let $X$ be a projective variety with isolated singular points. Then we have*

$$IH^i(X) = \begin{cases} H^i(X_{\mathrm{reg}}) & 0 \leq i < d_X \\ \mathrm{Im}[H^{d_X}(X) \longrightarrow H^{d_X}(X_{\mathrm{reg}})] & i = d_X \\ H^i(X) & d_X < i \leq 2d_X. \end{cases}$$

*In particular we have $H^i(X_{\mathrm{reg}}) \simeq H^{2d_X - i}(X)$ for any $0 \leq i < d_X$.*

*Proof.* Let $p_1, p_2, \ldots, p_k$ be the singular points of $X$ and $j : X_{\mathrm{reg}} \hookrightarrow X$ the embedding. Then $X = X_{\mathrm{reg}} \sqcup (\sqcup_{i=1}^k \{p_i\})$ is a Whitney stratification of $X$ and we have $\mathrm{IC}_X^{\cdot}[-d_X] \simeq \tau^{\leqslant d_X - 1}(Rj_*\mathbb{C}_{X_{\mathrm{reg}}})$. Hence we obtain a distinguished triangle

$$\mathrm{IC}_X^{\cdot}[-d_X] \longrightarrow Rj_*\mathbb{C}_{X_{\mathrm{reg}}} \longrightarrow \tau^{\geqslant d_X}(Rj_*\mathbb{C}_{X_{\mathrm{reg}}}) \xrightarrow{+1}.$$

Applying the functor $\mathrm{R}\Gamma(X, \bullet)$, we easily see that

$$IH^i(X) = H^i(X, \mathrm{IC}_X^{\cdot}[-d_X]) \simeq H^i(X_{\mathrm{reg}})$$

for $0 \leq i < d_X$ and the morphism

$$IH^{d_X}(X) = H^{d_X}(X, \mathrm{IC}_X^{\cdot}[-d_X]) \hookrightarrow H^{d_X}(X_{\mathrm{reg}})$$

is injective. Now let us embed the canonical morphism $\mathbb{C}_X \longrightarrow \mathrm{IC}_X^{\cdot}[-d_X]$ (Proposition 8.2.15) into a distinguished triangle

$$\mathbb{C}_X \longrightarrow \mathrm{IC}_X^{\cdot}[-d_X] \longrightarrow F^{\cdot} \stackrel{+1}{\longrightarrow} .$$

Then $F^{\cdot}$ is supported by the zero-dimensional subset $\sqcup_{i=1}^{k}\{p_i\}$ of $X$ and $H^i(F^{\cdot}) = 0$ for any $i \geq d_X$. Therefore, applying $\mathrm{R}\Gamma(X, \bullet)$ to this distinguished triangle, we obtain

$$H^i(X) \simeq IH^i(X) = H^i(X, \mathrm{IC}_X^{\cdot}[-d_X])$$

for $d_X < i \leq 2d_X$ and the morphism

$$H^{d_X}(X) \twoheadrightarrow IH^{d_X}(X) = H^{d_X}(X, \mathrm{IC}_X^{\cdot}[-d_X])$$

is surjective. This completes the proof. □

For some classes of varieties with mild singularities, intersection cohomology groups are isomorphic to the usual cohomology groups. For example, let us recall the following classical notion.

**Definition 8.2.20.** Let $X$ be an algebraic variety or an analytic space. We say that $X$ is *rationally smooth* (or a rational homology "manifold") if for any point $x \in X$ we have

$$H^i_{\{x\}}(X, \mathbb{C}_X) = \begin{cases} \mathbb{C} & i = 2d_X \\ \\ 0 & \text{otherwise.} \end{cases}$$

By definition, rationally smooth varieties are pure-dimensional. Smooth varieties are obviously rationally smooth. Typical examples of rationally smooth varieties with singularities are complex surfaces with Kleinian singularities and moduli spaces of algebraic curves. More generally $V$-manifolds are rationally smooth. It is known that the usual cohomology groups of a rationally smooth variety satisfy Poincaré duality (see Corollary 8.2.22 below) and the hard Lefschetz theorem.

**Proposition 8.2.21.** *Let $X$ be a rationally smooth irreducible analytic space. Then the canonical morphisms $\mathbb{C}_X \to \omega_X^{\cdot}[-2d_X]$ and $\mathbb{C}_X \to \mathrm{IC}_X^{\cdot}[-d_X]$ are isomorphisms.*

*Proof.* Let $i_{\{x\}} : \{x\} \hookrightarrow X$ be the embedding. Then we have

$$i_{\{x\}}^{-1}\omega_X^{\cdot} = i_{\{x\}}^{-1}\mathbf{D}_X(\mathbb{C}_X) \simeq \mathbf{D}_{\{x\}}i_{\{x\}}^{!}(\mathbb{C}_X) \simeq \mathrm{RHom}_{\mathbb{C}}(\mathrm{R}\Gamma_{\{x\}}(X, \mathbb{C}_X), \mathbb{C})$$

and hence an isomorphism

$$H^{j-2d_X}[i_{\{x\}}^{-1}\omega_X^{\cdot}] \simeq [H^{2d_X-j}_{\{x\}}(X, \mathbb{C}_X)]^*$$

for any $j \in \mathbb{Z}$. Then the isomorphism $\mathbb{C}_X \simeq \omega_X^{\cdot}[-2d_X]$ follows from these isomorphisms and the rationally smoothness of $X$. Now let us set $F^{\cdot} := \mathbb{C}_X[d_X] \simeq \omega_X^{\cdot}[-d_X] \in D_c^b(X)$. Then the complex $F^{\cdot}$ satisfies the condition $\mathbf{D}_X(F^{\cdot}) \simeq F^{\cdot}$. Therefore, by Proposition 8.2.5 we can easily show that $F^{\cdot}$ is isomorphic to the intersection cohomology complex $\mathrm{IC}_X^{\cdot}$. □

**Corollary 8.2.22.** *Let $X$ be a rationally smooth irreducible analytic space. Then we have an isomorphism $H^i(X) \simeq IH^i(X)$ for any $i \in \mathbb{Z}$.*

**Definition 8.2.23.** Let $X$ be an irreducible algebraic variety or an irreducible analytic space. Let $U$ be a Zariski open dense subset of $X_{\mathrm{reg}}$ and $j : U \to X$ the embedding. For a local system $L \in \mathrm{Loc}(U)$ on $U$ we set

$$\mathrm{IC}_X(L)^{\cdot} = {}^p j_{!*}(L[d_X]) \in \mathrm{Perv}(\mathbb{C}_X)$$

and call it a *twisted intersection cohomology complex* of $X$. We sometimes denote $\mathrm{IC}_X(L)^{\cdot}[-d_X]$ by ${}^{\pi}L^{\cdot}$.

Let $X$ be an algebraic variety or an analytic space. Consider an irreducible closed subvariety $Y$ of $X$ and a simple object $L$ of $\mathrm{Loc}(Y_0)$ (i.e., an irreducible local system on $Y_0$), where $Y_0$ is a smooth Zariski open dense subset of $Y$. Then the minimal extension $\mathrm{IC}_Y(L)^{\cdot}$ of the locally constant perverse sheaf $L[d_Y]$ to $Y$ can be naturally considered as a perverse sheaf on $X$ by Corollary 8.1.44 (ii). By Corollary 8.2.10 and Lemma 8.2.24 below this perverse sheaf on $X$ is a simple object in $\mathrm{Perv}(\mathbb{C}_X)$. Moreover, it is well known that any simple object in $\mathrm{Perv}(\mathbb{C}_X)$ can be obtained in this way (see [BBD]).

**Lemma 8.2.24.** *Let $L$ be an irreducible local system on a smooth irreducible variety $X$. Then the locally constant perverse sheaf $L[d_X]$ on $X$ is a simple object in* $\mathrm{Perv}(\mathbb{C}_X)$.

*Proof.* Let $0 \to F_1^{\cdot} \to L[d_X] \to F_2^{\cdot} \to 0$ be an exact sequence in $\mathrm{Perv}(\mathbb{C}_X)$. Choose a Zariski open dense subset $U$ of $X$ on which $F_1^{\cdot}$ and $F_2^{\cdot}$ have locally constant cohomology sheaves and set $j : U \hookrightarrow X$. Then by Lemma 8.1.23 there exist local systems $M_1, M_2$ on $U$ such that $F_1^{\cdot}|_U \simeq M_1[d_X]$, $F_2^{\cdot}|_U \simeq M_2[d_X]$. Hence we obtain an exact sequence $0 \to M_1 \to L|_U \to M_2 \to 0$ of local systems on $U$. Since $M_1$ can be extended to the local system $j_*M_1 \subset j_*(L|_U) \simeq L$ of the same rank on $X$, it follows from the irreducibility of $L$ that $M_1$ or $M_2$ is zero. Namely, $F_1^{\cdot}$ or $F_2^{\cdot}$ is supported by $Z = X \setminus U$. Since $L[d_X] \simeq {}^p j_{!*}(L|_U[d_X])$ by Corollary 8.2.6, $F_1^{\cdot}$ or $F_2^{\cdot}$ should be zero by Corollary 8.2.8. □

**Remark 8.2.25.** Assume that $X$ is a complex manifold. For a $\mathbb{C}^{\times}$-invariant Lagrangian analytic subset $\Lambda$ of the cotangent bundle $T^*X$ denote by $\mathrm{Perv}_{\Lambda}(\mathbb{C}_X)$ the subcategory of $\mathrm{Perv}(\mathbb{C}_X)$ consisting of objects whose micro-supports are contained in $\Lambda$. From the results in some simple cases it is generally expected that for any $\Lambda \subset T^*X$ as above there exists a finitely presented algebra $R$ such that the category $\mathrm{Perv}_{\Lambda}(\mathbb{C}_X)$ is equivalent to that of finite-dimensional representations

of $R$. In some special (but important) cases this algebraic (or quiver) description of the category $\mathrm{Perv}_\Lambda(\mathbb{C}_X)$ was established by [GGM]. Recently this problem has been completely solved for any smooth projective variety $X$ and any $\Lambda \subset T^*X$ by S. Gelfand–MacPherson–Vilonen [GMV].

Now we state the decomposition theorem due to Beilinson–Bernstein–Deligne–Gabber (see [BBD]) without proofs.

**Theorem 8.2.26 (Decomposition theorem).** *For a proper morphism $f : X \longrightarrow Y$ of algebraic varieties, we have*

$$Rf_*\big[\mathrm{IC}_X{}^{\cdot}\big] \simeq \bigoplus_{k}^{\text{finite}} i_{k*}\mathrm{IC}_{Y_k}(L_k)^{\cdot}[l_k]. \tag{8.2.1}$$

*Here for each $k$, $Y_k$ is an irreducible closed subvariety of $Y$, $i_k : Y_k \hookrightarrow Y$ is the embedding, $L_k \in \mathrm{Loc}(Y_k')$ for some smooth Zariski open subset $Y_k'$ of $Y_k$, and $l_k$ is an integer.*

The proof relies on a deep theory of weights for étale perverse sheaves in positive characteristics. This result can be extended to analytic situation via the theory of Hodge modules (see Section 8.3 below).

**Corollary 8.2.27.** *Let $X$ be a projective variety and $\pi : \widetilde{X} \to X$ a resolution of singularities of $X$. Then $IH^i(X)$ is a direct summand of $H^i(\widetilde{X})$ for any $i \in \mathbb{Z}$.*

*Proof.* By the decomposition theorem we have

$$R\pi_*(\mathbb{C}_{\widetilde{X}}[d_X]) \simeq \mathrm{IC}_X{}^{\cdot} \oplus F^{\cdot}$$

for some $F^{\cdot} \in D_c^b(X)$, from which the result follows. □

**Remark 8.2.28.** The decomposition theorem has various important applications in geometric representation theory of reductive algebraic groups. For details we refer the reader to Lusztig [L2] and Chriss–Ginzburg [CG].

In general the direct image of a perverse sheaf is not necessarily a perverse sheaf. We will give a sufficient condition on a morphism $f : X \to Y$ so that $Rf_*(\mathrm{IC}_X{}^{\cdot})$ is perverse.

**Definition 8.2.29.** Let $f : X \to Y$ be a dominant morphism of irreducible algebraic varieties. We say that $f$ is *small* (resp. *semismall*) if the condition $\mathrm{codim}_Y\{y \in Y \mid \dim f^{-1}(y) \geq k\} > 2k$ (resp. $\mathrm{codim}_Y\{y \in Y \mid \dim f^{-1}(y) \geq k\} \geq 2k$) is satisfied for any $k \geq 1$.

Note that if $f : X \to Y$ is semismall then there exists a smooth open dense subset $U \subset Y$ such that $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ is a finite morphism. In particular we have $d_X = d_Y$.

**Proposition 8.2.30.** *Let $f : X \to Y$ be a dominant proper morphism of irreducible algebraic varieties. Assume that $X$ is rationally smooth.*

(i) *Assume that $f$ is semismall. Then the direct image $Rf_*(\mathbb{C}_{X^{\mathrm{an}}}[d_X])$ of the constant perverse sheaf $\mathbb{C}_{X^{\mathrm{an}}}[d_X] \simeq \mathrm{IC}_X^{\cdot}$ is a perverse sheaf on $Y$.*

(ii) *Assume that $f$ is small. Then we have an isomorphism $Rf_*(\mathbb{C}_{X^{\mathrm{an}}}[d_X]) \simeq \mathrm{IC}_Y(L)^{\cdot}$ for some $L \in \mathrm{Loc}(U)$, where $U$ is a smooth open subset of $Y$.*

*Proof.* (i) By $\mathbf{D}_Y Rf_* \simeq Rf_* \mathbf{D}_X$ ($f$ is proper) it suffices to check the condition $Rf_*(\mathbb{C}_{X^{\mathrm{an}}}[d_X]) \in {}^pD_c^{\leqslant 0}(X)$. This follows easily from $R^j f_*(\mathbb{C}_{X^{\mathrm{an}}})_y = H^j(f^{-1}(y), \mathbb{C})$ and the fact that $H^j(f^{-1}(y), \mathbb{C}) = 0$ for $j > 2 \dim f^{-1}(y)$.

(ii) By our assumption there exists an open subset $U \subset Y$ such that $f|_{f^{-1}(U)}$: $f^{-1}(U) \to U$ is a finite morphism. If necessary, we can shrink $U$ so that $U$ is smooth and $f_* \mathbb{C}_{X^{\mathrm{an}}}|_{U^{\mathrm{an}}} \in \mathrm{Loc}(U)$. We denote this local system by $L$. Set $Z = Y \setminus U$ and let $i : Z \hookrightarrow Y$ be the embedding. By Proposition 8.2.5 we have only to show that $i^{-1} Rf_*(\mathbb{C}_{X^{\mathrm{an}}}[d_X]) \in {}^pD_c^{\leqslant -1}(Z)$ and $i^! Rf_*(\mathbb{C}_{X^{\mathrm{an}}}[d_X]) \in {}^pD_c^{\geqslant 1}(Z)$. Again by $\mathbf{D}_Y Rf_* \simeq Rf_* \mathbf{D}_X$ it is enough to check only the condition $i^{-1} Rf_*(\mathbb{C}_{X^{\mathrm{an}}}[d_X]) \in {}^pD_c^{\leqslant -1}(Z)$. This can be shown by the argument used in the proof of (i). □

Recall that the normalization $\pi : \widetilde{X} \to X$ of a projective variety $X$ is a finite map which induces an isomorphism $\pi|_{\pi^{-1} X_{\mathrm{reg}}} : \pi^{-1} X_{\mathrm{reg}} \xrightarrow{\sim} X_{\mathrm{reg}}$.

**Corollary 8.2.31.** *Let $X$ be a projective variety and $\pi : \widetilde{X} \to X$ its normalization. Then we have an isomorphism*

$$R\pi_* \big[\mathrm{IC}_{\widetilde{X}}^{\cdot}\big] \simeq \mathrm{IC}_X^{\cdot}.$$

*In particular, there exists an isomorphism $IH^i(\widetilde{X}) \simeq IH^i(X)$ for any $i \in \mathbb{Z}$.*

Since the normalization $\pi : \widetilde{C} \to C$ of an algebraic curve $C$ is smooth, we obtain an isomorphism $IH^i(C) \simeq H^i(\widetilde{C})$ for any $i \in \mathbb{Z}$. However, this is not always true in higher-dimensional cases.

**Example 8.2.32.** Let $C$ be an irreducible plane curve defined by $C = \{(x_0 : x_1 : x_2) \in \mathbb{P}^2(\mathbb{C}) | x_0^3 + x_1^3 = x_0 x_1 x_2\}$. Since $C$ has an isolated singular point, we obtain

$$IH^i(C) = \begin{cases} \mathbb{C} & i = 0 \\ 0 & i = 1 \\ \mathbb{C} & i = 2 \end{cases}$$

by Proposition 8.2.19. In this case the normalization $\widetilde{C}$ of $C$ is isomorphic to $\mathbb{P}^1(\mathbb{C})$ and hence we observe that $IH^i(C) \simeq H^i(\widetilde{C})$ for any $i \in \mathbb{Z}$.

**Remark 8.2.33.** Proposition 8.2.30 has important consequences in representation theory. For example, let $G$ be a semisimple algebraic group over $\mathbb{C}$ and $B \subset G$ a Borel subgroup (see Chapter 9 for the definitions). For the flag variety $X = G/B$ of $G$ let us set $\widetilde{G} = \{(g, x) \in G \times X \mid gx = x\} \subset G \times X$. Then $\widetilde{G}$ is a smooth complex manifold because the second projection $\widetilde{G} \to X$ is a fiber bundle on $X$. Furthermore, it turns out that the first projection $f : \widetilde{G} \to G$ is small (see Lusztig [L1]). Therefore, via the Riemann–Hilbert correspondence, Proposition 8.2.30 implies that

$$H^j \int_f \mathcal{O}_{\widetilde{G}} = 0 \quad \text{for} \quad j \neq 0.$$

The remaining non-zero term $H^0 \int_f \mathcal{O}_{\widetilde{G}}$ is a regular holonomic system on $G$ and coincides with the one satisfied by the characters of representations (invariant distributions) of a real form of $G$ (see [HK1]). Harish-Chandra obtained many important results in representation theory through the detailed study of this system of equations. We also note that Proposition 8.2.30 and Theorem 8.2.36 below play crucial roles in the recent progress of the geometric Langlands program (see, for example, [MV]).

We end this section by presenting a beautiful application of the decomposition theorem due to Borho–MacPherson [BM] on the explicit description of the direct images of constant perverse sheaves. Let $f : X \to Y$ be a dominant projective morphism of irreducible algebraic varieties. Then by a well-known result in analytic geometry (see, for example, Thom [Th, p. 276]), there exists a complex stratification $Y = \sqcup_{\alpha \in A} Y_\alpha$ of $Y$ by connected strata $Y_\alpha$'s such that $f|_{f^{-1}Y_\alpha} : f^{-1}Y_\alpha \to Y_\alpha$ is a topological fiber bundle with the fiber $F_\alpha := f^{-1}(y_\alpha)$ $(y_\alpha \in Y_\alpha)$. Let us assume, moreover, that $f$ is semismall. Then we have $\operatorname{codim}_Y Y_\alpha \geq 2 \dim F_\alpha$ for any $\alpha \in A$. Note that for any $i > 2 \dim F_\alpha$ we have $H^i(F_\alpha) = 0$. In particular $[H^i Rf_*(\mathbb{C}_X)]|_{Y_\alpha} = 0$ for any $i > \operatorname{codim}_Y Y_\alpha$. Set $c_\alpha = \operatorname{codim}_Y Y_\alpha$ and denote by $L_\alpha$ the local system $[H^{c_\alpha} Rf_*(\mathbb{C}_X)]|_{Y_\alpha}$ on $Y_\alpha$.

**Definition 8.2.34.** We say that a stratum $Y_\alpha$ is *relevant* if the condition $c_\alpha = 2 \dim F_\alpha$ holds.

We easily see that $f$ is small if and only if the only relevant stratum is the open dense one. Moreover, a stratum $Y_\alpha$ is relevant if and only if $L_\alpha \neq 0$. For a relevant stratum $Y_\alpha$ the top-dimensional cohomology group $H^{c_\alpha}(F_\alpha) \simeq (L_\alpha)_{y_\alpha}$ of the fiber $F_\alpha$ has a basis corresponding to the $d_{F_\alpha}$-dimensional irreducible components of $F_\alpha$. The fundamental group of $Y_\alpha$ acts on the $\mathbb{C}$-vector space $H^{c_\alpha}(F_\alpha) \simeq (L_\alpha)_{y_\alpha}$ by permutations of these irreducible components. This action completely determines the local system $L_\alpha$. For each $\alpha \in A$ let

$$L_\alpha = \bigoplus_\phi (L_\phi)^{\oplus m_\phi}$$

be the irreducible decomposition of the local system $L_\alpha$, where $\phi$ ranges through the set of all irreducible representations of the fundamental group of $Y_\alpha$ and $m_\phi \geq 0$ is the multiplicity of the irreducible local system $L_\phi$ corresponding to $\phi$.

**Definition 8.2.35.** We say that a pair $(Y_\alpha, \phi)$ of a stratum $Y_\alpha$ and an irreducible representation of the fundamental group of $Y_\alpha$ is *relevant* if the stratum $Y_\alpha$ is relevant and $m_\phi \neq 0$.

**Theorem 8.2.36 (Borho–MacPherson [BM]).** *In the situation as above, assume, moreover, that $X$ is rationally smooth. Then the direct image $Rf_*(\mathbb{C}_X[d_X])$ of the constant perverse sheaf $\mathbb{C}_X[d_X] \simeq \mathrm{IC}_X^\cdot$ is explicitly given by*

$$Rf_*(\mathbb{C}_X[d_X]) \simeq \bigoplus_{(Y_\alpha,\phi)} [i_{\alpha *}\mathrm{IC}_{Z_\alpha}(L_\phi)^\cdot]^{\oplus m_\phi},$$

*where $(Y_\alpha, \phi)$ ranges through the set of all relevant pairs, $Z_\alpha$ is the closure of $Y_\alpha$ and $i_\alpha : Z_\alpha \hookrightarrow Y$ is the embedding.*

*Proof.* By Proposition 8.2.30 the direct image $Rf_*(\mathbb{C}_X[d_X])$ is a perverse sheaf on $Y$. By the decomposition theorem we can prove recursively that on each strata $Y_\beta$ it is written more explicitly as

$$Rf_*(\mathbb{C}_X[d_X]) \simeq \bigoplus_{(Y_\alpha,\phi)} [i_{\alpha *}\mathrm{IC}_{Z_\alpha}(L_\phi)^\cdot]^{\oplus n_\phi},$$

where $(Y_\alpha, \phi)$ ranges through the set of all pairs of $Y_\alpha$ and irreducible representations $\phi$ of the fundamental group of $Y_\alpha$, and $n_\phi$ are some non-negative integers. Indeed, let $Y_{\alpha_0}$ be the unique open dense stratum in $Y$. Then on $Y_{\alpha_0}$ the direct image $Rf_*(\mathbb{C}_X[d_X])$ is obviously written as

$$Rf_*(\mathbb{C}_X[d_X]) \simeq \bigoplus_{(Y_{\alpha_0},\phi)} (L_\phi[d_X])^{\oplus m_\phi},$$

where $(Y_{\alpha_0}, \phi)$ ranges through the set of relevant pairs. Namely, for any pair $(Y_\alpha, \phi)$ such that $\alpha = \alpha_0$ we have $n_\phi = m_\phi$. Let $Y_{\alpha_1}$ be a stratum such that $\mathrm{codim}_Y Y_{\alpha_1} = 1$. Since $H^i Rf_*(\mathbb{C}_X[d_X]) = 0$ on $Y_{\alpha_1}$ for $i \neq -d_X$ by the semismallness of $f$, for any pair $(Y_\alpha, \phi)$ such that $\alpha = \alpha_1$ we must have $n_\phi = 0$. Therefore, on $Y_{\alpha_0} \sqcup Y_{\alpha_1}$ we obtain an isomorphism

$$Rf_*(\mathbb{C}_X[d_X]) \simeq \bigoplus_{(Y_{\alpha_0},\phi)} [i_{\alpha_0 *}\mathrm{IC}_{Z_{\alpha_0}}(L_\phi)^\cdot]^{\oplus m_\phi},$$

where $(Y_{\alpha_0}, \phi)$ ranges through the set of relevant pairs. By repeating this argument, we can finally prove the theorem. □

**Remark 8.2.37.** By Theorem 8.2.36 we can geometrically construct and study representations of Weyl groups of semisimple algebraic groups. This is the so-called theory of Springer representations. For this very important application of Theorem 8.2.36, see Borho–MacPherson [BM]. As another subject where Theorem 8.2.36 is applied effectively we also point out the work by Göttsche [Go] computing the Betti numbers of Hilbert schemes of points of algebraic surfaces. Inspired by this result Nakajima [Na1] and Grojnowski [Gr] found a beautiful symmetry in the cohomology groups of these Hilbert schemes (see [Na2] for the details).

## 8.3 Hodge modules

### 8.3.1 Motivation

Let $k$ an the algebraic closure of the prime field $\mathbb{F}_p$ of characteristic $p$. Starting from a given separated scheme $X$ of finite type over $\mathbb{Z}$, we can construct by base changes, the schemes $X_{\mathbb{C}} = X \otimes_{\mathbb{Z}} \mathbb{C}$ over $\mathbb{C}$ and $X_k = X \otimes_{\mathbb{Z}} k$ over $k$. The scheme $X_k$ is regarded as the counterpart of $X_{\mathbb{C}}$ in positive characteristic. We have cohomology groups $H^*(X_{\mathbb{C}}^{\mathrm{an}}, \mathbb{Q})$, $H_c^*(X_{\mathbb{C}}^{\mathrm{an}}, \mathbb{Q})$ for the underlying analytic space $X_{\mathbb{C}}^{\mathrm{an}}$ of $X_{\mathbb{C}}$; the corresponding notions in positive characteristic are the so-called $l$-adic étale cohomology groups $H^*(X_k, \bar{\mathbb{Q}}_l)$, $H_c^*(X_k, \bar{\mathbb{Q}}_l)$ ($l$ is a prime number different from $p$). More precisely, under some suitable conditions on $X$ over $\mathbb{Z}$ we have the isomorphisms

$$H^*(X_{\mathbb{C}}^{\mathrm{an}}, \mathbb{Q}) \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}_l \simeq H^*(X_k, \bar{\mathbb{Q}}_l), \quad H_c^*(X_{\mathbb{C}}^{\mathrm{an}}, \mathbb{Q}) \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}_l \simeq H_c^*(X_k, \bar{\mathbb{Q}}_l),$$

which show that the étale topology for $X_k$ corresponds to the classical topology for $X_{\mathbb{C}}$. This correspondence can be extended to the level of local systems and perverse sheaves on $X_{\mathbb{C}}^{\mathrm{an}}$ and $X_k$.

For $X_k$ we have the Frobenius morphism, which is an operation peculiar to the case of positive characteristic. This allows us, compared with the case of $X_{\mathbb{C}}$, to develop a more detailed theory on $X_k$ by considering étale local systems and étale perverse sheaves endowed with the action of the Frobenius morphisms (Weil sheaves). That is the theory of weights for étale sheaves, which played an important role in the proof of the Weil conjecture [De3], [BBD].

However, Grothendieck's philosophy predicted the existence of the theory of weights for objects over $\mathbb{C}$. In the case of local systems, the theory of the variation of Hodge structures had been known as a realization of such a theory over $\mathbb{C}$ [De2]. In the case of general perverse sheaves, a theory of weight which is based on the Riemann–Hilbert correspondence was constructed, and gave the final answer to this problem. This is the theory of Hodge modules due to Morihiko Saito [Sa1], [Sa3].

In this section, we present a brief survey on the theory of Hodge modules.

### 8.3.2 Hodge structures and their variations

In this subsection we discuss standard notions on Hodge structures. For more precise explanations, refer to [De2], [GS].

Let $X$ be a smooth projective algebraic variety. The complexifications $H_{\mathbb{C}} = H^n(X^{\mathrm{an}}, \mathbb{C})$ of its rational cohomology groups $H = H^n(X^{\mathrm{an}}, \mathbb{Q})$ ($n \in \mathbb{Z}$) can be naturally identified with the de Rham cohomology groups. Moreover, in such cases, a certain family $\{ H^{p,q} \mid p, q \in \mathbb{N}, p + q = n \}$ of subspaces of $H_{\mathbb{C}}$ is defined by the theory of harmonic forms so that we have the Hodge decomposition

$$H_{\mathbb{C}} = \bigoplus_{p+q=n} H^{p,q}, \quad \bar{H}^{p,q} = H^{q,p}.$$

Here we denote the complex conjugation map of $H_{\mathbb{C}}$ with respect to $H$ by $h \mapsto \bar{h}$ $(h \in H_{\mathbb{C}})$. Let us set $F^p(H_{\mathbb{C}}) := \bigoplus_{i \geq p} H^{i,n-i}$. Then $F$ defines a decreasing filtration of $H_{\mathbb{C}}$ and the equality $H_{\mathbb{C}} = F^p \oplus \bar{F}^{n-p+1}$ holds for every $p$. We call it the Hodge filtration of $H_{\mathbb{C}}$. Note that we can reconstruct the subspaces $H^{p,q}$ from the Hodge filtration by using $H^{p,q} = F^p \cap \bar{F}^q$.

Therefore, it would be natural to define the notion of Hodge structures in the following way. Let $H$ be a finite-dimensional vector space over $\mathbb{Q}$ and $H_{\mathbb{C}}$ its complexification. Denote by $h \mapsto \bar{h}$ $(h \in H_{\mathbb{C}})$ the complex conjugation map of $H_{\mathbb{C}}$ and consider a finite decreasing filtration $F = \{ F^p(H_{\mathbb{C}}) \}_{p \in \mathbb{Z}}$ by subspaces in $H_{\mathbb{C}}$. That is, we assume that $F^p(H_{\mathbb{C}})$'s are subspaces of $H_{\mathbb{C}}$ satisfying $F^p(H_{\mathbb{C}}) \supset F^{p+1}(H_{\mathbb{C}})$ $(^\forall p)$ and $F^p(H_{\mathbb{C}}) = \{0\}$, $F^{-p}(H_{\mathbb{C}}) = H_{\mathbb{C}}$ $(p \gg 0)$. For an integer $n$, we say that the pair $(H,\ F)$ is a *Hodge structure* of *weight n* if the condition

$$H_{\mathbb{C}} = F^p \oplus \bar{F}^{n-p+1}$$

holds for any $p$. The filtration $F$ is called the *Hodge filtration*. In this case, if we set $H^{p,q} = F^p \cap \bar{F}^q$, then we obtain the Hodge decomposition

$$H_{\mathbb{C}} = \bigoplus_{p+q=n} H^{p,q}, \quad \bar{H}^{p,q} = H^{q,p}.$$

We can naturally define the morphisms between Hodge structures as follows. Given two Hodge structures $(H,\ F)$, $(H',\ F)$ of the same weight $n$, a linear map $f : H \to H'$ is called a *morphism of Hodge structures* if it satisfies the condition

$$f(F^p(H_{\mathbb{C}})) \subset F^p(H'_{\mathbb{C}})$$

for any $p$. Here we used the same symbol $f$ for the complexified map $H_{\mathbb{C}} \to H'_{\mathbb{C}}$ of $f$. Thus we have defined the category $SH(n)$ of the Hodge structures of weight $n$. The morphisms in $SH(n)$ are *strict* with respect to the Hodge filtration $F$. Namely, for $f \in \mathrm{Hom}_{SH(n)}\big((H,\ F),\ (H',\ F)\big)$ we always have

$$f(F^p(H_{\mathbb{C}})) = f(H_{\mathbb{C}}) \cap F^p(H'_{\mathbb{C}})$$

for any $p \in \mathbb{Z}$, from which we see that $SH(n)$ is an abelian category.

Next let us explain the *polarizations of Hodge structures*. We say a Hodge structure $(H,\ F) \in SH(n)$ is *polarizable* if there exists a bilinear form $S$ on $H_{\mathbb{C}}$ satisfying the following properties:

(i) If $n$ is even, $S$ is symmetric. If $n$ is odd, $S$ is anti-symmetric.
(ii) If $p + p' \neq n$, we have $S(H^{p,n-p},\ H^{p',n-p'}) = 0$.
(iii) For any $v \in H^{p,n-p}$ such that $v \neq 0$ we have $(\sqrt{-1})^{n-2p} S(v,\ \bar{v}) > 0$.

Denote by $SH(n)^p$ the full subcategory of $SH(n)$ consisting of polarizable Hodge structures of weight $n$. Then it turns out that $SH(n)^p$ is an abelian category, and any object from it can be expressed as a direct sum of irreducible objects.

As we have explained above, for a smooth projective algebraic variety $X$, a natural Hodge structure of weight $n$ can be defined on $H^n(X^{\mathrm{an}},\ \mathbb{Q})$; however, the situation is

much more complicated for non-projective varieties with singularities. In such cases, the cohomology group $H^n(X^{\mathrm{an}}, \mathbb{Q})$ is a sort of mixture of the Hodge structures of various weights. This is the theory of mixed Hodge structures due to Deligne.

Let us give the definition of mixed Hodge structures. Let $H$ be a finite-dimensional vector space over $\mathbb{Q}$, $F$ a finite decreasing filtration of $H_{\mathbb{C}}$ and $W = \{W_n(H)\}_{n\in\mathbb{Z}}$ a finite increasing filtration of $H$. By complexifying $W$, we get an increasing filtration of $H_{\mathbb{C}}$, which we also denote by $W$. Then the complexification of $\mathrm{gr}_n^W(H) = W_n(H)/W_{n-1}(H)$ is identified with $W_n(H_{\mathbb{C}})/W_{n-1}(H_{\mathbb{C}})$ and its decreasing filtration is defined by

$$\tilde{F}^p(W_n(H_{\mathbb{C}})/W_{n-1}(H_{\mathbb{C}})) = (W_n(H_{\mathbb{C}}) \cap F^p(H_{\mathbb{C}}) + W_{n-1}(H_{\mathbb{C}}))/W_{n-1}(H_{\mathbb{C}}).$$

We say that a triplet $(H,\ F,\ W)$ is a *mixed Hodge structure* if for any $n$ the filtered vector space $\mathrm{gr}_n^W(H,\ F) := (\mathrm{gr}_n^W(H),\ \tilde{F})$ is a Hodge structure of weight $n$. We can naturally define the morphisms between mixed Hodge structures, and hence the category $SHM$ of mixed Hodge structures is defined. Let $SHM^p$ be the full subcategory of $SHM$ consisting of objects $(H,\ F,\ W) \in SHM$ such that $\mathrm{gr}_n^W(H,\ F) \in SH(n)^p$ for any $n$. They are abelian categories.

Next we explain the notion of the variations of Hodge structures, which naturally appears in the study of deformation (moduli) theory of algebraic varieties.

Let $f : Y \to X$ be a smooth projective morphism between two smooth algebraic varieties. Then the $n$th higher direct image sheaf $R^n f_*^{\mathrm{an}}(\mathbb{Q}_{Y^{\mathrm{an}}})$ is a local system on $X^{\mathrm{an}}$ whose stalk at $x \in X$ is isomorphic to $H^n(f^{-1}(x)^{\mathrm{an}}, \mathbb{Q})$. Since the fiber $f^{-1}(x)$ is a smooth projective variety, there exists a Hodge structure of weight $n$ on $H^n(f^{-1}(x)^{\mathrm{an}}, \mathbb{Q})$. Namely, the sheaf $R^n f_*^{\mathrm{an}}(\mathbb{Q}_{Y^{\mathrm{an}}})$ is a local system whose stalks are Hodge structures of weight $n$. Extracting properties of this local system, we come to the definition of the variations of Hodge structures as follows.

Let $X$ be a smooth algebraic variety and $H$ a $\mathbb{Q}$-local system on $X^{\mathrm{an}}$. Then by a theorem of Deligne (Theorem 5.3.8), there is a unique (up to isomorphisms) regular integrable connection $\mathcal{M}$ on $X$ such that $DR(\mathcal{M}) = \mathbb{C} \otimes_{\mathbb{Q}} H\,[\dim X]$. Let us denote this regular integrable connection $\mathcal{M}$ by $\mathcal{M}(H)$. Assume that $F = \{F^p(\mathcal{M}(H))\}_{p\in\mathbb{Z}}$ is a finite decreasing filtration of $\mathcal{M}(H)$ by $\mathcal{O}_X$-submodules such that $F^p/F^{p+1}$ is a locally free $\mathcal{O}_X$-module for any $p$. Namely, $F$ corresponds to a filtration of the associated complex vector bundle by its subbundles. Since the complexification $(H_x)_{\mathbb{C}}$ of the stalk $H_x$ of $H$ at $x \in X$ coincides with $\mathbb{C} \otimes_{\mathcal{O}_{X,x}} \mathcal{M}(H)_x$, a decreasing filtration $F(x)$ of $(H_x)_{\mathbb{C}}$ is naturally defined by $F$. Now we say that the pair $(H,\ F)$ is a *variation of Hodge structures* of weight $n$ if it satisfies the conditions

(i) For any $x \in X$, $(H_x, F(x)) \in SH(n)$.
(ii) For any $p \in \mathbb{Z}$, we have $\Theta_X \cdot F^p(\mathcal{M}(H)) \subset F^{p-1}(\mathcal{M}(H))$, where $\Theta_X$ stands for the sheaf of holomorphic vector fields on $X$.

The last condition is called *Griffiths transversality*. Also, we say that the variation of Hodge structures $(H,\ F)$ is *polarizable* if there exists a morphism

$$S : H \otimes_{\mathbb{Q}_{X^{\mathrm{an}}}} H \longrightarrow \mathbb{Q}_{X^{\mathrm{an}}}$$

of local systems which defines a polarization of $(H_x, F(x))$ at each point $x \in X$. We denote by $VSH(X, n)$ the category of the variations of Hodge structures on $X$ of weight $n$. Its full subcategory consisting of polarizable objects is denoted by $VSH(X, n)^p$. The categories $VSHM(X)$, $VSHM(X)^p$ of the variations of mixed Hodge structures on $X$ can be defined in the same way as $SHM$, $SHM^p$. All these categories $VSH(X, n)$, $VSH(X, n)^p$, $VSHM(X)$ and $VSHM(X)^p$ are abelian categories.

### 8.3.3 Hodge modules

The theory of variations of Hodge structures in the previous subsection may be regarded as the theory of weights for local systems. Our problem here is to extend it to the theory of weights for general perverse sheaves. The $\mathbb{Q}$-local systems $H$ appearing in the variations of Hodge structures should be replaced with perverse sheaves $K$ (over $\mathbb{Q}$). In this situation, a substitute for the regular integrable connection $\mathcal{M}(H)$ is a regular holonomic system $\mathcal{M}$ such that

$$DR(\mathcal{M}) = \mathbb{C} \otimes_{\mathbb{Q}} K.$$

Now, what is the Hodge filtration in this generalized setting? Instead of the decreasing Hodge filtration $F = \{F^p\}$ of $\mathcal{M}$, set $F_p = F^{-p}$ and let us now consider the increasing filtration $\{F_p\}$. Then Griffiths transversality can be rephrased as $\Theta_X \cdot F_p \subset F_{p+1}$. Therefore, for a general regular holonomic system $\mathcal{M}$, we can consider good filtrations in the sense of Section 2.1 as a substitute of Hodge filtrations.

Now, for a smooth algebraic variety $X$ denote by $MF_{rh}(D_X)$ the category of the pairs $(\mathcal{M}, F)$ of $\mathcal{M} \in \mathrm{Mod}_{rh}(D_X)$ and a good filtration $F$ of $M$. We also denote by $MF_{rh}(D_X, \mathbb{Q})$ the category of the triplets $(\mathcal{M}, F, K)$ consisting of $(\mathcal{M}, F) \in MF_{rh}(D_X)$ and a perverse sheaf $K \in \mathrm{Perv}(\mathbb{Q}_X)$ over $\mathbb{Q}$ such that $DR(\mathcal{M}) = \mathbb{C} \otimes_{\mathbb{Q}} K$. Also $MF_{rh}W(D_X, \mathbb{Q})$ stands for the category of quadruplets $(\mathcal{M}, F, K, W)$ consisting of $(\mathcal{M}, F, K) \in MF_{rh}(D_X, \mathbb{Q})$ and its finite increasing filtration $W = \{W_n\}$ in the category $MF_{rh}(D_X, \mathbb{Q})$. Although these categories are not abelian, they are additive categories. Under these definitions, we can show that $VSH(X, n)$ (resp. $VSHM(X)$) is a full subcategory of $MF_{rh}(D_X, \mathbb{Q})$ (resp. $MF_{rh}W(D_X, \mathbb{Q})$). Indeed, by sending $(H, F) \in VSH(X, n)$ (resp. $(H, F, W) \in VSHM(X)$) to $(\mathcal{M}(H), F, H[\dim X]) \in MF_{rh}(D_X, \mathbb{Q})$ (resp. $(\mathcal{M}(H), F, H[\dim X], W) \in MF_{rh}W(D_X, \mathbb{Q})$) we get the inclusions

$$\phi_X^n : VSH(X, n) \longrightarrow MF_{rh}(D_X, \mathbb{Q}),$$
$$\phi_X : VSHM(X) \longrightarrow MF_{rh}W(D_X, \mathbb{Q})$$

of categories. So our problem is now summarized in the following three parts:

(1) Define a full abelian subcategory of $MF_{rh}(D_X, \mathbb{Q})$ (resp. $MF_{rh}W(D_X, \mathbb{Q})$) consisting of objects of weight $n$ (resp. objects of mixed weights).
(2) Define the various operations, such as direct image, inverse image, duality functor for (the derived categories of) the abelian categories defined in (1).

(3) Show that the notions defined in (1), (2) satisfy the properties that deserves the name of the theory of weights. That is, prove basic theorems similar to those in the theory of weights in positive characteristic.

Morihiko Saito tackled these problems for several years and gave a decisive answer. In [Sa1], Saito defined the full abelian subcategory $MH(X, n)$ of $MF_{rh}(D_X, \mathbb{Q})$ consisting of the Hodge modules of weight $n$. He also gave the definition of the full abelian subcategory $MH(X, n)^p$ of $MH(X, n)$ consisting of polarizable objects and proved its stability through the direct images associated to projective morphisms. Next, in [Sa3], he defined the category $MHM(X)$ of mixed Hodge modules as a subcategory of $MF_{rh}W(D_X, \mathbb{Q})$ and settled the above problems (2), (3). Since the definition of *Hodge modules* requires many steps, we do not give their definition here and explain only their properties.

We first present some basic properties of the categories $MH(X, n)$ and $MH(X, n)^p$ following [Sa1]:

(p1) $MH(X, n)$, $MH(X, n)^p$ are full subcategories of $MF_{rh}(D_X, \mathbb{Q})$, and we have $MH(X, n)^p \subset MH(X, n)$.

(p2) (locality) Consider an open covering $X = \bigcup_{\lambda\in\Lambda} U_\lambda$ of $X$. For $\mathcal{V} \in MF_{rh}(D_X, \mathbb{Q})$ we have $\mathcal{V} \in MH(X, n)$ (resp. $MH(X, n)^p$) if and only if $\mathcal{V}|_{U_\lambda} \in MH(U_\lambda, n)$ (resp. $\mathcal{V}|_{U_\lambda} \in MH(U_\lambda, n)^p$).

(p3) All morphisms in $MH(X, n)$ and $MH(X, n)^p$ are strict with respect to the filtrations $F$.

We see from (p3) that $MH(X, n)$ and $MH(X, n)^p$ are abelian categories. For $\mathcal{V} = (\mathcal{M}, F, K) \in MH(X, n)$ the support $\mathrm{supp}(\mathcal{M})$ of $\mathcal{M}$ is called the support of $\mathcal{V}$ and we denote it by $\mathrm{supp}(\mathcal{V})$. This is a closed subvariety of $X$. Now let $Z$ be an irreducible closed subvariety of $X$. We say $\mathcal{V} = (\mathcal{M}, F, K) \in MH(X, n)$ has the *strict support* $Z$ if the support of $\mathcal{V}$ is $Z$ and there is neither subobject nor quotient object of $\mathcal{V}$ whose support is a non-empty proper subvariety of $Z$. Let us denote by $MH_Z(X, n)$ the full subcategory of $MH(X, n)$ consisting of objects having the strict support $Z$. We also set $MH_Z(X, n)^p = MH_Z(X, n) \cap MH(X, n)^p$. Then we have

(p4) $MH(X, n) = \bigoplus_Z MH_Z(X, n)$, $MH(X, n)^p = \bigoplus_Z MH_Z(X, n)^p$, where $Z$ ranges over the family of irreducible closed subvarieties of $X$.

If one wants to find a category satisfying only the conditions (p1)–(p4) one can set $MH(X, n) = \{0\}$ for all $X$; however, the category $MH(X, n)$ is really not trivial, as we will explain below.

For $\mathcal{V} = (\mathcal{M}, F, K) \in MF_{rh}(D_X, \mathbb{Q})$ and an integer $m$ we define $\mathcal{V}(m) \in MF_{rh}(D_X, \mathbb{Q})$ by

$$\mathcal{V}(m) = (\mathcal{M} \otimes_{\mathbb{Q}} \mathbb{Q}(m),\ F[m],\ K \otimes_{\mathbb{Q}} \mathbb{Q}(m))$$

(the *Tate twist* of $\mathcal{V}$), where $\mathbb{Q}(m) = (2\pi\sqrt{-1})^m\mathbb{Q} \subset \mathbb{C}$ and $F[m]_p = F_{p-m}$. The readers might feel it strange to write $\otimes_{\mathbb{Q}}\mathbb{Q}(m)$; however, this notation is natural from the viewpoint of Hodge theory.

(p5) $\phi_X^n(VSH(X,\,n)^p) \subset MH_X(X,\,n+\dim X)^p$.
(p6) If $\mathcal{V} \in MH(X,\,n)$ (resp. $MH(X,\,n)^p$), then $\mathcal{V}(m) \in MH(X,\,n-2m)$ (resp. $MH(X,\,n-2m)^p$).

From (p5), (p6) and the stability through projective direct images to be stated below, we see that $MH(X,\,n)$ contains many non-trivial objects (see also (m13) below).

Let $f : X \to Y$ be a projective morphism between two smooth algebraic varieties. Then the derived direct image $f_\bigstar(\mathcal{M},\,F)$ of $(\mathcal{M},\,F) \in MF_{rh}(D_X)$ is defined as a complex of filtered modules (more precisely it is an object of a certain derived category). As a complex of $D$-modules it is the ordinary direct image $\int_f \mathcal{M}$.

(p7) If $(\mathcal{M},\,F,\,K) \in MH(X,\,n)^p$, then the complex $f_\bigstar(\mathcal{M},\,F)$ is strict with respect to the filtrations and we have

$$(H^j f_\bigstar(\mathcal{M},\,F),\ {}^pH^j f_*(K)) \in MH(Y,\,n+j)^p.$$

for any $j \in \mathbb{Z}$ (${}^pH^j f_*(K)$ is the $j$th perverse part of the direct image complex $f_*(K)$, see Section 8.1).

Let us explain how the filtered complex $f_\bigstar(\mathcal{M},\,F)$ can be defined. First consider the case where $f$ is a closed embedding. In this case, for $j \neq 0$ we have $H^j \int_f \mathcal{M} = 0$ and $H^0 \int_f \mathcal{M} = f_*(D_{Y\leftarrow X} \otimes_{D_X} \mathcal{M})$. Using the filtration on $D_{Y\leftarrow X}$ induced from the one on $D_Y$, let us define a filtration on $\int_f \mathcal{M}$ by

$$F_p\Big(\int_f \mathcal{M}\Big) := f_*\Big(\sum_q F_q(D_{Y\leftarrow X}) \otimes F_{p-q+\dim X-\dim Y}(\mathcal{M})\Big).$$

Then this is the filtered complex $f_\bigstar(\mathcal{M},\,F)$ of sheaves. Next consider the case where $f$ is a projection $X = Y \times Z \to Y$ ($Z$ is a smooth projective variety of dimension $m$). Now $D_{Y\leftarrow X} \otimes^L_{D_X} \mathcal{M}$ can be expressed as the relative de Rham complex

$$DR_{X/Y}(\mathcal{M}) = [\Omega^0_{X/Y} \otimes_{\mathcal{O}_X} \mathcal{M} \to \Omega^1_{X/Y} \otimes_{\mathcal{O}_X} \mathcal{M} \to \cdots \to \Omega^m_{X/Y} \otimes_{\mathcal{O}_X} \mathcal{M}]$$

(here $\Omega^m_{X/Y} \otimes_{\mathcal{O}_X} \mathcal{M}$ is in degree 0). Hence a filtration of $DR_{X/Y}(\mathcal{M})$ as a complex of $f^{-1}D_Y$-modules is defined by

$$F_p(DR_{X/Y}(\mathcal{M})) = \big[\Omega^0_{X/Y} \otimes_{\mathcal{O}_X} F_p(\mathcal{M}) \to \Omega^1_{X/Y} \otimes_{\mathcal{O}_X} F_{p+1}(\mathcal{M}) \to \cdots \to \Omega^m \otimes_{\mathcal{O}_X} F_{p+m}(\mathcal{M})\big],$$

and its sheaf-theoretical direct image is $f_\bigstar(\mathcal{M},\,F)$. Note that the complex $f_\bigstar(\mathcal{M},\,F)$ is strict if and only if the morphism

$$H^j\big(Rf_*(F_p(DR_{X/Y}(\mathcal{M})))\big) \longrightarrow H^j\big(Rf_*(DR_{X/Y}(\mathcal{M}))\big) = H^j \int_f \mathcal{M}$$

is injective for any $j$ and $p$. If this is the case, $H^j \int_f \mathcal{M}$ becomes a filtered module by the filtration

$$F_p\Big(H^j\int_f \mathcal{M}\Big) = H^j\big(Rf_*(F_p(DR_{X/Y}(\mathcal{M})))\big).$$

We next present some basic properties of mixed Hodge modules following [Sa3]:

(m1) $MHM(X)$ is a full subcategory of $MF_{rh}W(D_X, \mathbb{Q})$, and subobjects and quotient objects (in the category $MF_{rh}W(D_X, \mathbb{Q})$) of an object of $MHM(X)$ are again in $MHM(X)$. That is, the category $MHM(X)$ is stable under the operation of taking subquotients in $MF_{rh}W(D_X, \mathbb{Q})$.

(m2) If $\mathcal{V} = (\mathcal{M}, F, K, W) \in MHM(X)$, then $\mathrm{gr}_n^W \mathcal{V} = \mathrm{gr}_n^W(\mathcal{M}, F, K) \in MH(X, n)^p$. Furthermore, if $\mathcal{V} \in MF_{rh}W(D_X, \mathbb{Q})$ satisfies $\mathrm{gr}_k^W \mathcal{V} = 0$ $(k \neq n)$, $\mathrm{gr}_n^W \mathcal{V} \in MH(X, n)^p$, then we have $\mathcal{V} \in MHM(X)$.

(m3) (locality) Let $X = \bigcup_{\lambda\in\Lambda} U_\lambda$ be an open covering of $X$. Then for $\mathcal{V} \in MF_{rh}W(D_X, \mathbb{Q})$ we have $\mathcal{V} \in MHM(X)$ if and only if $\mathcal{V}|_{U_\lambda} \in MHM(U_\lambda)$ for any $\lambda \in \Lambda$.

(m4) All morphisms in $MHM(X)$ are strict with respect to the filtrations $F$, $W$.

(m5) If $\mathcal{V} = (\mathcal{M}, F, K, W) \in MHM(X)$, then $\mathcal{V}(m) = (\mathcal{M} \otimes_{\mathbb{Q}} \mathbb{Q}(m), F[m], K \otimes_{\mathbb{Q}} \mathbb{Q}(m), W[-2m]) \in MHM(X)$.

It follows from (m4) that $MHM(X)$ is an abelian category. Also by (m2) the object

$$\mathbb{Q}_X^H[\dim X] := (\mathcal{O}_X, F, \mathbb{Q}_{X^{\mathrm{an}}}[\dim X], W)$$
$$(\mathrm{gr}_p^F = 0\ (p \neq 0),\ \mathrm{gr}_n^W = 0\ (n \neq \dim X))$$

in $MF_{rh}W(D_X, \mathbb{Q})$ belongs to $MHM(X)$. Let us define a functor

$$\mathrm{rat} : MHM(X) \longrightarrow \mathrm{Perv}(\mathbb{Q}_X)$$

by assigning $K \in \mathrm{Perv}(\mathbb{Q}_X)$ to $(\mathcal{M}, F, K, W) \in MHM(X)$. Then the functor rat induces also a functor

$$\mathrm{rat} : D^bMHM(X) \longrightarrow D^b(\mathrm{Perv}(\mathbb{Q}_X)) \simeq D_c^b(\mathbb{Q}_X)$$

of triangulated categories (i.e., it sends distinguished triangles in $D^bMHM(X)$ to those in $D_c^b(\mathbb{Q}_X)$), where we used the isomorphism $D^b(\mathrm{Perv}(\mathbb{Q}_X)) \simeq D_c^b(\mathbb{Q}_X)$ proved by [BBD] and [Bei]. In [Sa3] the functors

$$\mathbb{D} : MHM(X) \longrightarrow MHM(X)^{\mathrm{op}},$$
$$f_\bigstar,\ f_! : D^bMHM(X) \longrightarrow D^bMHM(Y),$$
$$f^\bigstar,\ f^! : D^bMHM(Y) \longrightarrow D^bMHM(X)$$

were defined in the derived categories of mixed Hodge modules and he proved various desired properties ($f : X \to Y$ is a morphism of algebraic varieties). Namely, we have

(m6) $\mathbb{D} \circ \mathbb{D} = \mathrm{Id}$, $\mathbb{D} \circ f_\bigstar = f_! \circ \mathbb{D}$, $\mathbb{D} \circ f^\bigstar = f^! \circ \mathbb{D}$.

(m7) $\mathbf{D} \circ \mathrm{rat} = \mathrm{rat} \circ \mathbb{D}$, $f_* \circ \mathrm{rat} = \mathrm{rat} \circ f_\bigstar$, $f_! \circ \mathrm{rat} = \mathrm{rat} \circ f_!$, $f^* \circ \mathrm{rat} = \mathrm{rat} \circ f^\bigstar$, $f^! \circ \mathrm{rat} = \mathrm{rat} \circ f^!$.

(m8) If $f$ is a projective morphism, then $f_\bigstar = f_!$.

Let $\mathcal{V} \in D^b MHM(X)$. We say that $\mathcal{V}$ has *mixed weights* $\leq n$ (resp. $\geq n$) if $\mathrm{gr}_i^W H^j \mathcal{V} = 0$ $(i > j + n)$ (resp. $\mathrm{gr}_i^W H^j \mathcal{V} = 0$ $(i < j + n)$). Also $\mathcal{V}$ is said to have a *pure weight* $n$ if $\mathrm{gr}_i^W H^j \mathcal{V} = 0$ $(i \neq j + n)$ holds. Now we have the following results, which justify the name "theory of weights":

(m9) If $\mathcal{V}$ has mixed weights $\leq n$ (resp. $\geq n$), then $\mathbb{D}\mathcal{V}$ has mixed weights $\geq -n$ (resp. $\leq -n$).

(m10) If $\mathcal{V}$ has mixed weights $\leq n$, then $f_!\mathcal{V}$, $f^\bigstar \mathcal{V}$ have mixed weights $\leq n$.

(m11) If $\mathcal{V}$ has mixed weights $\geq n$, then $f_\bigstar \mathcal{V}$, $f^!\mathcal{V}$ have mixed weights $\geq n$.

Next we will explain the relation with the variations of mixed Hodge structures.

(m12) Let $\mathcal{H} \in VSHM(X)^p$. Then $\phi_X(\mathcal{H}) \in MF_{rh}W(D_X, \mathbb{Q})$ is an object of $MHM(X)$ if and only if $\mathcal{H}$ is *admissible* in the sense of Kashiwara [Kas13].

This implies in particular that $MHM(\mathrm{pt}) = SHM^p$ (the case when $X$ is a one-point variety $\{\mathrm{pt}\}$).

Finally, let us describe the objects in $MH_Z(X, n)^p$ by using direct images ($Z$ is an irreducible subvariety of $X$). Let $U$ be a smooth open subset of $Z$ and assume that the inclusion map $j : U \hookrightarrow X$ is an affine morphism. For $\mathcal{H} \in VSH(U, n)^p$, $j_!\phi_U^n \mathcal{H}$ and $j_*\phi_U^n \mathcal{H}$ are objects in $MHM(X)$ having weights $\leq n$, $\geq n$, respectively. Therefore, if we set

$$j_{!*}\phi_U^n \mathcal{H} = \mathrm{Im}\big[j_!\phi_U^n \mathcal{H} \longrightarrow j_*\phi_U^n \mathcal{H}\big],$$

then $j_{!*}\phi_U^n \mathcal{H}$ is an object in $MHM(X)$ having the pure weight $n + \dim Z$ (this functor $j_{!*}$ is a Hodge-theoretical version of the functor of minimal extensions defined in Section 8.2). Hence $\mathrm{gr}^W_{n+\dim Z} j_{!*}\phi_U^n \mathcal{H}$ is an object in $MH(X, n + \dim Z)^p$. Furthermore, it turns out that this object belongs to $MH_Z(X, n)^p$. Let us denote by $VSH(Z, n)^p_{\mathrm{gen}}$ the category of polarizable variations of Hodge structures of weight $n$ defined on some smooth Zariski open subsets of $Z$. Then by the above correspondence, we obtain an equivalence

(m13) $VSH(Z, n)^p_{\mathrm{gen}} \xrightarrow{\sim} MH_Z(X, n + \dim Z)^p$.

of categories. Now consider the trivial variation of Hodge structures

$$(\mathcal{O}_{Z_{\mathrm{reg}}}, F, \mathbb{Q}_{Z^{\mathrm{an}}_{\mathrm{reg}}}) \in VSH(Z_{\mathrm{reg}}, 0)^p$$

defined over the regular part $Z_{\mathrm{reg}}$ of $Z$ ($\mathrm{gr}_p^F = 0$ $(p \neq 0)$). Let us denote by $\mathrm{IC}_Z^H$ the corresponding object in $MH_Z(X, \dim Z)^p$ obtained by (m13). If we define the filtration $W$ on it by the condition $\mathrm{gr}_n^W = 0$ $(n \neq \dim Z)$, then we see that $\mathrm{IC}_Z^H \in MHM(X)$. The underlying $D$-module (resp. perverse sheaf) of $\mathrm{IC}_Z^H$ is the minimal extension $L(Z_{\mathrm{reg}}, \mathcal{O}_{Z_{\mathrm{reg}}})$ (resp. the intersection cohomology complex $\mathrm{IC}(\mathbb{Q}_{Z^{\mathrm{an}}_{\mathrm{reg}}})$).

Also in the case of $Z = X$, we have $\mathrm{IC}_X^H = \mathbb{Q}_X^H[\dim X]$. Let $a_X : X \to \mathrm{pt}$ be the unique map and for $\mathcal{V} \in MHM(X)$ set

$$H^k(X, \mathcal{V}) = H^k\big((a_X)_*\mathcal{V}\big), \quad H_c^k(X, \mathcal{V}) = H^k\big((a_X)_!\mathcal{V}\big).$$

Then these are objects in $MHM(\mathrm{pt}) = SHM^p$. When $X$ is a projective variety, by (m10) and (m11) we get in particular

$$H^k(X, \mathrm{IC}_Z^H) = H_c^k(X, \mathrm{IC}_Z^H) \in SH(k + \dim Z)^p.$$

This result shows that the global intersection cohomology groups of $Z$ have Hodge structures with pure weights.

# Part II

# Representation Theory

# 9

# Algebraic Groups and Lie Algebras

In this chapter we summarize basic notions of algebraic groups and Lie algebras. For details we refer to other textbooks such as Humphreys [Hu1], Springer [Sp1], Humphreys [Hu2]. We do not give any proof here; however, various examples are presented so that inexperienced readers can also follow the rest of this book.

## 9.1 Lie algebras and their enveloping algebras

Let $k$ be a field. A *Lie algebra* $\mathfrak{g}$ over $k$ is a vector space over $k$ endowed with a bilinear operation (called a Lie bracket)

$$\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} \qquad ((x, y) \mapsto [x, y]) \tag{9.1.1}$$

satisfying the following axioms (9.1.2), (9.1.3):

$$[x, x] = 0 \quad (x \in \mathfrak{g}), \tag{9.1.2}$$

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \quad (x, y, z \in \mathfrak{g}). \tag{9.1.3}$$

We call $[\bullet, \bullet]$ the bracket product of $\mathfrak{g}$.

For example, any vector space $\mathfrak{g}$ with the bracket product

$$[x, y] = 0 \quad (x, y \in \mathfrak{g})$$

is a Lie algebra. We call such a Lie algebra *commutative*.

Any associative algebra $A$ over $k$ can be regarded as a Lie algebra by setting

$$[x, y] = xy - yx \quad (x, y \in A). \tag{9.1.4}$$

In particular, the algebra $\mathrm{End}(V)$ consisting of linear transformations of a vector space $V$ is naturally a Lie algebra. We usually denote this Lie algebra by $\mathfrak{gl}(V)$. When $V = k^n$, it is identified with the set of $n \times n$-matrices and we write it as $\mathfrak{gl}_n(k)$.

A vector subspace $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ which is closed under the bracket product of $\mathfrak{g}$ is called a (Lie) subalgebra of $\mathfrak{g}$. A subalgebra of a Lie algebra is naturally a Lie algebra. The following are all subalgebras of $\mathfrak{gl}_n(k)$):

$$\mathfrak{sl}_n(k) = \{x \in \mathfrak{gl}_n(k) \mid \mathrm{Tr}(x) = 0\}, \tag{9.1.5}$$

$$\mathfrak{so}_n(k) = \{x \in \mathfrak{gl}_n(k) \mid {}^t x + x = 0\}, \tag{9.1.6}$$

$$\mathfrak{sp}_{2m}(k) = \{x \in \mathfrak{gl}_{2m}(k) \mid {}^t xJ + Jx = 0\} \quad (n = 2m), \tag{9.1.7}$$

where we put

$$J = \left(\begin{array}{c|c} O & E \\ \hline -E & O \end{array}\right) \qquad (E \text{ is the unit matrix of size } m). \tag{9.1.8}$$

Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be Lie algebras. A linear map $f : \mathfrak{g}_1 \to \mathfrak{g}_2$ preserving the bracket products is called a homomorphism of Lie algebras.

Let $\mathfrak{g}$ be a Lie algebra. Then there exists uniquely up to isomorphisms a pair $(A, i)$ of an associative algebra $A$ and a homomorphism $i : \mathfrak{g} \to A$ of Lie algebras satisfying the following universal property:

$$\left\{\begin{array}{l} \text{For any such pair } (A', i') \text{ there exists a unique homomorphism} \\ f : A \to A' \text{ of associative algebras such that } f \circ i = i'. \end{array}\right. \tag{9.1.9}$$

We denote this associative algebra by $U(\mathfrak{g})$ and call it the *enveloping algebra* of $\mathfrak{g}$. It can be realized as the quotient of the tensor algebra $T(\mathfrak{g})$ of $\mathfrak{g}$ by the two-sided ideal generated by the elements $xy - yx - [x, y]$ $(x, y \in \mathfrak{g})$. The following theorem due to *Poincaré–Birkhoff–Witt* (we call it *PBW* for short hereafter) is fundamental in the theory of Lie algebras.

**Theorem 9.1.1.** *If $x_1, \ldots, x_n$ is a basis of the $k$-vector space $\mathfrak{g}$, then the elements*

$$i(x_1)^{m_1} \cdots i(x_n)^{m_n} \quad (m_1, \ldots, m_n \in \mathbb{N} = \{0, 1, 2, \ldots\})$$

*in $U(\mathfrak{g})$ form a basis of $U(\mathfrak{g})$.*

It follows from PBW that $i : \mathfrak{g} \to U(\mathfrak{g})$ is injective. Hereafter, we consider $\mathfrak{g}$ as embedded into $U(\mathfrak{g})$ and omit the injection $i$.

Let $\mathfrak{g}$ be a Lie algebra and let $V$ be a vector space. We call a homomorphism

$$\sigma : \mathfrak{g} \to \mathfrak{gl}(V) \tag{9.1.10}$$

of Lie algebras a *representation* of $\mathfrak{g}$ on $V$. In this case we also say that $V$ is a $\mathfrak{g}$-*module*. By the universal property of $U(\mathfrak{g})$ giving a $\mathfrak{g}$-module structure on a vector space $V$ is equivalent to giving a homomorphism $U(\mathfrak{g}) \to \mathrm{End}(V)$ of associative algebras. Namely, a $\mathfrak{g}$-module is a (left) $U(\mathfrak{g})$-module, and vice versa. Universal enveloping algebras are indispensable tools in the representation theory of Lie algebras.

## 9.2 Semisimple Lie algebras (1)

From now on we assume that Lie algebras are finite dimensional unless otherwise stated. A subspace $\mathfrak{a}$ of a Lie algebra $\mathfrak{g}$ satisfying the condition $[\mathfrak{g}, \mathfrak{a}] \subset \mathfrak{a}$ is called an ideal of $\mathfrak{g}$. In this case $\mathfrak{a}$ is clearly a Lie subalgebra of $\mathfrak{g}$. Moreover, the quotient space $\mathfrak{g}/\mathfrak{a}$ is naturally a Lie algebra.

If there exists a sequence of ideals of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{a}_0 \supset \mathfrak{a}_1 \supset \mathfrak{a}_2 \supset \cdots\cdots \supset \mathfrak{a}_r = \{0\}$$

such that $[\mathfrak{a}_i/\mathfrak{a}_{i+1}, \mathfrak{a}_i/\mathfrak{a}_{i+1}] = 0$ (i.e., $\mathfrak{a}_i/\mathfrak{a}_{i+1}$ is commutative), we call $\mathfrak{g}$ a *solvable Lie algebra*. For a Lie algebra $\mathfrak{g}$ there always exists a unique maximal solvable ideal $\mathfrak{r}(\mathfrak{g})$ of $\mathfrak{g}$, called the *radical* of $\mathfrak{g}$. A Lie algebra $\mathfrak{g}$ is called *semisimple* if $\mathfrak{r}(\mathfrak{g}) = 0$. For any Lie algebra $\mathfrak{g}$ the quotient Lie algebra $\mathfrak{g}/\mathfrak{r}(\mathfrak{g})$ is semisimple. Moreover, it is known that there exists a semisimple subalgebra $\mathfrak{s}$ of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{r}(\mathfrak{g})$ (Levi's theorem).

A Lie algebra $\mathfrak{g}$ is called *simple* if it is non-commutative and contains no non-trivial ideals (ideals other than $\{0\}$ and $\mathfrak{g}$ itself). Simple Lie algebras are semisimple. If the characteristic of the ground field $k$ is zero, any semisimple Lie algebra is isomorphic to a direct sum of simple Lie algebras.

For $x \in \mathfrak{g}$ we define a linear map $\mathrm{ad}(x) : \mathfrak{g} \to \mathfrak{g}$ by

$$(\mathrm{ad}(x))(y) = [x, y] \qquad (y \in \mathfrak{g}). \tag{9.2.1}$$

Then

$$\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}) \tag{9.2.2}$$

is a homomorphism of Lie algebras, and hence it gives a representation of $\mathfrak{g}$. We call it the *adjoint representation* of $\mathfrak{g}$.

We define a symmetric bilinear form $B_{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g} \to k$ on $\mathfrak{g}$ by

$$B_{\mathfrak{g}}(x, y) = \mathrm{Tr}(\mathrm{ad}(x)\,\mathrm{ad}(y)) \quad (x, y \in \mathfrak{g}). \tag{9.2.3}$$

This is called the *Killing form* of $\mathfrak{g}$.

**Theorem 9.2.1 (Cartan's criterion).** *We assume that the characteristic of $k$ is 0. Then a Lie algebra $\mathfrak{g}$ is semisimple if and only if its Killing form $B_{\mathfrak{g}}$ is non-degenerate.*

By this theorem we can check the semisimplicity of the Lie algebras $\mathfrak{sl}_n(k)$ ($n \geqq 2$), $\mathfrak{so}_n(k)$ ($n \geqq 3$), $\mathfrak{sp}_{2n}(k)$ ($n \geqq 2$).

In the rest of this section we assume that $k$ is an algebraically closed field of characteristic zero and $\mathfrak{g}$ is a semisimple Lie algebra over $k$.

**Definition 9.2.2.** We say that a subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is a *Cartan subalgebra* if it is maximal among the subalgebras of $\mathfrak{g}$ satisfying the following two conditions:

$$\mathfrak{h} \text{ is commutative.} \tag{9.2.4}$$

$$\text{For any } h \in \mathfrak{h} \text{ the linear transformation } \mathrm{ad}(h) \text{ is semisimple.} \tag{9.2.5}$$

**Example 9.2.3.** Let $\mathfrak{g} = \mathfrak{sl}_n(k)$. Set

$$d(a_1, \ldots, a_n) = \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix},$$

$$\mathfrak{h} = \left\{ d(a_1, \ldots, a_n) \;\middle|\; a_1, \ldots, a_n \in \mathbb{C}, \quad \sum_{i=1}^{n} a_i = 0 \right\}.$$

Then $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. Indeed, it is easily seen that $\mathfrak{h}$ satisfies (9.2.4), (9.2.5). Moreover, the subalgebra

$$\mathfrak{z}_{\mathfrak{g}}(\mathfrak{h}) = \{x \in \mathfrak{g} \mid [\mathfrak{h}, x] = \{0\}\}$$

coincides with $\mathfrak{h}$, and hence $\mathfrak{h}$ is a maximal commutative subalgebra.

We denote by $\mathrm{Aut}(\mathfrak{g})$ the group of automorphisms of $\mathfrak{g}$:

$$\mathrm{Aut}(\mathfrak{g}) = \{g \in GL(\mathfrak{g}) \mid [g(x), g(y)] = g([x, y]) \quad (x, y \in \mathfrak{g})\}.$$

**Theorem 9.2.4.** *If $\mathfrak{h}_1$, $\mathfrak{h}_2$ are Cartan subalgebras of $\mathfrak{g}$, then there exists some $g \in \mathrm{Aut}(\mathfrak{g})$ such that $g(\mathfrak{h}_1) = \mathfrak{h}_2$.*

In what follows we choose and fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. For each $\lambda \in \mathfrak{h}^*$ we set

$$\mathfrak{g}_\lambda = \{x \in \mathfrak{g} \mid \mathrm{ad}(h)x = \lambda(h)x \quad (h \in \mathfrak{h})\}. \tag{9.2.6}$$

The linear transformations $\mathrm{ad}(h)$ $(h \in \mathfrak{h})$ are simultaneously diagonalizable, and hence we have $\mathfrak{g} = \bigoplus_{\lambda \in \mathfrak{h}^*} \mathfrak{g}_\lambda$. We set

$$\Delta = \{\lambda \in \mathfrak{h}^* \mid \mathfrak{g}_\lambda \neq \{0\}\} \backslash \{0\}. \tag{9.2.7}$$

**Theorem 9.2.5.**

(i) $\mathfrak{g}_0 = \mathfrak{h}$.

(ii) *For any $\alpha \in \Delta$ we have* $\dim \mathfrak{g}_\alpha = 1$.

(iii) *The set $\Delta$ is a root system in $\mathfrak{h}^*$.* (*The definition of a root systems will be given in Section* 9.3).

We call $\Delta$ the *root system* of $\mathfrak{g}$ (with respect to $\mathfrak{h}$).

**Example 9.2.6.** Set $\mathfrak{g} = \mathfrak{sl}_n(k)$ and consider the Cartan subalgebra $\mathfrak{h}$ in Example 9.2.3. For $i = 1, \ldots, n$, we define $\lambda_i \in \mathfrak{h}^*$ by

$$\lambda_i(d(a_1, \ldots, a_n)) = a_i.$$

Then we have

$$\Delta = \{\lambda_i - \lambda_j \mid i \neq j\},$$
$$\mathfrak{g}_{\lambda_i - \lambda_j} = ke_{ij} \quad (i \neq j),$$

where $e_{ij}$ is the matrix whose $(k, l)$th entry is $\delta_{ik}\delta_{jl}$.

## 9.3 Root systems

Let $k$ be a field of characteristic zero and $V$ a vector space over $k$. For $\alpha \in V$ and $\alpha^\vee \in V^*$ such that $\langle \alpha^\vee, \alpha \rangle = 2$, we define a linear map $s_{\alpha,\alpha^\vee} : V \to V$ by

$$s_{\alpha,\alpha^\vee}(v) = v - \langle \alpha^\vee, v \rangle \alpha. \tag{9.3.1}$$

By setting $H = \mathrm{Ker}(\alpha^\vee)$ we have $V = H \oplus k\alpha$, $s_{\alpha,\alpha^\vee}(\alpha) = -\alpha$ and $s_{\alpha,\alpha^\vee}|_H = \mathrm{id}$. If $\Delta$ is a finite subset of $V$ spanning $V$, then for $\alpha \in \Delta$ we have at most one $\alpha^\vee \in V^*$ satisfying $\langle \alpha^\vee, \alpha \rangle = 2$ and $s_{\alpha,\alpha^\vee}(\Delta) = \Delta$.

**Definition 9.3.1.** A *root system* (in $V$) is a finite subset $\Delta$ of $V$ which satisfies the following conditions:

(i) $\Delta \not\ni 0$.
(ii) The set $\Delta$ spans $V$.
(iii) For any $\alpha \in \Delta$, there exists a unique $\alpha^\vee \in V^*$ such that $\langle \alpha^\vee, \alpha \rangle = 2$ and $s_{\alpha,\alpha^\vee}(\Delta) = \Delta$.
(iv) For $\alpha, \beta \in \Delta$ we have $\langle \alpha^\vee, \beta \rangle \in \mathbb{Z}$.
(v) If $c\alpha \in \Delta$ for some $\alpha \in \Delta$ and $c \in k$, then $c = \pm 1$.

An element of $\Delta$ is called a root.

For root systems $\Delta_1$ and $\Delta_2$ in $V_1$ and $V_2$, respectively, we have a root system $\Delta_1 \sqcup \Delta_2$ in $V_1 \oplus V_2$. We call this root system the direct sum of $\Delta_1$ and $\Delta_2$. We say that a root system is *irreducible* if there exists no non-trivial direct sum decomposition.

Assume that $\Delta$ is a root system in $V$. For a field extension $k_1$ of $k$, $\Delta$ is also a root system in $k_1 \otimes V$. Furthermore, for a subfield $k_2$ of $k$, setting $V' = \sum_{\alpha \in \Delta} k_2 \alpha$, we see $k \otimes V' = V$ and $\Delta$ is a root system of $V'$. This means that the classification of root systems is independent of the choice of base fields.

**Theorem 9.3.2.** *Assume that $k$ is an algebraically closed field of characteristic 0. For any root system $\Delta$ there exists uniquely up to isomorphisms a semisimple Lie algebra $\mathfrak{g}(\Delta)$ over $k$ whose root system is isomorphic to $\Delta$. Moreover, the Lie algebra $\mathfrak{g}(\Delta)$ is simple if and only if $\Delta$ is an irreducible root system.*

Hence semisimple Lie algebras over an algebraically closed field of characteristic zero are classified by their root systems. In particular, the classification does not depend on the choice of the base field $k$.

Henceforth we assume that $\Delta$ is a root system in the vector space $V$ over a field $k$ of characteristic 0.

It follows from the simple fact ${}^t s_{\alpha,\alpha^\vee} = s_{\alpha^\vee,\alpha}$ that

$$\Delta^\vee = \{\alpha^\vee \mid \alpha \in \Delta\} \subset V^* \tag{9.3.2}$$

is also a root system in $V^*$. We call it the *coroot system* of $\Delta$.

The subgroup $W$ of $GL(V)$ defined by

$$W = \langle s_{\alpha,\alpha^\vee} \mid \alpha \in \Delta \rangle \tag{9.3.3}$$

is called the *Weyl group* of $\Delta$. It is a finite group since it is regarded as a subgroup of the permutation group of the finite set $\Delta$. The Weyl group of $\Delta$ and that of $\Delta^\vee$ are naturally identified via $w \leftrightarrow {}^t w^{-1}$. Hence we may also regard $W$ as a subgroup of $GL(V^*)$. In what follows we write $s_{\alpha,\alpha^\vee} \in W$ simply as $s_\alpha$.

We can always take a subset $\Delta^+$ of $\Delta$ with the following properties:

$$\Delta = \Delta^+ \cup (-\Delta^+), \quad \Delta^+ \cap (-\Delta^+) = \emptyset. \tag{9.3.4}$$

$$\text{If } \alpha, \beta \in \Delta^+, \quad \alpha + \beta \in \Delta, \quad \text{then } \alpha + \beta \in \Delta^+. \tag{9.3.5}$$

We call such a subset $\Delta^+$ a *positive root system* in $\Delta$.

**Proposition 9.3.3.** *The Weyl group $W$ acts simply transitively on the set of positive root systems in $\Delta$.*

From now on we fix a positive root system $\Delta^+$. An element of $\Delta^+$ is called a positive root.

A positive root which cannot be expressed as a sum of two positive roots is called a *simple root*. We denote the set of simple roots by

$$\Pi = \{\alpha_1, \ldots, \alpha_l\}. \tag{9.3.6}$$

We also set $s_i = s_{\alpha_i}$ $(i = 1, \ldots, l)$.

**Proposition 9.3.4.**
(i) *$\Pi$ is a basis of the vector space $V$.*
(ii) *Any positive root $\alpha \in \Delta^+$ can be uniquely written as $\alpha = \sum_{i=1}^l n_i \alpha_i$ $(n_i \in \mathbb{N})$.*
(iii) $W(\Pi) = \Delta$.
(iv) $W = \langle s_i \mid i = 1, \ldots, l \rangle$.

We call the square matrix $A = (a_{ij})$ of size $l$ defined by

$$a_{ij} = \langle \alpha_i^\vee, \alpha_j \rangle \tag{9.3.7}$$

the *Cartan matrix* of $\Delta$. By virtue of Proposition 9.3.3 the Cartan matrix is uniquely determined from the root system $\Delta$ up to permutations of the indices $1, \ldots, l$. Moreover, one can reconstruct the root system $\Delta$ from its Cartan matrix using Proposition 9.3.4 (iii) and (iv), because $s_i(\alpha_j) = \alpha_j - a_{ij}\alpha_i$. Therefore, in order to classify all root systems, it suffices to classify all possible Cartan matrices associated to root systems. It follows directly from the definition that such a Cartan matrix satisfy

$$a_{ii} = 2. \tag{9.3.8}$$

It can also be shown that

$$a_{ij} \leqq 0 \quad (i \neq j), \tag{9.3.9}$$

$$a_{ij} = 0 \Longleftrightarrow a_{ji} = 0. \tag{9.3.10}$$

Moreover, we see from an observation in the case of $l = 2$ that

$$\text{if } i \neq j, \text{ then } a_{ij}a_{ji} \text{ takes one of the values } 0, 1, 2, 3. \tag{9.3.11}$$

The diagram determined by the following is called the *Dynkin diagram* of the root system $\Delta$:

**Vertices:** For each $i = 1, \ldots, l$ we write a vertex $\overset{i}{\circ}$.

**Edges:** Whenever $i \neq j$, we draw a line (edge) between the vertices $\overset{i}{\circ}$ and $\overset{j}{\circ}$ by the following rules:

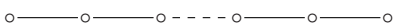
(i) If $a_{ij} = a_{ji} = 0$, $\overset{i}{\circ}\quad\overset{j}{\circ}$

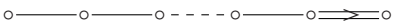
(ii) If $a_{ij} = a_{ji} = -1$, $\overset{i}{\circ}\!-\!\overset{j}{\circ}$

(iii) If $a_{ij} = -1, a_{ji} = -2$, $\overset{i}{\circ}\!\Rightarrow\!\overset{j}{\circ}$

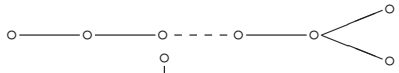
(iv) If $a_{ij} = -1, a_{ji} = -3$, $\overset{i}{\circ}\!\Rightarrow\!\overset{j}{\circ}$

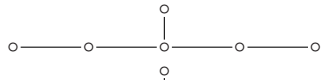
**Theorem 9.3.5.** *The Dynkin diagram of any irreducible root system is one of the diagrams* $(A_l), \ldots, (G_2)$ *listed below. Furthermore, for each Dynkin diagram in the list, there exists a unique* (*up to isomorphisms*) *irreducible root system corresponding to it.*

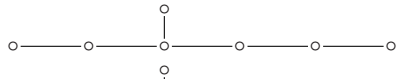
$(A_l)$  (*the number of vertices* $l \geqq 1$)

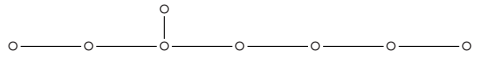
$(B_l)$  (*the number of vertices* $l \geqq 2$)

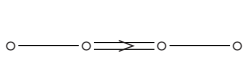
$(C_l)$  (*the number of vertices* $l \geqq 3$)

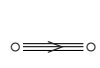
$(D_l)$  (*the number of vertices* $l \geqq 4$)

$(E_6)$ 

$(E_7)$ 

$(E_8)$ 

$(F_4)$ 

$(G_2)$ 

**Example 9.3.6.** Let $\widetilde{V}$ be a vector space endowed with a basis $e_1, \ldots, e_n$ and set $V = \widetilde{V}/k(e_1 + \cdots + e_n)$. We denote by $\lambda_i$ the image of $e_i$ in $V$. Then $\Delta = \{\lambda_i - \lambda_j \mid i \neq j\}$ is a root system in $V$. By using the dual basis $e_1^*, \ldots, e_n^* \in \widetilde{V}^*$ of $e_1, \ldots, e_n$ we identify $V^*$ with

$$\left\{\sum_{i=1}^{n} a_i e_i^* \in \widetilde{V}^* \;\middle|\; \sum_{i=1}^{n} a_i = 0\right\}.$$

Then we have $(\lambda_i - \lambda_j)^\vee = e_i^* - e_j^*$. The Weyl group $W$ of this root system is identified with the symmetric group $S_n$ by the action $\lambda_i \mapsto \lambda_{\sigma(i)}$ of $\sigma \in S_n$ on $V$. We can take a positive root system

$$\Delta^+ = \{\lambda_i - \lambda_j \mid i < j\}$$

$\Delta$, for which the corresponding simple root system is given by

$$\Pi = \{\alpha_1, \ldots, \alpha_{n-1}\}, \qquad \alpha_i = \lambda_i - \lambda_{i+1} \quad (i = 1, 2, \ldots, n-1).$$

Therefore, the Dynkin diagram of $\Delta$ is of type $(A_{n-1})$.

Let us introduce several basic notation concerning root systems and Weyl groups. For $i = 1, \ldots, l$, we define the fundamental weight $\pi_i \in V$ by the equation

$$\langle \alpha_j^\vee, \pi_i \rangle = \delta_{ij}. \tag{9.3.12}$$

We also set

$$Q = \sum_{\alpha \in \Delta} \mathbb{Z}\alpha = \bigoplus_{i=1}^{l} \mathbb{Z}\alpha_i, \tag{9.3.13}$$

$$Q^+ = \sum_{\alpha \in \Delta^+} \mathbb{N}\alpha = \bigoplus_{i=1}^{l} \mathbb{N}\alpha_i, \tag{9.3.14}$$

$$P = \{\lambda \in V \mid \langle \alpha^\vee, \lambda \rangle \in \mathbb{Z} \ \ (\alpha \in \Delta)\} = \bigoplus_{i=1}^{l} \mathbb{Z}\pi_i, \tag{9.3.15}$$

$$P^+ = \{\lambda \in V \mid \langle \alpha^\vee, \lambda \rangle \in \mathbb{N} \ \ (\alpha \in \Delta^+)\} = \bigoplus_{i=1}^{l} \mathbb{N}\pi_i. \tag{9.3.16}$$

Then $Q$ and $P$ are lattices ($\mathbb{Z}$-lattice) in $V$. We call $Q$ (resp. $P$) the *root lattice* (resp. *weight lattice*).

We define a partial ordering $\geqq$ on $V$ by

$$\lambda \geqq \mu \Longleftrightarrow \lambda - \mu \in Q^+. \tag{9.3.17}$$

By Proposition 9.3.4 (iv) any $w \in W$ can be written as

$$w = s_{i_1} \cdot \cdots \cdot s_{i_k}. \tag{9.3.18}$$

We denote by $l(w)$ the minimal number $k$ required for such expressions and call it the *length* of $w$. The expression (9.3.18) is called a *reduced expression* of $w$ if $k = l(w)$. Let us consider two elements $y, w \in W$. If we have reduced expressions $y = s_j \cdots s_{j_r}$ and $w = s_{i_1} \cdots s_{i_k}$ such that $(j_1, \ldots, j_r)$ is a subsequence of $(i_1, \ldots, i_k)$, we write $y \leqq w$. It defines a partial ordering on $W$ called the *Bruhat ordering* of $W$. It is well known that there exists a unique maximal element $w_0$ (*longest element*) with respect to this Bruhat ordering. It is characterized as the unique element of $W$ with the largest length.

## 9.4 Semisimple Lie algebras (2)

In this section $k$ denotes an algebraically closed field of characteristic zero.

According to Theorem 9.3.2 and Theorem 9.3.5 any simple Lie algebras over $k$ is either a member of the four infinite series $A_l$, $B_l$, $C_l$, $D_l$ or of $E_6$, $E_7$, $E_8$, $F_4$, $G_2$. The simple Lie algebras appearing in the four infinite series can be explicitly described using matrices as follows:

$$\begin{aligned} A_l &: \mathfrak{sl}_{l+1}(k), \\ B_l &: \mathfrak{so}_{2l+1}(k), \\ C_l &: \mathfrak{sp}_{2l}(k), \\ D_l &: \mathfrak{so}_{2l}(k) \end{aligned}$$

are called simple Lie algebras of classical type. The remaining five simple Lie algebras, called of exceptional type, can also be described by using Cayley algebras or Jordan algebras. Here, we describe Serre's construction of general semisimple Lie algebras by means of generators and fundamental relations. Let $\mathfrak{g}$ be a semisimple Lie algebra over $k$, $\mathfrak{h}$ its Cartan subalgebra, and $\Delta$ the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$. We use the notation in Section 9.3 concerning root systems such as $\Delta^+$, $\Pi = \{\alpha_1, \ldots, \alpha_l\}$. For $\alpha, \beta \in \Delta$ we have

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \begin{cases} \mathfrak{g}_{\alpha+\beta} & (\alpha + \beta \in \Delta) \\ k\alpha^\vee & (\alpha + \beta = 0) \\ 0 & (\alpha + \beta \notin \Delta \cup \{0\}). \end{cases}$$

Let us set $h_i = \alpha_i^\vee \in \mathfrak{h}$ for $i = 1, \ldots, l$. Then we can take $e_i \in \mathfrak{g}_{\alpha_i}$ and $f_i \in \mathfrak{g}_{-\alpha_i}$ so that $[e_i, f_i] = h_i$. It follows from the commutativity of $\mathfrak{h}$ that

$$[h_i, h_j] = 0. \tag{9.4.1}$$

Since $e_i \in \mathfrak{g}_{\alpha_i}$ and $f_i \in \mathfrak{g}_{-\alpha_i}$, we also have

$$[h_i, e_j] = a_{ij} e_j, \quad [h_i, f_j] = -a_{ij} f_j. \tag{9.4.2}$$

The fact that $\pm(\alpha_i - \alpha_j) \notin \Delta \cup \{0\}$ for $i \neq j$ implies

$$[e_i, f_j] = \delta_{ij} h_i. \tag{9.4.3}$$

Finally, by the theory of root systems we have $\pm(\alpha_j + (-a_{ij} + 1)\alpha_i) \notin \Delta \cup \{0\}$ for $i \neq j$, from which we obtain

$$(\mathrm{ad}(e_i))^{-a_{ij}+1}(e_j) = 0, \quad (\mathrm{ad}(f_i))^{-a_{ij}+1}(f_j) = 0 \quad (i \neq j). \tag{9.4.4}$$

**Theorem 9.4.1.** *The semisimple Lie algebra $\mathfrak{g}$ is generated by $h_1, \ldots, h_l$, $e_1, \ldots, e_l$, $f_1, \ldots, f_l$, and it has* (9.4.1),…,(9.4.4) *as its fundamental relations.*

**Remark 9.4.2.** Note that (9.4.1),…,(9.4.4) are written only in terms of the Cartan matrix $A = (a_{ij})$. More generally, a square matrix $A = (a_{ij})$ with integer entries satisfying the constraints (9.3.8), (9.3.9) and (9.3.10) is called a *generalized Cartan matrix*. The Kac–Moody Lie algebra associated to a generalized Cartan matrix $A$

is the Lie algebra $\mathfrak{g}$ defined by the generators $h_1, \ldots, h_l, e_1, \ldots, e_l, f_1, \ldots, f_l$ and the relations (9.4.1),…,(9.4.4), where $a_{ij}$ are the entries of $A$ (more precisely, it is in fact a quotient of $\mathfrak{g}$ by a certain ideal $\mathfrak{r}$ which is conjectured to be 0). The Kac–Moody Lie algebras are natural generalization of semisimple Lie algebras although they are infinite dimensional in general. They are important objects that appear in many branches of contemporary mathematics.

Recall that a semisimple Lie algebra $\mathfrak{g}$ has a direct sum decomposition (with respect to the simultaneous eigenspaces of the action of its Cartan subalgebra $\mathfrak{h}$):

$$\mathfrak{g} = \mathfrak{h} \oplus \left( \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha \right).$$

Under this decomposition the subspaces

$$\mathfrak{n} = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha, \qquad \mathfrak{n}^- = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_{-\alpha}, \tag{9.4.5}$$

$$\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}, \qquad \mathfrak{b}^- = \mathfrak{h} \oplus \mathfrak{n}^- \tag{9.4.6}$$

turn out to be subalgebras of $\mathfrak{g}$. The Lie algebras $\mathfrak{b}$ and $\mathfrak{b}^-$ are maximal solvable subalgebras of $\mathfrak{g}$. In general any maximal solvable Lie subalgebra of $\mathfrak{g}$ is called a *Borel subalgebra*. It is known that for two Borel subalgebras $\mathfrak{b}_1$, $\mathfrak{b}_2$ of $\mathfrak{g}$ there exists an automorphism $g \in \mathrm{Aut}(\mathfrak{g})$ such that $g(\mathfrak{b}_1) = \mathfrak{b}_2$.

Let us give a description of the center $\mathfrak{z}$ of the universal enveloping algebra $U(\mathfrak{g})$. Since $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$, it follows from PBW (Theorem 9.1.1) that

$$U(\mathfrak{g}) = U(\mathfrak{h}) \oplus (\mathfrak{n}^- U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}). \tag{9.4.7}$$

In this decomposition $U(\mathfrak{h})$ is naturally isomorphic to the symmetric algebra $S(\mathfrak{h})$ because $\mathfrak{h}$ is abelian. We define the *Weyl vector* $\rho \in \mathfrak{h}^*$ by

$$\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha \tag{9.4.8}$$

(it is known that $\rho = \sum_{i=1}^{l} \pi_i$). Let us consider the first projection $p : U(\mathfrak{g}) \to U(\mathfrak{h})$ with respect to the direct sum decomposition (9.4.7) and the automorphism $f$ of $U(\mathfrak{h}) = S(\mathfrak{h})$ defined by

$$f(h) = h - \rho(h)1 \qquad (h \in \mathfrak{h}).$$

We denote by $\gamma : \mathfrak{z} \to U(\mathfrak{h})$ the restriction of $f \circ p$ to $\mathfrak{z}$.

**Theorem 9.4.3.**

(i) *The map $\gamma$ is a homomorphism of associative algebras.*

(ii) *The homomorphism $\gamma$ is injective, and its image coincides with the set $U(\mathfrak{h})^W$ of $W$-invariant elements of $U(\mathfrak{h})$. ($W$ acts on $\mathfrak{h}$. Hence it acts also on $U(\mathfrak{h}) = S(\mathfrak{h})$.)*

(iii) *The homomorphism $\gamma$ does not depend on the choice of $\Delta^+$. In particular, for the projection $p' : U(\mathfrak{g}) = U(\mathfrak{h}) \oplus (\mathfrak{n}U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}^-) \to U(\mathfrak{h})$ and the automorphism $f'$ of $U(\mathfrak{h})$ defined by $f'(h) = h + \rho(h)1$ $(h \in \mathfrak{h})$ we have $\gamma = f' \circ p'$.*

We call $\gamma$ the *Harish-Chandra homomorphism*.

We note the following fact, which is an easy consequence of the PBW theorem and

$$\mathfrak{z} \subset \{u \in U(\mathfrak{g}) \mid hu - uh = 0 \ (\forall h \in \mathfrak{h})\}.$$

**Lemma 9.4.4.** *Let us decompose $z \in \mathfrak{z}$ as $z = u_1 + u_2$ by $u_1 \in U(\mathfrak{h})$ and $u_2 \in \mathfrak{n}^-U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}$. Then we have $u_2 \in \mathfrak{n}^-U(\mathfrak{g}) \cap U(\mathfrak{g})\mathfrak{n}$.*

Algebra homomorphisms from $\mathfrak{z}$ to $k$ are called *central characters*. For each $\lambda \in \mathfrak{h}^*$ we define a central character $\chi_\lambda : \mathfrak{z} \to k$ by

$$\chi_\lambda(z) = (\gamma(z))(\lambda) \qquad (z \in \mathfrak{z}), \tag{9.4.9}$$

where we identify $U(\mathfrak{h})(\simeq S(\mathfrak{h}))$ with the algebra of polynomial functions on $\mathfrak{h}^*$. The next proposition is a consequence of Theorem 9.4.3.

**Proposition 9.4.5.**

(i) *Any central character coincides with $\chi_\lambda$ for some $\lambda \in \mathfrak{h}^*$.*

(ii) *$\chi_\lambda = \chi_\mu$ if and only if $\lambda$ and $\mu$ are in the same $W$-orbit.*

## 9.5 Finite-dimensional representations of semisimple Lie algebras

In this section $k$ is an algebraically closed field of characteristic zero and $\mathfrak{g}$ denotes a semisimple Lie algebra over $k$.

The following theorem is fundamental.

**Theorem 9.5.1.** *Any finite-dimensional representation of $\mathfrak{g}$ is completely reducible* (*i.e., is a direct sum of irreducible representations*).

Therefore, main problems in the study of finite-dimensional representations of semisimple Lie algebras are to classify all irreducible representations and to study their properties.

For a finite-dimensional representation $\sigma : \mathfrak{g} \to \mathfrak{gl}(V)$ of $\mathfrak{g}$ set

$$V_\lambda = \left\{v \in V \mid (\sigma(h))(v) = \lambda(h)v \quad (h \in \mathfrak{h})\right\} \quad (\lambda \in \mathfrak{h}^*). \tag{9.5.1}$$

Then it is known that we have $V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda$. When $V_\lambda \neq \{0\}$, we say that $\lambda$ is a *weight* of the $\mathfrak{g}$-module $V$. The vector space $V_\lambda$ is called the *weight space* of $V$ with weight $\lambda$.

**Theorem 9.5.2.**

(i) *Let $V$ be a finite-dimensional irreducible $\mathfrak{g}$-module. Then there exists a unique weight $\lambda$ (resp. $\mu$) that is maximal (resp. minimal) with respect to the partial ordering* (9.3.17) *of $\mathfrak{h}^*$. We call $\lambda$ (resp. $\mu$) the highest (resp. lowest) weight of $V$. Then we have $\lambda \in P^+$, $\mu \in -P^+$, and $\mu = w_0(\lambda)$, where $w_0$ is the element of $W$ with the largest length.*

(ii) *Assume that $\lambda \in P^+$ and $\mu \in -P^+$. Then there exists an irreducible $\mathfrak{g}$-module $L^+(\lambda)$ (resp. $L^-(\mu)$) with highest weight $\lambda$ (resp. lowest weight $\mu$). Such a $\mathfrak{g}$-module is unique up to isomorphisms.*

In particular, any weight of a finite-dimensional $\mathfrak{g}$-module belongs to the weight lattice $P$.

**Example 9.5.3.** Let us consider the case of $\mathfrak{g} = \mathfrak{sl}_2(k)$. Set

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then we get a basis $h$, $e$, $f$ of $\mathfrak{g}$. The one-dimensional subspace $\mathfrak{h} = kh \subset \mathfrak{g}$ is a Cartan subalgebra. Define $\alpha \in \mathfrak{h}^*$ by $\alpha(h) = 2$. Then we have

$$\begin{cases} \Delta = \{\pm\alpha\}, \quad \Delta^+ = \Pi = \{\alpha\}, \quad W = \{\pm 1\} \\ \rho = \dfrac{\alpha}{2}, \quad P = \mathbb{Z}\rho, \quad P^+ = \mathbb{N}\rho. \end{cases}$$

By the canonical injection $\mathfrak{g} \hookrightarrow \mathfrak{gl}_2(k)$ we obtain a two-dimensional $\mathfrak{g}$-module $V$. Using this $\mathfrak{g}$-module $V$ we can uniquely define a $\mathfrak{g}$-module structure on the symmetric algebra $S(V)$ of $V$ by

$$a \cdot (fg) = (a \cdot f)g + f(a \cdot g) \quad (a \in \mathfrak{g},\ f, g \in S(V)).$$

Then the set $S^n(V)$ of elements of degree $n$ in $S(V)$ turns out to be a $\mathfrak{g}$-submodule. Moreover, as a $\mathfrak{g}$-module we have the following isomorphisms:

$$L^+(n\rho) = L^-(-n\rho) = S^n(V) \qquad (n \in \mathbb{N}).$$

For a finite-dimensional $\mathfrak{g}$-module $V$ we define its *character* $\mathrm{ch}(V)$ by

$$\mathrm{ch}(V) = \sum_{\lambda \in P} (\dim V_\lambda) e^\lambda. \tag{9.5.2}$$

This is an element of the group algebra $\mathbb{Z}[P] = \bigoplus_{\lambda \in P} \mathbb{Z} e^\lambda$ of $P$.

**Theorem 9.5.4 (Weyl's character formula).** *For $\lambda \in P^+$ and $\mu \in -P^+$ we have*

$$\mathrm{ch}(L^+(\lambda)) = \frac{\sum_{w \in W} (-1)^{l(w)} e^{w(\lambda+\rho)-\rho}}{\prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})},$$

$$\mathrm{ch}(L^-(\mu)) = \frac{\sum_{w \in W} (-1)^{l(w)} e^{w(\mu-\rho)+\rho}}{\prod_{\alpha \in \Delta^+} (1 - e^{\alpha})}.$$

Since $L^+(0)$ is a one-dimensional (trivial) representation of $\mathfrak{g}$, we have $\mathrm{ch}(L^+(0)) = e^0$. Hence we get

$$\prod_{\alpha\in\Delta^+}(1-e^{-\alpha}) = \sum_{w\in W}(-1)^{l(w)}e^{w\rho-\rho} \tag{9.5.3}$$

(*Weyl's denominator formula*) by Theorem 9.5.4.

## 9.6 Algebraic groups and their Lie algebras

In this section $k$ is an algebraically closed field. If an algebraic variety $G$ over $k$ is endowed with a group structure and its group operations

$$\begin{aligned} G\times G &\to G \qquad ((g_1,g_2)\mapsto g_1g_2),\\ G &\to G \qquad (g\mapsto g^{-1}) \end{aligned}$$

are morphisms of algebraic varieties, then we call $G$ an *algebraic group* over $k$.

For two algebraic groups $G_1$ and $G_2$ a morphism $f : G_1 \to G_2$ of algebraic varieties which is also a group homomorphism is called a homomorphism of algebraic groups.

The additive group $k$ and the multiplicative group $k^\times$ of $k$ are obviously algebraic groups. They are denoted by $\mathbb{G}_\mathrm{a}$ and $\mathbb{G}_\mathrm{m}$, respectively, when they are regarded as algebraic groups. Abelian varieties and the general linear group

$$\begin{aligned} GL_n(k) &= \{g\in M_n(k)\mid \det(g)\in k^\times\} \\ &= \mathrm{Spec}\, k[x_{ij},\det(x_{ij})^{-1}] \end{aligned} \tag{9.6.1}$$

are also basic examples of algebraic groups.

It is known that an algebraic group $G$ is affine if and only if it is isomorphic to a closed subgroup of $GL_n(k)$ for some $n$. In this case we call $G$ a *linear algebraic group*. Note that $\mathbb{G}_\mathrm{a}$ and $\mathbb{G}_\mathrm{m}$ are linear algebraic groups by

$$\mathbb{G}_\mathrm{a}\hookrightarrow GL_2(k)\ \left(a\mapsto\begin{pmatrix}1&a\\0&1\end{pmatrix}\right),\qquad \mathbb{G}_\mathrm{m}\simeq GL_1(k).$$

In this book we are only concerned with linear algebraic groups. It is known that any algebraic group is an extension of an abelian variety by a linear algebraic group (Chevalley–Rosenlicht's theorem).

We can define the Lie algebra $\mathrm{Lie}(G)$ of an algebraic group $G$ similarly to the case of Lie groups as follows. For $g\in G$ we define the left translation $l_g : G\to G$ and the right translation $r_g : G\to G$ by $l_g(x) = gx$ and $r_g(x) = xg$, respectively. We denote their derivations at $x\in G$ by

$$(dl_g)_x : T_xG \to T_{l_g(x)}G,\quad (dr_g)_x : T_xG\to T_{r_g(x)}G,$$

where $T_xG$ denotes the tangent space $G$ at $x$. A vector field $Z$ on $G$ is said to be left-invariant (resp. right-invariant) if for any $g, x \in G$ the condition

$$(dl_g)_x(Z_x) = Z_{l_g(x)} \qquad (\text{resp. } (dr_g)_x(Z_x) = Z_{r_g(x)})$$

is satisfied. We denote the vector space of left-invariant (resp. right-invariant) vector fields on $G$ by $\mathrm{Lie}(G)_l$ (resp. $\mathrm{Lie}(G)_r$). These are Lie algebras by the Lie bracket operation

$$[Z_1, Z_2](f) = Z_1(Z_2(f)) - Z_2(Z_1(f))$$
$$(Z_1, Z_2 \text{ are vector fields and } f \in \mathcal{O}_G).$$

Define linear maps $\varphi_l : \mathrm{Lie}(G)_l \to T_eG$ and $\varphi_r : \mathrm{Lie}(G)_r \to T_eG$ by

$$\varphi_l(Z) = Z_e, \quad \varphi_r(Z) = Z_e.$$

Then $\varphi_l$ and $\varphi_r$ are isomorphisms of vector spaces, and $-\varphi_r^{-1} \circ \varphi_l$ is an isomorphism of Lie algebras. The Lie algebra $\mathrm{Lie}(G)_l \cong \mathrm{Lie}(G)_r$ defined in this way is called the *Lie algebra* of $G$ and denoted by $\mathrm{Lie}(G)$.

We will occasionally identify $\mathrm{Lie}(G)$ with $\mathrm{Lie}(G)_l$ or $\mathrm{Lie}(G)_r$. Whenever we use the identification of $\mathrm{Lie}(G)$ with invariant vector fields, we will always specify whether the invariance is the left or right one. Identifying $T_e(GL_n(k))$ with $M_n(k) = \mathfrak{gl}_n(k)$ we see that $\varphi_l : \mathrm{Lie}(GL_n(k)) = \mathrm{Lie}(GL_n(k))_l \to \mathfrak{gl}_n(k)$ is an isomorphism of Lie algebras. We will identify $\mathrm{Lie}(GL_n(k))$ with $\mathfrak{gl}_n(k)$ through $\varphi_l$ in the following. In general, the Lie algebra $\mathrm{Lie}(H)_l$ of a closed subgroup $H$ of $G$ is identified with the subalgebra $\{Z \in \mathrm{Lie}(G)_l \mid Z_e \in T_eH\}$ of $\mathrm{Lie}(G)_l$. In particular, an embedding of $G$ into $GL_n(k)$ gives an identification of $\mathrm{Lie}(G)$ as a subalgebra of $\mathfrak{gl}_n(k)$. For a homomorphism $f : G_1 \to G_2$ of algebraic groups we obtain as the composite of

$$\mathrm{Lie}(G_1) = \mathrm{Lie}(G_1)_l \xrightarrow{\varphi_l}_{\sim} T_eG_1 \xrightarrow{(df)_e} T_eG_2 \xleftarrow[\sim]{\varphi_l} \mathrm{Lie}(G_2)_l = \mathrm{Lie}(G_2)$$

a homomorphism

$$df : \mathrm{Lie}(G_1) \to \mathrm{Lie}(G_2) \tag{9.6.2}$$

of Lie algebras. Hence the operation $\mathrm{Lie}(\bullet)$ defines a functor from the category of algebraic groups to that of Lie algebras. Let $V$ be a finite-dimensional vector space over $k$. A homomorphism of algebraic groups

$$\sigma : G \to GL(V) \tag{9.6.3}$$

is called a *representation* of $G$. In this case we also say that $V$ is a *$G$-module*. By differentiating $\sigma$ we get a representation

$$d\sigma : \mathrm{Lie}(G) \to \mathfrak{gl}(V) \tag{9.6.4}$$

of the Lie algebra $\mathrm{Lie}(G)$ This defines a functor from the category of $G$-modules to that of $\mathrm{Lie}(G)$-modules. More generally, assume that we are given a homomorphism

(9.6.3) of abstract groups, where $V$ is a (not necessarily finite-dimensional) vector space. Then we say that $V$ is a $G$-module if there exists a family $V^\lambda$ ($\lambda \in \Lambda$) of finite-dimensional $G$-invariant subspaces of $V$ satisfying the properties:

$$V = \sum_{\lambda \in \Lambda} V^\lambda, \tag{9.6.5}$$

$$G \to GL(V^\lambda) \text{ is a homomorphism of algebraic groups.} \tag{9.6.6}$$

In this case we also have the associated Lie algebra homomorphism (9.6.4).

For $g \in G$ define an automorphism $i_g : G \to G$ by $i_g(x) = gxg^{-1}$ and set

$$\mathrm{Ad}(g) = di_g \in GL(\mathrm{Lie}(G)). \tag{9.6.7}$$

Then

$$\mathrm{Ad} : G \to GL(\mathrm{Lie}(G)) \tag{9.6.8}$$

is a representation of $G$. We call it the *adjoint representation* of $G$. The associated Lie algebra homomorphism

$$d(\mathrm{Ad}) : \mathrm{Lie}(G) \to \mathrm{Lie}(GL(\mathrm{Lie}(G)))(= \mathfrak{gl}(\mathrm{Lie}(G)))$$

coincides with the adjoint representation (9.2.2) of the Lie algebra $\mathrm{Lie}(G)$. When $G$ is a closed subgroup of $GL_n(k)$, the adjoint representation of $G$ on $\mathrm{Lie}(G)(\subset \mathfrak{gl}_n(k))$ is described by matrices as follows:

$$\mathrm{Ad}(g)x = gxg^{-1} \qquad (g \in G, \quad x \in \mathrm{Lie}(G)). \tag{9.6.9}$$

## 9.7 Semisimple algebraic groups

In this section $k$ is an algebraically closed field.

A matrix $g \in GL_n(k)$ is said to be semisimple if it is diagonalizable. We also say that $g \in GL_n(k)$ is unipotent if all of its eigenvalues are 1. According to the theory of Jordan normal forms any matrix $g \in GL_n(k)$ can be uniquely decomposed as follows:

$$\begin{cases} g = su & (s, u \in GL_n(k)) \\ s\text{: semisimple, } u\text{: unipotent and } su = us. \end{cases} \tag{9.7.1}$$

This decomposition is called the Jordan decomposition.

Let $G$ be a linear algebraic group over $k$ and fix an embedding of $G$ into $GL_n(k)$. It is known that in the decomposition (9.7.1) of an element $g \in G$ inside the general linear group $GL_n(k)$ we have $s, u \in G$. It is also known that the decomposition $g = su$ does not depend on the choice of an embedding of $G$ into a general linear group. Therefore, we can define the notions of *semisimple elements*, *unipotent elements* and *Jordan decompositions* also for any linear algebraic group $G$. These notions are preserved by homomorphisms of linear algebraic groups.

For a linear algebraic group $G$ there exists among all connected solvable normal closed subgroups of $G$ a unique maximal one $R(G)$, called the radical of $G$. The unipotent elements in $R(G)$ form a connected normal closed subgroup $R_u(G)$ of $G$, and is called the unipotent radical of $G$. We say that $G$ is *semisimple* (resp. *reductive*) if $R(G) = \{1\}$ (resp. $R_u(G) = \{1\}$).

**Example 9.7.1.** The general linear group $GL_n(k)$ is reductive and its radical $R(GL_n(k))$ is the subgroup consisting of scalar matrices. The following closed subgroups of $GL_n(k)$ are semisimple (except the case of $n = 2$ in (9.7.3)):

$$SL_n(k) = \{g \in GL_n(k) \mid \det(g) = 1\}, \tag{9.7.2}$$

$$SO_n(k) = \{g \in SL_n(k) \mid {}^t gg = 1\}, \tag{9.7.3}$$

$$Sp_{2m}(k) = \{g \in GL_{2m}(k) \mid {}^t gJg = J\} \quad (n = 2m), \tag{9.7.4}$$

where the matrix $J$ was defined by (9.1.8). In these cases we have $\mathrm{Lie}(SL_n(k)) = \mathfrak{sl}_n(k)$, $\mathrm{Lie}(SO_n(k)) = \mathfrak{so}_n(k)$ and $\mathrm{Lie}(Sp_{2m}(k)) = \mathfrak{sp}_{2m}(k)$.

A direct product of finitely many copies of the multiplicative group $\mathbb{G}_{\mathrm{m}}$ is called a torus. Any reductive algebraic group is "almost isomorphic to" a direct product of a torus and a semisimple algebraic group. Hence the essential part of the theory of reductive algebraic groups is played by semisimple ones. It is sometimes more convenient to develop the theory in the framework of reductive algebraic groups rather than just dealing with semisimple ones; however, we mainly restrict ourselves to the case of semisimple algebraic groups in order to simplify notation.

From now on we denote by $G$ a connected semisimple algebraic group over $k$ and set $\mathfrak{g} = \mathrm{Lie}(G)$.

Maximal elements among the closed subgroups of $G$, which is isomorphic to a torus, is called a *maximal torus* of $G$.

**Theorem 9.7.2.** *For two maximal tori $H_1$ and $H_2$ of $G$ there exists some $g \in G$ such that $gH_1g^{-1} = H_2$.*

In what follows we fix a maximal torus $H$ of $G$ and set $\mathfrak{h} = \mathrm{Lie}(H)$. Let $N_G(H)$ (resp. $Z_G(H)$) be the normalizer (resp. centralizer) of $H$ in $G$. Then we have $Z_G(H) = H$ and

$$W = N_G(H)/H \tag{9.7.5}$$

turns out to be a finite group. This group $W$ naturally acts on $H$. Since $H$ is isomorphic to a direct sum of copies of $\mathbb{G}_{\mathrm{m}}$, the character group

$$L = \mathrm{Hom}(H, k^\times) \tag{9.7.6}$$

is a free abelian group of rank $\dim H$. Set

$$L_{\mathbb{Q}} = \mathbb{Q} \otimes_{\mathbb{Z}} L \tag{9.7.7}$$

and regard $L$ as an additive subgroup of the $\mathbb{Q}$-vector space $L_{\mathbb{Q}}$. In order to avoid confusion we denote by $e^\lambda$ the character of $H$ which corresponds to $\lambda \in L \subset L_{\mathbb{Q}}$.

Since $H$ is a torus, any representation of $H$ is a direct sum of one-dimensional representations. Equivalently, for an $H$-module $V$ we have

$$V = \bigoplus_{\lambda \in L} V_\lambda, \tag{9.7.8}$$

where

$$V_\lambda = \{v \in V \mid h \cdot v = e^\lambda(h)v \quad (h \in H)\} \qquad (\lambda \in L). \tag{9.7.9}$$

By applying this to the restriction of the adjoint representation $\mathrm{Ad} : G \to GL(\mathfrak{g})$ to $H$, we have

$$\mathfrak{g} = \bigoplus_{\lambda \in L} \mathfrak{g}_\lambda, \tag{9.7.10}$$

where

$$\mathfrak{g}_\lambda = \{x \in \mathfrak{g} \mid \mathrm{Ad}(h)x = e^\lambda(h)x \quad (h \in H)\} \qquad (\lambda \in L). \tag{9.7.11}$$

We define a finite subset $\Delta$ of $L$ by

$$\Delta = \{\alpha \in L \mid \alpha \neq 0, \quad \mathfrak{g}_\alpha \neq \{0\}\}. \tag{9.7.12}$$

**Theorem 9.7.3.**

(i) $\mathfrak{g}_0 = \mathfrak{h}$.

(ii) *For any* $\alpha \in \Delta$ *we have* $\dim \mathfrak{g}_\alpha = 1$.

(iii) *The set* $\Delta$ *is a root system in the* $\mathbb{Q}$*-vector space* $L_\mathbb{Q}$*, and* $W$ *is naturally identified with the Weyl group of* $\Delta$ ($W$ *acts on* $L$*, and hence on* $L_\mathbb{Q}$).

(iv) *If* $P$ *and* $Q$ *are the weight lattice and the root lattice of* $\Delta$ (*see Section* 9.3), *respectively, then we have* $Q \subset L \subset P$.

We call $\Delta$ the *root system* (with respect to $H$) of $G$. The pair $(\Delta, L)$ is called the *root datum* (with respect to $H$) of $G$. By Theorem 9.7.2 the pair $(\Delta, L)$ is uniquely determined up to isomorphisms regardless of the choice of $H$.

**Example 9.7.4.** Let $G = SL_n(k)$. Set

$$d(a_1, \ldots, a_n) = \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix}.$$

Then the closed subgroup

$$H = \left\{ d(a_1, \ldots, a_n) \;\middle|\; a_i \in k^\times, \;\; \prod_{i=1}^{n} a_i = 1 \right\} \cong (k^\times)^{n-1}$$

of $G$ is a maximal torus of $G$. In this case, the normalizer $N_G(H)$ of $H$ in $G$ is the subgroup consisting of matrices so that each column and each row contains exactly one non-zero entry. Namely, we have

$$N_G(H) = \left\{ (a_i \delta_{i\sigma(j)})_{i,j=1,\dots,n} \;\middle|\; \sigma \in S_n,\ a_i \in k^\times,\ \left(\prod_{i=1}^n a_i\right) \mathrm{sgn}(\sigma) = 1 \right\}.$$

Therefore, the Weyl group $W = N_G(H)/H$ is isomorphic to $S_n$ by $(a_i \delta_{i\sigma(j)})H \leftrightarrow \sigma$.

Let us define, for $i = 1, \dots, n$, a character $\lambda_i$ of $H$ by

$$e^{\lambda_i}(d(a_1, \dots, a_n)) = a_i.$$

Then we have an isomorphism

$$\left(\bigoplus_{i=1}^n \mathbb{Z} e_i\right) \Big/ \mathbb{Z}\left(\sum_{i=1}^n e_i\right) \xleftrightarrow{\sim} L = \sum_{i=1}^n \mathbb{Z}\lambda_i \qquad (e_i \leftrightarrow \lambda_i).$$

We can also easily see that

$$\Delta = \{\lambda_i - \lambda_j \mid i \neq j\}.$$

This is the root system of type $(A_{n-1})$. Moreover, $L = P$ holds.

The action of $W = S_n$ on $H$ is given by

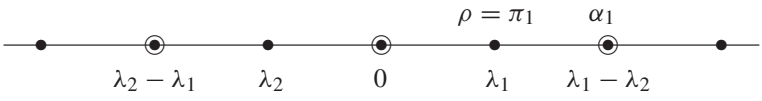
$$\sigma(d(a_1, \dots, a_n)) = d(a_{\sigma^{-1}(1)}, \dots, a_{\sigma^{-1}(n)}),$$

and we have

$$\sigma(\lambda_i) = \lambda_{\sigma(i)}.$$

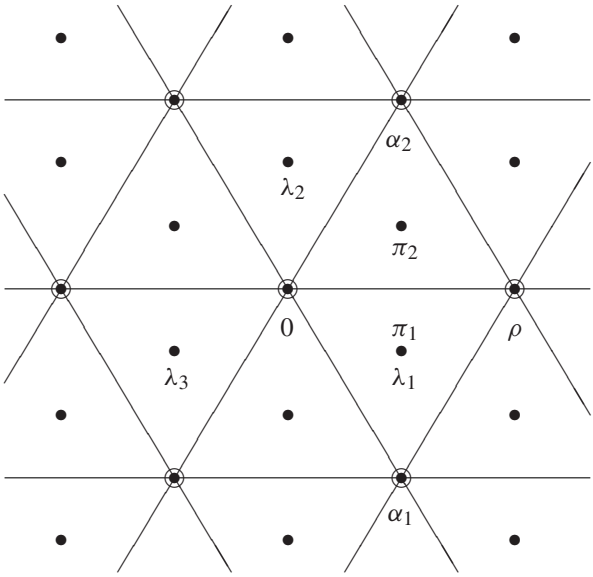
**Example 9.7.5.** In the following examples, white circles ○ (resp. black dots •) are the points in the root lattice $Q$ (resp. the weight lattice $P$).

(i) $G = SL_2(\mathbb{C})$



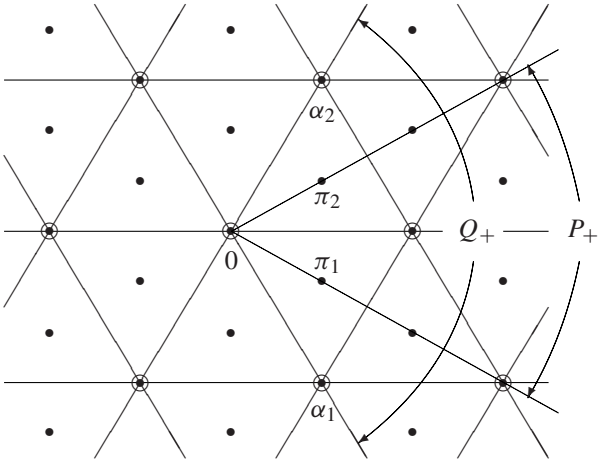

$$\Delta = \{\lambda_1 - \lambda_2, \lambda_2 - \lambda_1\} \supset \Delta_+ = \{\alpha_1 := \lambda_1 - \lambda_2\}$$
$$\Longrightarrow \rho = \frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha = \frac{1}{2}\alpha_1 = \pi_1.$$

(ii) $G = SL_3(\mathbb{C})$




$$\Delta = \{\lambda_i - \lambda_j \mid i \neq j\}$$
$$\supset \Delta_+ = \{\alpha_1 := \lambda_1 - \lambda_2,\ \alpha_2 := \lambda_2 - \lambda_3,\ \alpha_1 + \alpha_2 = \lambda_1 - \lambda_3\}$$
$$\Longrightarrow \rho = \frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha = \alpha_1 + \alpha_2 = \pi_1 + \pi_2.$$

Although $Q \subset P$, there is no inclusion relation between $Q_+$ and $P_+$ in this case as we see in the following figure:



**Theorem 9.7.6.** *Let $\Delta$ be a root system and $P$ (resp. $Q$) the weight lattice (resp. root lattice) of $\Delta$. We take an arbitrary subgroup $L$ of $P$ which contains $Q$. Then there exists uniquely up to isomorphism a connected semisimple algebraic group $G(\Delta, L)$ over $k$ whose root datum is isomorphic to $(\Delta, L)$. Moreover, $G(\Delta, L)$ contains no non-trivial connected normal closed subgroup if and only if the root system $\Delta$ is irreducible.*

For example, if $\Delta$ is of type $(A_{n-1})$, then $P/Q \simeq \mathbb{Z}/n\mathbb{Z}$ and the number of subgroups $L$ in $P$ containing $Q$ coincides with that of divisors of $n$.

For $G = G(\Delta, L)$ the subgroup $\mathrm{Ad}(G)$ of $\mathrm{Aut}(\mathfrak{g})$ is the identity component subgroup of $\mathrm{Aut}(\mathfrak{g})$ and is isomorphic to $G(\Delta, Q)$. Therefore, we call $G(\Delta, Q)$ the adjoint group of the root system $\Delta$. When $k = \mathbb{C}$, the fundamental group of $G(\Delta, L)$ with respect to the classical topology is isomorphic to $P/L$. In particular, $G(\Delta, P)$ is simply connected if $k = \mathbb{C}$. For this reason we call $G(\Delta, P)$ the simply connected (and connected) semisimple algebraic group associated to the root system $\Delta$ even when $k$ is a general algebraically closed field.

Finally, we explain the relation between semisimple algebraic groups and semisimple Lie algebras. If $k$ is a field of characteristic zero, then $\mathfrak{g} = \mathrm{Lie}(G)$ is a semisimple Lie algebra, and $\mathfrak{h} = \mathrm{Lie}(H)$ is its Cartan subalgebra. Hence Theorems 9.7.2, 9.7.3, 9.7.6 correspond to Theorem 9.2.4, 9.2.5, 9.3.2, respectively (Theorem 9.2.4 holds even if we replace $\mathrm{Aut}(\mathfrak{g})$ with $\mathrm{Ad}(G)$). We assumed that the base field $k$ is of characteristic zero in the case of semisimple Lie algebras; however, in the case of semisimple algebraic groups this assumption is not necessary. So we have a larger class of objects of study in the theory of semisimple algebraic groups. We note that the classification of semisimple Lie algebras over fields of positive characteristics is different from the classification in the case of characteristic zero.

## 9.8 Representations of semisimple algebraic groups

Let $G$ be a semisimple algebraic group over an algebraically closed field $k$ and let $H$ be a maximal torus of $G$. Set $\mathfrak{g} = \mathrm{Lie}(G)$ and $\mathfrak{h} = \mathrm{Lie}(H)$. We denote by $(\Delta, L)$ the root datum of $G$ with respect to $H$. We fix a positive root system $\Delta^+$ in $\Delta$ and use the notation in Section 9.3.

For a representation $\sigma : G \to GL(V)$ of $G$ we consider the direct sum decomposition (9.7.8), (9.7.9) of $V$ as an $H$-module. We say that $\lambda \in L$ is a weight of the $G$-module $V$ if $V_\lambda \neq \{0\}$.

**Theorem 9.8.1.**

(i) *Let $V$ be a finite-dimensional irreducible $G$-module. Then there exists a unique weight $\lambda$ (resp. $\mu$) of $V$ which is maximal (resp. minimal) with respect to the partial ordering* (9.3.17) *($\lambda$ and $\mu$ are called the highest weight and the lowest weight of $V$, respectively). Moreover, we have $\lambda \in L \cap P^+$, $\mu \in L \cap (-P^+)$ and $\mu = w_0(\lambda)$.*

(ii) *Assume that* $\lambda \in L \cap P^+$ *and* $\mu \in L \cap (-P^+)$. *Then there exists uniquely up to isomorphisms an irreducible* $G$*-module* $L^+(\lambda)$ (*resp.* $L^-(\mu)$) *whose highest* (*resp. lowest*) *weight is* $\lambda$ (*resp.* $\mu$).

This theorem corresponds to Theorem 9.5.2. However, an analogue of Theorem 9.5.1 is true only for base fields of characteristic zero.

**Theorem 9.8.2.** *Assume that the characteristic of* $k$ *is zero. Then any* $G$*-module is completely reducible.*

The corresponding problem in positive characteristics is related to Mumford's conjecture (Haboush's theorem).

**Example 9.8.3.** Let $G = SL_2(k)$. Then the subgroup

$$H = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \;\middle|\; a \in k^\times \right\}$$

is a maximal torus of $G$. Define a character $\rho \in L$ by

$$e^\rho \left( \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \right) = a$$

and set $\alpha = 2\rho$. Then we have

$$\begin{cases} \Delta = \{\pm\alpha\}, \quad \Delta^+ = \Pi = \{\alpha\}, \quad W = \{\pm 1\} \\ L = P = \mathbb{Z}\rho, \quad P^+ = \mathbb{N}\rho. \end{cases}$$

Let $V$ be the two-dimensional $G$-module given by the natural embedding of $G$ into $GL_2(k)$. The action of $G$ on $V$ is extended to the action on the symmetric algebra $S(V)$ of $V$, and the set $S^n(V)$ of elements of degree $n$ turns out to be a $G$-module. If $k$ is a field of characteristic zero, we have the isomorphisms

$$L^+(n\rho) = L^-(-n\rho) = S^n(V) \qquad (n \in \mathbb{N})$$

of $G$-modules. For a field $k$ of positive characteristic, the $G$-module $S^n(V)$ is not always irreducible and $L^+(n\rho) = L^-(-n\rho)$ is a $G$-submodule of $S^n(V)$.

For a finite-dimensional representation $\sigma : G \to GL(V)$ of $G$ we define its *character* by

$$\mathrm{ch}(V) = \sum_{\lambda \in L} (\dim V_\lambda) e^\lambda. \tag{9.8.1}$$

We can regard $\mathrm{ch}(V)$ as a function on $H$ or as an element of the group algebra of $L$. Indeed, the coordinate ring $k[H]$ of $H$ is a vector space spanned by the elements $\{e^\lambda\}_{\lambda \in L}$. If we regard the character $\mathrm{ch}(V)$ as a function on $H$, we have obviously

$$\mathrm{Tr}(\sigma(h)) = (\mathrm{ch}(V))(h) \qquad (h \in H). \tag{9.8.2}$$

Since the trace $\mathrm{Tr}(\sigma(g))$ is a class function on $G$ and $\bigcup_{x \in G} xHx^{-1}$ is dense in $G$, the character $\mathrm{Tr}(\sigma(g))$ (in the original sense) is completely determined by $\mathrm{ch}(V)$.

**Theorem 9.8.4.** *Assume that the characteristic of $k$ is zero. For $\lambda \in L \cap P^+$ and $\mu \in L \cap (-P^+)$ the $\mathfrak{g}$-modules associated to the $G$-modules $L^+(\lambda)$ and $L^-(\mu)$ coincide (see* (9.6.4)*) with $L^+(\lambda)$ and $L^-(\mu)$ in Section* 9.5*, respectively. Consequently we obtain*

$$\mathrm{ch}(L^+(\lambda)) = \frac{\sum_{w\in W}(-1)^{l(w)}e^{w(\lambda+\rho)-\rho}}{\prod_{\alpha\in\Delta^+}(1-e^{-\alpha})},$$
$$\mathrm{ch}(L^-(\mu)) = \frac{\sum_{w\in W}(-1)^{l(w)}e^{w(\mu-\rho)+\rho}}{\prod_{\alpha\in\Delta^+}(1-e^{\alpha})}.$$

As we see from Example 9.8.3, the character formula in the case of positive characteristics cannot be expressed as in the theorem above. The character formula for fields of positive characteristics was a long-standing problem, and there was a significant progress on it in the 1990s (see the last part of the introduction).

## 9.9 Flag manifolds

In this section we follow the notation in Section 9.8. It is known that there exist closed connected subgroups $B$ and $B^-$ of $G$ satisfying the conditions

$$\mathrm{Lie}(B) = \mathfrak{h} \oplus \left( \bigoplus_{\alpha\in\Delta^+} \mathfrak{g}_\alpha \right), \tag{9.9.1}$$

$$\mathrm{Lie}(B^-) = \mathfrak{h} \oplus \left( \bigoplus_{\alpha\in\Delta^+} \mathfrak{g}_{-\alpha} \right). \tag{9.9.2}$$

Set $N = R_u(B)$ and $N^- = R_u(B^-)$. Then we have

$$\mathrm{Lie}(N) = \bigoplus_{\alpha\in\Delta^+} \mathfrak{g}_\alpha, \tag{9.9.3}$$

$$\mathrm{Lie}(N^-) = \bigoplus_{\alpha\in\Delta^+} \mathfrak{g}_{-\alpha}, \tag{9.9.4}$$

$$B = HN, \qquad H \cap N = \{1\}, \tag{9.9.5}$$

$$B^- = HN^-, \qquad H \cap N^- = \{1\}. \tag{9.9.6}$$

We say that a subgroup of $G$ is a *Borel subgroup* if it is maximal in the family of connected solvable closed subgroups of $G$. The subgroups $B$ and $B^-$ of $G$ are Borel subgroups. If the characteristic of $k$ is zero, then the Lie algebra of a Borel subgroup is a Borel subalgebra of $\mathfrak{g}$.

**Theorem 9.9.1.**

(i) *For any pair $B_1$ and $B_2$ of Borel subgroups of $G$, there exists an element $g \in G$ such that $gB_1g^{-1} = B_2$.*

(ii) *The normalizer of a Borel subgroup $B$ of $G$ is $B$ itself.*

Denote by $X$ the set of all Borel subgroups of $G$ (if $k$ is a field of characteristic zero, $X$ can be identified with the set of Borel subalgebras of $\mathfrak{g}$). The group $G$ acts on $X$ by conjugation. Then for a Borel subgroup $B \in X$ the stabilizer of $B$ in $G$ is $B$ itself by Theorem 9.9.1. Thus we obtain a bijection

$$X \xleftarrow{\sim} G/B. \tag{9.9.7}$$

In general, the homogeneous space $K/K'$ obtained from an algebraic group $K$ and a closed subgroup $K'$ of $K$ is also an algebraic variety. So the above set $X$ has a natural structure of an algebraic variety. We call $X$ the *flag variety* (or the flag manifold) of $G$. For $g \in G$ the morphism $N^- \to G/B$ ($n \mapsto gnB$) is an open embedding, and the flag variety $X$ is covered by the open subsets $gN^-B/B \subset X$ which are isomorphic to $N^- \simeq k^{|\Delta^+|}$:

$$X = \bigcup_{g \in G} gN^-B/B. \tag{9.9.8}$$

**Theorem 9.9.2.** *The flag variety $X$ is a projective variety.*

**Example 9.9.3.** Let $G = SL_n(k)$. If we choose the positive root system $\Delta^+ = \{\lambda_i - \lambda_j \mid i < j\}$ in Example 9.7.4, then we get

$$B = \left\{ \begin{pmatrix} a_1 & & * \\ & \ddots & \\ 0 & & a_n \end{pmatrix} \middle| \prod_{i=1}^{n} a_i = 1 \right\},$$

$$B^- = \left\{ \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ * & & a_n \end{pmatrix} \middle| \prod_{i=1}^{n} a_i = 1 \right\}.$$

In this case, $N$ (resp. $N^-$) is the subgroup of $G$ consisting of upper (resp. lower) triangular matrices whose diagonal entries are 1. Moreover, the flag manifold $X = G/B$ can be identified with the set of flags in $k^n$:

$$\left\{ (V_i)_{i=0}^n \mid V_i \text{ is an } i\text{-dimensional subspace of } k^n,\ V_i \subset V_{i+1} \right\}.$$

Indeed, consider the natural action of $G$ on the set of flags in $k^n$. Then the stabilizer of the reference flag $(V_i^0)_{i=0}^n$ defined by

$$V_i^0 = ke_1 \oplus \cdots \oplus ke_i \quad (e_1, \ldots, e_n \text{ are the unit vectors of } k^n)$$

coincides with $B$. If $n = 2$, then $X = \mathbb{P}^1$ and the action of $G = SL_2(k)$ on the flag manifold $X = \mathbb{P}^1$ is given by the linear fractional (Möbius) transformation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (z) = \left( \frac{az+b}{cz+d} \right),$$
$$\left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(k),\ (z) \in \mathbb{P}^1 = k \cup \{\infty\} \right).$$

The double coset decomposition of $G$ by $B$ is given by the following theorem (the *Bruhat decomposition*).

**Theorem 9.9.4.** $G = \coprod_{w \in W} BwB$ (*more precisely, for each* $w \in W = N_G(H)/H$ *we choose a representative* $\dot{w} \in N_G(H)$ *of* $w$ *and consider the double coset* $B\dot{w}B$. *Since* $B\dot{w}B$ *is independent of the choice of a representative* $\dot{w}$ *of* $w$, *we simply denote it by* $BwB$).

Hence, if we set

$$X_w = BwB/B \subset X \tag{9.9.9}$$

for $w \in W$, then we have

$$X = \coprod_{w \in W} X_w \tag{9.9.10}$$

**Theorem 9.9.5.**

(i) *For any* $w \in W$ $X_w$ *is a locally closed submanifold of* $X$. *Furthermore,* $X_w$ *is isomorphic to* $k^{l(w)}$ ($l(w)$ *is the length of* $w$ *defined in Section* 9.3).

(ii) *The closure* $\overline{X}_w$ *of* $X_w$ *in* $X$ *coincides with* $\coprod_{y \leqq w} X_y$ (*here* $\leqq$ *is the Bruhat ordering defined in Section* 9.3).

We call $X_w$ (resp. $\overline{X}_w$) a *Schubert cell* (resp. a *Schubert variety*).

**Example 9.9.6.** Let $G = SL_n(k)$ and let us follow the notation in Examples 9.7.4 and 9.9.3. Then for $\sigma \in S_n = W$ we have

$$\begin{aligned} X_\sigma &= \{(V_i) \in X \mid \dim(V_j \cap V_i^0) = \#(\sigma[1,j] \cap [1,i]) \quad (i,j = 1,\dots,n)\} \\ &= \{(V_i) \in X \mid \dim \mathrm{Im}(V_j \to k^n/V_i^0) = \#(\sigma[1,j] \cap [i+1,n])(i,j = 1,\dots,n)\} \\ &= \left\{ (V_i) \in X \;\middle|\; \dim \frac{\mathrm{Im}(V_j \to k^n/V_i^0)}{\mathrm{Im}(V_{j-1} \to k^n/V_i^0)} = \begin{pmatrix} 1 & i < \sigma(j) \\ 0 & i \geqq \sigma(j) \end{pmatrix} \quad (j = 1,\dots,n) \right\}, \end{aligned}$$

where $[1,j] = \{1,2,\dots,j\}$, $[j+1,n] = \{j+1, j+2, \dots, n\}$. Furthermore, for a given $g = (g_{ij}) \in G$, we can determine the Schubert cell $X_\sigma$ which contains the point $gB$ by the following procedure. Define a sequence $(i_1,\dots,i_n)$ of natural numbers inductively by the following rules:

(i) $i_1$ is the largest number such that $g_{i_1 1} \neq 0$.
(ii) If the numbers $i_1, \ldots, i_{r-1}$ are given, then $i_r$ is the largest number such that

$$\det(g_{i_p q})_{p,q=1,\ldots,r} \neq 0.$$

Then we get $g \in B\sigma B$, where $\sigma \in S_n$ is given by $\sigma(r) = i_r$.

## 9.10 Equivariant vector bundles

In this section $G$ denotes any linear algebraic group over an algebraically closed field $k$. Assume that we are given an algebraic variety $X$ over $k$ endowed with an algebraic $G$-action.

**Definition 9.10.1.** Let $V \to X$ be an algebraic vector bundle on $X$. We say that $V$ is a *$G$-equivariant vector bundle* if we are given an action of the algebraic group $G$ on the algebraic variety $V$ satisfying the following condition:

$$\begin{aligned} &\text{For } g \in G, x \in X, \text{ we have } g(V_x) = V_{gx}, \\ &\text{and } g : V_x \to V_{gx} \text{ is a linear isomorphism,} \end{aligned} \tag{9.10.1}$$

where $V_x$ denotes the fiber of $V$ at $x \in X$.

For a vector bundle $V \to X$ (of finite rank) on $X$ denote by $\mathcal{O}_X(V)$ the sheaf of $\mathcal{O}_X$-modules consisting of algebraic sections of $V$. Then the correspondence $V \mapsto \mathcal{O}_X(V)$ gives an equivalence of categories:

$$\boxed{\text{vector bundles on } X} \xrightarrow{\sim} \boxed{\text{locally free } \mathcal{O}_X\text{-modules of finite rank}}.$$

Let us interpret the notion of equivariant vector bundles in the language of $\mathcal{O}_X$-modules.

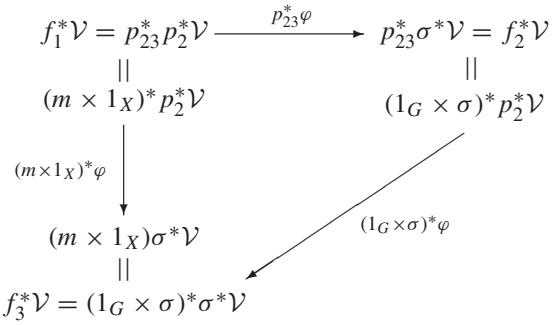
**Notation 9.10.2.**

$$m : \quad G \times G \to G, \qquad m(g_1, g_2) = g_1 g_2, \tag{9.10.2}$$
$$\sigma : \quad G \times X \to X, \qquad \sigma(g, x) = gx, \tag{9.10.3}$$
$$p_2 : \quad G \times X \to X, \qquad p_2(g, x) = x, \tag{9.10.4}$$
$$p_{23} : G \times G \times X \to G \times X, \qquad p_{23}(g_1, g_2, x) = (g_2, x), \tag{9.10.5}$$
$$f_1 : G \times G \times X \to X, \qquad f_1(g_1, g_2, x) = x, \tag{9.10.6}$$
$$f_2 : G \times G \times X \to X, \qquad f_2(g_1, g_2, x) = g_2 x, \tag{9.10.7}$$
$$f_3 : G \times G \times X \to X, \qquad f_3(g_1, g_2, x) = g_1 g_2 x, \tag{9.10.8}$$

**Definition 9.10.3.** Assume that $\mathcal{V}$ is a locally free $\mathcal{O}_X$-module of finite rank. We say that $\mathcal{V}$ is *$G$-equivariant* if we are given an isomorphism

$$\varphi : p_2^* \mathcal{V} \xrightarrow{\sim} \sigma^* \mathcal{V} \tag{9.10.9}$$

of $\mathcal{O}_{G\times X}$-modules for which the following diagram is commutative:




(9.10.10)

The commutativity of (9.10.10) is called the *cocycle condition*.

Note that giving a $G$-equivariant structure on $V$ is equivalent to giving a $G$-equivariant structure on the corresponding $\mathcal{O}_X$-module $\mathcal{V} = \mathcal{O}_X(V)$. Indeed, we obtain an isomorphism $V_x \xrightarrow{\sim} V_{gx}$ from (9.10.9) by taking the fibers at $(g, x) \in G \times X$, ant it gives the action of $g \in G$ on $V$.

Hereafter $V$ is a $G$-equivariant vector bundle on $X$ and we set $\mathcal{V} = \mathcal{O}_X(V)$. Then we get a natural linear $G$-action on the vector space $\Gamma(X, \mathcal{V})$ by

$$(gs)(x) = g(s(g^{-1}x)) \quad (g \in G,\ x \in X,\ s \in \Gamma(X, \mathcal{V})). \tag{9.10.11}$$

In terms of the $G$-equivariant structure on $\mathcal{V}$, it can be described as follows. By (9.10.9) we obtain an isomorphism

$$\Gamma(G \times X, p_2^*\mathcal{V}) \xrightarrow{\sim} \Gamma(G \times X, \sigma^*\mathcal{V}). \tag{9.10.12}$$

If we denote the coordinate ring of $G$ by $k[G]$ (since $G$ is a linear algebraic group, $G$ is affine), we have

$$\Gamma(G \times X, p_2^*\mathcal{V}) = \Gamma(G \times X, \mathcal{O}_G \boxtimes \mathcal{V}) = k[G] \otimes \Gamma(X, \mathcal{V}). \tag{9.10.13}$$

Now consider the morphisms $\varepsilon_i : G \times X \xrightarrow{\sim} G \times X$ $(i = 1, 2)$ defined by $\varepsilon_1(g, x) = (g, gx)$, $\varepsilon_2(g, x) = (g, g^{-1}x)$. Then $\varepsilon_1 = \varepsilon_2^{-1}$, $p_2 \circ \varepsilon_1 = \sigma$ and

$$\begin{aligned} \Gamma(G \times X, \sigma^*\mathcal{V}) &= \Gamma(G \times X, \varepsilon_1^* p_2^*\mathcal{V}) \\ &= \Gamma(G \times X, \varepsilon_{2*} p_2^*\mathcal{V}) \\ &= \Gamma(G \times X, p_2^*\mathcal{V}) \\ &= k[G] \otimes \Gamma(X, \mathcal{V}). \end{aligned} \tag{9.10.14}$$

Therefore, by combining this with (9.10.12) we get

$$k[G] \otimes \Gamma(X, \mathcal{V}) \xrightarrow{\sim} k[G] \otimes \Gamma(X, \mathcal{V}). \tag{9.10.15}$$

If we restrict it to $\Gamma(X, \mathcal{V}) = 1 \otimes \Gamma(X, \mathcal{V})$, we finally obtain a linear map

$$\widetilde{\varphi} : \Gamma(X, \mathcal{V}) \to k[G] \otimes \Gamma(X, \mathcal{V}). \tag{9.10.16}$$

Assume that the image of $s \in \Gamma(X, \mathcal{V})$ by $\widetilde{\varphi}$ is given by

$$\widetilde{\varphi}(s) = \sum_i f_i \otimes s_i \qquad (f_i \in k[G] \text{ and } s_i \in \Gamma(X, \mathcal{V})).$$

Then the $G$-action (9.10.11) on $\Gamma(X, \mathcal{V})$ is given by

$$gs = \sum_i f_i(g) s_i \quad (g \in G).$$

In particular, this $G$-action is algebraic.

Replacing $\Gamma(G \times X, \bullet)$, $\Gamma(X, \bullet)$ with $H^i(G \times X, \bullet)$, $H^i(X, \bullet)$, respectively, in the above arguments, we obtain a linear map

$$\widetilde{\varphi} : H^i(X, \mathcal{V}) \to k[G] \otimes H^i(X, \mathcal{V}) \tag{9.10.17}$$

similarly (since $G$ is affine, we have $H^i(G, \mathcal{O}_G) = 0$ for $i > 0$). Thus the cohomology groups $H^i(X, \mathcal{V})$ are also endowed with structures of $G$-modules in the sense of Section 9.6. Indeed, the cocycle condition (9.10.10) implies that it actually gives an action of the group $G$, and the algebraicity of this action follows from (9.10.17).

The construction of representations via equivariant vector bundles explained above is a fundamental technique in representation theory.

## 9.11 The Borel–Weil–Bott theorem

In this section $G$ is a connected semisimple algebraic group over $k$ and $X$ denotes its flag variety. We will follow the notation of Sections 9.8 and 9.9.

For a $G$-equivariant vector bundle $V$ on $X = G/B$ its fiber $V_B$ of $V$ at $B \in G/B$ is a $B$-module. Conversely, for any finite-dimensional $B$-module $U$ we can construct a $G$-equivariant vector bundle $V$ on $X$ such that $V_B = U$ as follows. Consider the locally free $B$-action on the trivial vector bundle $G \times U$ on $G$ given by

$$b \cdot (g, u) = (gb^{-1}, bu) \qquad (b \in B,\ (g, u) \in G \times U). \tag{9.11.1}$$

Then the quotient space $V = B\backslash(G \times U)$ obtained by this action is an algebraic vector bundle on $X = G/B$. Indeed, we can show that it is locally trivial by using the affine covering of $X$ in (9.9.8). Moreover, the action of $G$ on $G \times U$ given by

$$g_1 : (g, u) \mapsto (g_1 g, u) \qquad (g_1 \in G,\ (g, u) \in G \times U)$$

induces an action of $G$ on $V$ for which $V$ turns out to be a $G$-equivariant vector bundle on $X$ with $V_B = U$. This $G$-equivariant vector bundle $V$ is denoted by $\Lambda(U)$. W have obtained a one-to-one correspondence between $G$-equivariant vector

bundles on $X$ and finite-dimensional $B$-modules. In particular, a $G$-equivariant line bundle on $X$ corresponds to a one-dimensional $B$-module. Since the actions of the unipotent radical $N = R_u(B)$ of $B$ on one-dimensional $B$-modules are trivial (Jordan decompositions are preserved by homomorphisms of algebraic groups), $G$-equivariant line bundles on $X$ correspond to characters $\lambda \in L$ of $H = B/N$. For a character $\lambda \in L$ we denote the corresponding $G$-equivariant line bundle on $X$ by $\Lambda(\lambda)$ and set $\mathcal{L}(\lambda) = \mathcal{O}_X(\Lambda(\lambda))$.

**Example 9.11.1.** Let $G = SL_2(k)$. Then $X = \mathbb{P}^1 = k \cup \{\infty\}$ and $L = P = \mathbb{Z}\rho$ as we have already seen in Example 9.9.3. We can easily see that

$$\mathcal{L}(n\rho) = \mathcal{O}_{\mathbb{P}^1}(-n), \tag{9.11.2}$$

where $\mathcal{O}_{\mathbb{P}^1}(-n)$ is Serre's twisted sheaf. Let us consider the two open subsets $U_1 = k$, $U_2 = k^\times \cup \{\infty\} \cong k$ of $X$ and take a natural coordinate $(z)$ (resp. $x = 1/z$) of $U_1$ (resp. $U_2$). Then the line bundle $\Lambda(n\rho)$ can be obtained by gluing the two trivial line bundles $U_1 \times k$ and $U_2 \times k$ (on $U_1$ and $U_2$, respectively) by the identification

$$(z, u) = (x, v) \Longleftrightarrow x = \frac{1}{z}, \quad v = x^{-n}u. \tag{9.11.3}$$

The action of $G$ on $\Lambda(n\rho)$ is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (z, u) = \left( \frac{az + b}{cz + d},\ (cz + d)^n u \right). \tag{9.11.4}$$

Note that each cohomology group $H^i(X, \mathcal{L}(\lambda))$ is finite dimensional because $X$ is a projective variety. In order to describe the $G$-module structure of $H^i(X, \mathcal{L}(\lambda))$ let us introduce some notation. First set

$$\begin{aligned} P_{\text{sing}} &= \{\lambda \in P \mid {}^{\exists}\alpha \in \Delta \text{ such that } \langle \lambda - \rho,\ \alpha^\vee \rangle = 0\}, \\ P_{\text{reg}} &= P \setminus P_{\text{sing}}. \end{aligned} \tag{9.11.5}$$

We define a shifted action of $W$ on $P$ by

$$w \bigstar \lambda = w(\lambda - \rho) + \rho \quad (w \in W, \lambda \in P). \tag{9.11.6}$$

Then $W \bigstar (P_{\text{sing}}) = P_{\text{sing}}$ and we see that the anti-dominant part $-P^+$ of $P$ is a fundamental domain with respect to this shifted action (9.11.6) of $W$ on $P_{\text{reg}}$.

**Theorem 9.11.2.** *Let $\lambda \in L (\subset P)$. Then we have*

(i) *If $\langle \lambda, \alpha^\vee \rangle \leqq 0$ for any $\alpha \in \Delta^+$ (i.e., $\lambda \in -P^+$), then the line bundle $\mathcal{L}(\lambda)$ is generated by global sections. Namely, the natural morphism*

$$\mathcal{O}_X \otimes_k \Gamma(X, \mathcal{L}(\lambda)) \to \mathcal{L}(\lambda)$$

*is surjective.*

(ii) *The line bundle $\mathcal{L}(\lambda)$ is ample if and only of $\langle \lambda, \alpha^\vee \rangle < 0$ for any $\alpha \in \Delta^+$.*

(iii) *Assume that the characteristic of k is zero.*

(a) *If* $\lambda \in P_{\mathrm{sing}}$, *then* $H^i(X, \mathcal{L}(\lambda)) = 0 \quad (i \geqq 0)$.

(b) *Let* $\lambda \in P_{\mathrm{reg}}$ *and take* $w \in W$ *such that* $w\bigstar\lambda \in -P^+$. *Then we have*

$$H^i(X, \mathcal{L}(\lambda)) = \begin{cases} L^-(w\bigstar\lambda) & (i = l(w)) \\ 0 & (i \neq l(w)). \end{cases}$$

It is an open problem to determine the $G$-module structures of $H^i(X, \mathcal{L}(\lambda))$ for fields of positive characteristics.

# 10

# Conjugacy Classes of Semisimple Lie Algebras

In this chapter we discuss conjugacy classes in semisimple Lie algebras using the theory of invariant polynomials for the adjoint representations. In particular, we will give a parametrization of conjugacy classes and present certain geometric properties of them. Those results will be used in the next chapter to establish the Beilinson–Bernstein correspondence.

In the rest of this book we will always work over the complex number field $\mathbb{C}$ although the arguments below work over any algebraically closed field of characteristic zero except for certain points where the Riemann–Hilbert correspondence is used.

We denote by $G$ a connected, simply connected, semisimple algebraic group over $\mathbb{C}$ and fix a maximal torus $H$ of $G$. We set $\mathfrak{g} = \mathrm{Lie}(G)$ and $\mathfrak{h} = \mathrm{Lie}(H)$. We denote by $\Delta$ the root system of $G$ with respect to $H$ and fix a positive root system $\Delta^+$. We also use the notation in Chapter 9. For example, $B$ stands for the Borel subgroup of $G$ such that $\mathrm{Lie}(B) = \mathfrak{h} \oplus (\bigoplus_{\alpha\in\Delta^+} \mathfrak{g}_\alpha)$ and we denote the flag variety $G/B$ by $X$. This notation will be fixed until the end of this book.

## 10.1 The theory of invariant polynomials

Let $K$ be a group acting linearly on a vector space $V$. Then $K$ acts also on the space $\mathbb{C}[V]$ of polynomial functions on $V$:

$$(k \cdot f)(v) = f(k^{-1}v) \qquad (k \in K,\ f \in \mathbb{C}[V],\ v \in V). \tag{10.1.1}$$

The polynomials belonging to

$$\mathbb{C}[V]^K := \{f \in \mathbb{C}[V] \mid k \cdot f = f \quad (k \in K)\} \tag{10.1.2}$$

are called *invariant polynomials*. The set $\mathbb{C}[V]^K$ of all invariant polynomials is a subring of $\mathbb{C}[V]$, and is called the *invariant polynomial ring*. The aim of the theory of invariant polynomials is to study the structure of the invariant polynomial rings $\mathbb{C}[V]^K$ for various $K$'s and $V$'s. Note that the ring $\mathbb{C}[V]^K$ is not necessarily finitely generated as a $\mathbb{C}$-algebra (Nagata's counterexample to Hilbert's 14th problem). However, if the

group $K$ is reductive, the ring $\mathbb{C}[V]^K$ is finitely generated and hence, the theory of invariant polynomials is reduced to the geometric study of the quotient algebraic variety

$$K\backslash V := \operatorname{Spec} \mathbb{C}[V]^K. \tag{10.1.3}$$

If $K$ is a finite group, then $K\backslash V$ coincides with the set-theoretical quotient of $V$; however, for a general group $K$, distinct orbits may correspond to the same point in $K\backslash V$.

In what follows, we study the structure of the invariant polynomial ring $\mathbb{C}[\mathfrak{g}]^G$ with respect to the adjoint representation $\mathrm{Ad} : G \to GL(\mathfrak{g})$ of $G$. The following theorem is fundamental.

**Theorem 10.1.1 (Chevalley's restriction theorem).** *The restriction of the natural map* $\mathbb{C}[\mathfrak{g}] \to \mathbb{C}[\mathfrak{h}]$ *to* $\mathbb{C}[\mathfrak{g}]^G$ :

$$\mathrm{rest} : \mathbb{C}[\mathfrak{g}]^G \to \mathbb{C}[\mathfrak{h}] \tag{10.1.4}$$

*is injective, and its image coincides with the invariant polynomial ring* $\mathbb{C}[\mathfrak{h}]^W$ *with respect to the action of the Weyl group* $W$ *on* $\mathfrak{h}$.

*Proof.* First, let us prove the injectivity of the map rest. Let $f \in \mathrm{Ker}(\mathrm{rest})$. Then it follows from $f|_{\mathfrak{h}} = 0$ that $f|_{\mathrm{Ad}(G)\mathfrak{h}} = 0$ by $f \in \mathbb{C}[\mathfrak{g}]^G$. Hence it is sufficient to show that $\mathrm{Ad}(G)\mathfrak{h}$ is dense in $\mathfrak{g}$. Note that $\mathrm{Ad}(G)\mathfrak{h}$ coincides with the image of the morphism $\varphi : G/H \times \mathfrak{h} \to \mathfrak{g}$ given by $\varphi(gH, h) = \mathrm{Ad}(g)h$. Hence by $\dim(G/H \times \mathfrak{h}) = \dim \mathfrak{g}$ we have only to prove that the tangent map $d\varphi$ of $\varphi$ at $(eH, h_0)$ is an isomorphism for some $h_0 \in \mathfrak{h}$. Under the identifications $T_{eH}(G/H) = \mathfrak{g}/\mathfrak{h}$, $T_{h_0}(\mathfrak{h}) = \mathfrak{h}$, $T_{h_0}(\mathfrak{g}) = \mathfrak{g}$ the map $d\varphi : \mathfrak{g}/\mathfrak{h} \times \mathfrak{h} \to \mathfrak{g}$ is given by

$$d\varphi(\overline{x}, h) = [x, h_0] + h. \qquad (x \in \mathfrak{g},\ h \in \mathfrak{h}).$$

Hence, if we take $h_0 \in \mathfrak{h}$ satisfying $\alpha(h_0) \neq 0$ for any $\alpha \in \Delta$, then $d\varphi : \mathfrak{g}/\mathfrak{h} \times \mathfrak{h} \to \mathfrak{g}$ is an isomorphism. The proof of the injectivity of rest is complete.

Let us show $\mathrm{Im}(\mathrm{rest}) \subset \mathbb{C}[\mathfrak{h}]^W$. Recall that $W = N_G(H)/H$. For $f \in \mathbb{C}[\mathfrak{g}]$ and a representative $\dot{w}$ of $w \in W$ in $N_G(H)$ we have $\dot{w} \cdot f|_{\mathfrak{h}} = w \cdot (f|_{\mathfrak{h}})$, and hence $\mathrm{rest}(f) \in \mathbb{C}[\mathfrak{h}]^W$ for any $f \in \mathbb{C}[\mathfrak{g}]^G$.

We finally prove the converse $\mathrm{Im}(\mathrm{rest}) \supset \mathbb{C}[\mathfrak{h}]^W$. Recall that finite-dimensional irreducible representations of $\mathfrak{g}$ are parameterized by $P^+$ (Section 9.5). Denote the irreducible representation with highest weight $\lambda \in P^+$ by $\sigma_\lambda : \mathfrak{g} \to \mathfrak{gl}(L^+(\lambda))$, and for each natural number $m$ define a polynomial $f_{\lambda,m} \in \mathbb{C}[\mathfrak{g}]$ by $f_{\lambda,m}(x) = \mathrm{Tr}(\sigma_\lambda(x)^m)$. Then we have obviously $f_{\lambda,m} \in \mathbb{C}[\mathfrak{g}]^G$. Therefore, it suffices to show that the set $\{\mathrm{rest}(f_{\lambda,m}) \mid \lambda \in P^+,\ m \in \mathbb{N}\}$ spans $\mathbb{C}[\mathfrak{h}]^W$. Note that $\mathrm{rest}(f_{\lambda,m})$ can be obtained from the character $\mathrm{ch}(L^+(\lambda)) \in \mathbb{Z}[P]^W \subset \mathbb{C}[P]^W$ of $L^+(\lambda)$ by

$$\mathrm{ch}(L^+(\lambda)) = \sum_\mu a_{\mu\lambda} e^\mu \Longrightarrow \mathrm{rest}(f_{\lambda,m}) = \sum_\mu a_{\mu\lambda} \mu^m. \tag{10.1.5}$$

Denote the set of homogeneous polynomials of degree $m$ on $\mathfrak{h}$ by $\mathbb{C}[\mathfrak{h}]_m$. Then we have $\mathbb{C}[\mathfrak{h}] = \bigoplus_{m=0}^\infty \mathbb{C}[\mathfrak{h}]_m$, and its completion (the ring of formal power series on $\mathfrak{h}$)

is given by $\mathbb{C}[[\mathfrak{h}]] = \prod_{m=0}^{\infty} \mathbb{C}[\mathfrak{h}]_m$. We embed $\mathbb{C}[P]$ in $\mathbb{C}[[\mathfrak{h}]]$ by $e^{\mu} \mapsto \sum_{m=0}^{\infty} \frac{1}{m!}\mu^m$ and regard it as a subring of $\mathbb{C}[[\mathfrak{h}]]$. Let $p_m : \mathbb{C}[[\mathfrak{h}]] \to \mathbb{C}[\mathfrak{h}]_m$ be the projection. To complete the proof it is enough to prove the following assertions:

$$\{\mathrm{ch}\, L^+(\lambda) \mid \lambda \in P^+\} \text{ is a basis of } \mathbb{C}[P]^W, \tag{10.1.6}$$

$$p_m(\mathbb{C}[P]^W) = \mathbb{C}[\mathfrak{h}]_m^W. \tag{10.1.7}$$

Let us prove (10.1.6). Set $S_\lambda = \sum_{\mu \in W(\lambda)} e^{\mu}$ for $\lambda \in P^+$. For any $\mu \in P$ there exists a unique $\lambda \in P^+$ such that $\mu \in W(\lambda)$, and hence the set $\{S_\lambda \mid \lambda \in P^+\}$ is a basis of $\mathbb{C}[P]^W$. On the other hand by the Weyl character formula we obtain

$$\mathrm{ch}\, L^+(\lambda) \in S_\lambda + \sum_{\substack{\mu \in P^+ \\ \mu < \lambda}} \mathbb{Z} S_\mu,$$

from which (10.1.6) follows immediately. It remains to show (10.1.7). Since $P$ is a $\mathbb{Z}$-lattice in $\mathfrak{h}^*$, the set $\{\mu^m \mid \mu \in P\}$ spans $\mathbb{C}[\mathfrak{h}]_m$ and hence we get $p_m(\mathbb{C}[P]) = \mathbb{C}[\mathfrak{h}]_m$. Let $f \in \mathbb{C}[\mathfrak{h}]_m^W$. We take $\overline{f} \in \mathbb{C}[P]$ satisfying $p_m(\overline{f}) = f$ and set $\widetilde{f} = \frac{1}{|W|} \sum_{w \in W} w \cdot \overline{f}$. Then we have $\widetilde{f} \in \mathbb{C}[P]^W$ and $p_m(\widetilde{f}) = f$. This completes the proof of (10.1.7). □

Theorem 10.1.1 asserts that $\mathbb{C}[\mathfrak{g}]^G$ is isomorphic to $\mathbb{C}[\mathfrak{h}]^W$. Therefore, our problem is to determine the structure of the invariant polynomial ring $\mathbb{C}[\mathfrak{h}]^W$ with respect to the $W$-action. The ring $\mathbb{C}[\mathfrak{h}]^W$ has the following remarkable property, which follows from the fact that the Weyl group $W$ is generated by reflections. For the proof, see Bourbaki [Bou].

**Theorem 10.1.2.** *The ring $\mathbb{C}[\mathfrak{h}]^W$ is generated by $l$ $(= \dim \mathfrak{h})$ algebraically independent homogeneous polynomials over $\mathbb{C}$. In particular, $\mathbb{C}[\mathfrak{h}]^W$ is isomorphic to a polynomial ring of $l$ variables.*

Combining this result with Theorem 10.1.1, we get

$$G\backslash\mathfrak{g} \cong W\backslash\mathfrak{h} \cong \mathbb{A}^l. \tag{10.1.8}$$

Moreover, the degrees of the $l$ independent generators in Theorem 10.1.2 can be explicitly described in terms of root systems as follows. Set

$$c = s_{\alpha_1} \cdots s_{\alpha_l} \in W, \tag{10.1.9}$$

where $\Pi = \{\alpha_1, \ldots, \alpha_l\}$ is the set of simple roots. It is called the *Coxeter transformation*. The conjugacy class of $c$ does not depend either on the choice of $\Pi$ or on the numbering of $\alpha_1, \ldots, \alpha_l$. First assume that the root system $\Delta$ is irreducible. Set

$$h = (\text{the order of } c) = (\textit{Coxeter number}) \tag{10.1.10}$$

and write the eigenvalues of the operator $c$ on $\mathfrak{h}$ by

$$\exp(2\pi\sqrt{-1}m_1/h), \ldots, \exp(2\pi\sqrt{-1}m_l/h),$$
$$(0 \leqq m_1 \leqq m_2 \leqq \cdots \leqq m_l < h).$$

Then it is known that

$$0 < m_1 = 1 < m_2 \leqq \cdots \leqq m_l < h, \quad m_i + m_{l-i+1} = h \tag{10.1.11}$$

$$\sum_{i=1}^{l} m_i = |\Delta^+|. \tag{10.1.12}$$

We call these numbers $m_1, \ldots, m_l$ the *exponents* of the irreducible root system $\Delta$. If the root system $\Delta$ is not irreducible, the exponents of $\Delta$ are defined to be the union of those in its irreducible components (this definition is different from Bourbaki's). Note that the multiset of exponents of a root system $\Delta$ consists of $l$ natural numbers and the sum of all exponents is equal to the cardinality $|\Delta^+|$ of the set of positive roots.

**Theorem 10.1.3.** *Denote the exponents of the root system $\Delta$ by $m_1, \ldots, m_l$. Then the degrees of the $l$ algebraically independent homogeneous polynomials in Theorem* 10.1.2 *are given by $m_1 + 1, \ldots, m_l + 1$.*

**Example 10.1.4.** Let $G = SL_n(\mathbb{C})$, $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ and choose a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ as follows:

$$\mathfrak{h} = \left\{ d(a_1, \ldots, a_n) \;\middle|\; \sum_{i=1}^{n} a_i = 0 \right\},$$

where

$$d(a_1, \ldots, a_n) = \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix}.$$

For $i = 1, \ldots, n$, define $\lambda_i \in \mathfrak{h}^*$ by $\lambda_i(d(a_1, \ldots, a_n)) = a_i$. Then $\mathbb{C}[\mathfrak{h}] = \mathbb{C}[\lambda_1, \ldots, \lambda_n]/(\lambda_1 + \cdots + \lambda_n)$. Since the action of $W = S_n$ on $\mathfrak{h}$ is given by $\sigma(\lambda_i) = \lambda_{\sigma(i)}$, $\mathbb{C}[\mathfrak{h}]^W$ is the ring of symmetric polynomials in $\lambda_1, \lambda_2, \ldots, \lambda_n$. Let us consider the following elementary symmetric polynomials of $\lambda_1, \ldots, \lambda_n$:

$$\sigma_1 = \sum_{i=1}^{n} \lambda_i, \ \sigma_2 = \sum_{i<j} \lambda_i \lambda_j, \ldots\ldots, \sigma_n = \lambda_1 \cdots \lambda_n.$$

Then the polynomial functions $\sigma_2, \ldots, \sigma_n$ on $\mathfrak{h}$ are algebraically independent generators of $\mathbb{C}[\mathfrak{h}]^W$ ($\sigma_1 = 0$ on $\mathfrak{h}$). Since the eigenvalues of the Coxeter transformation $(12)(23)\cdots(n-1, n) = (1\ 2\ 3 \cdots n) \in W = S_n$ are $\{\exp(2\pi\sqrt{-1}k/n)\}_{k=1}^{n-1}$ and the Coxeter number is $h = n$, the exponents are $1, \ldots, n-1$. This agrees with the fact that the degrees of $\sigma_2, \ldots, \sigma_n$ are $2, \ldots, n$, respectively. Now let us define polynomial functions $\overline{\sigma}_k \in \mathbb{C}[\mathfrak{g}]$ by $\det(t1 - x) = t^n + \sum_{k=1}^{n} \overline{\sigma}_k(x) t^{n-k}$ ($x \in \mathfrak{g}$). Then we have $\overline{\sigma}_k \in \mathbb{C}[\mathfrak{g}]^G$ and $\overline{\sigma}_1 = 0$, $\mathrm{rest}(\overline{\sigma}_k) = \sigma_k$ ($k = 2, \ldots, n$). This implies that $\overline{\sigma}_2, \ldots, \overline{\sigma}_n$ are algebraically independent homogeneous generators of $\mathbb{C}[\mathfrak{g}]^G$.

## 10.2 Classification of conjugacy classes

The $G$-orbit through $x \in \mathfrak{g}$ for the adjoint action of $G$ on $\mathfrak{g}$ is called the *conjugacy class* of $x$. For the study of conjugacy classes, the notion of Jordan decompositions is important. We say that $x \in \mathfrak{g}$ is semisimple (resp. nilpotent) if the linear operator $\mathrm{ad}(x) \in \mathfrak{gl}(\mathfrak{g})$ is semisimple (resp. nilpotent).

**Theorem 10.2.1.**

(i) *For any $x \in \mathfrak{g}$ there exists a unique pair of elements $(x_s, x_n)$ where $x_s$ is semisimple and $x_n$ nilpotent, such that*

$$x = x_s + x_n, \quad [x_s, x_n] = 0 \tag{10.2.1}$$

(*We call the decomposition* (10.2.1) *the* **Jordan decomposition** *of $x$. The elements $x_s$ and $x_n$ are called the semisimple part and the nilpotent part of $x$, respectively*).

(ii) *Let $\sigma : \mathfrak{g} \to \mathfrak{gl}(V)$ be a finite-dimensional representation of $\mathfrak{g}$. If $x \in \mathfrak{g}$ is semisimple* (*resp. nilpotent*), *then $\sigma(x)$ is semisimple* (*resp. nilpotent*).

For the proof, see Humphreys [Hu2].

A conjugacy class consisting of semisimple elements (resp. nilpotent elements) is called a *semisimple conjugacy class* (resp. *nilpotent conjugacy class*). The classification of semisimple conjugacy classes is given by the following.

**Theorem 10.2.2.** *For any semisimple conjugacy class $O$ the intersection $O \cap \mathfrak{h}$ with $\mathfrak{h}$ is a $W$-orbit in $\mathfrak{h}$. Hence the set of semisimple conjugacy classes in $\mathfrak{g}$ is parameterized by the set $W\backslash\mathfrak{h}$ of $W$-orbits in $\mathfrak{h}$.*

*Proof.* Suppose that $x \in \mathfrak{g}$ is a semisimple element. Then it follows directly from the definition of Cartan subalgebras that there exists a Cartan subalgebra $\mathfrak{h}'$ containing $x$. By the remark at the end of Section 9.7 we can take $g \in G$ so that $\mathrm{Ad}(g)\mathfrak{h}' = \mathfrak{h}$. Thus we have shown $O_G(x) \cap \mathfrak{h} \neq \emptyset$. Since the action of the Weyl group $W$ on $\mathfrak{h}$ is the restriction of the adjoint action of $N_G(H) = N_G(\mathfrak{h}) = \{g \in G \mid \mathrm{Ad}(g)\mathfrak{h} = \mathfrak{h}\}$ on $\mathfrak{g}$, $O_G(x) \cap \mathfrak{h}$ is a union of $W$-orbits. It remains to prove that it is a single orbit. Assume that two points $h_1, h_2 \in \mathfrak{h}$ are conjugate in $\mathfrak{g}$. We will show that these two points lie in the same $W$-orbit. By virtue of Theorem 10.1.1 $h_1$ and $h_2$ are mapped to the same point in $W\backslash\mathfrak{h} = \mathrm{Spec}(\mathbb{C}[\mathfrak{h}]^W)$. Note that $W\backslash\mathfrak{h}$ coincides with the set-theoretical quotient since $W$ is a finite group. This means that $h_1$ and $h_2$ are conjugate under the $W$-action. □

In order to classify general conjugacy classes (which are not necessarily semisimple) we use Jordan decompositions. Assume that two elements $x, y \in \mathfrak{g}$ are conjugate. Then their semisimple parts $x_s, y_s$ are also conjugate (by the uniqueness of Jordan decompositions). Hence we may assume that $x_s = y_s$ from the beginning. Furthermore, thanks to Theorem 10.2.2, we can reduce the situation to the case of $h = x_s = y_s \in \mathfrak{h}$. Therefore, setting

$$\mathcal{N}_h = \{z \in \mathfrak{g} \mid z \text{ is a nilpotent element such that } [h, z] = 0\}$$

for $h \in \mathfrak{h}$, it suffices to determine when two elements $h + z$ and $h + z'$ ($z, z' \in \mathcal{N}_h$) are conjugate to each other. If $\mathrm{Ad}(g)(h+z) = h+z'$ for some $g \in G$, then it follows from the uniqueness of Jordan decompositions that $\mathrm{Ad}(g)h = h$, i.e.,

$$g \in Z_G(h) := \{g \in G \mid \mathrm{Ad}(g)h = h\}.$$

Hence we reduced the problem to the classification of orbits for the adjoint $Z_G(h)$-action on $\mathcal{N}_h$.

**Theorem 10.2.3.** *For $h \in \mathfrak{h}$, we denote by $\mathfrak{k}_h$ the derived Lie algebra $[\mathfrak{z}_\mathfrak{g}(h), \mathfrak{z}_\mathfrak{g}(h)]$ of the centralizer $\mathfrak{z}_\mathfrak{g}(h) = \{x \in \mathfrak{g} \mid [x, h] = 0\}$ of $h$. Then we have the following:*

(i) *The Lie algebra $\mathfrak{k}_h$ is semisimple and $\mathcal{N}_h$ coincides with the set of nilpotent elements in $\mathfrak{k}_h$. In particular, we have $\mathcal{N}_h \subset \mathfrak{k}_h$.*

(ii) *The algebraic group $Z_G(h)$ is connected, and the orbits for the adjoint action of $Z_G(h)$ on $\mathcal{N}_h$ are exactly the nilpotent conjugacy classes in the semisimple Lie algebra $\mathfrak{k}_h$.*

*Proof.* Among connected closed subgroups $H'$ of $H$ satisfying $\mathrm{Lie}(H') \ni h$ there exists the smallest one $S$ (if we regard $G$ as a complex Lie group, $S$ is the closure of the one-parameter subgroup generated by $h$). Then $Z_G(h) = Z_G(S)$ holds. Since $S$ is a torus, $Z_G(S)$ is connected and reductive from a general fact on semisimple algebraic groups. Hence if we denote the center of $\mathfrak{z}_\mathfrak{g}(h) = \mathrm{Lie}(Z_G(h))$ by $\mathfrak{c}$, we have $\mathfrak{z}_\mathfrak{g}(h) = \mathfrak{c} \oplus \mathfrak{k}_h$ and $\mathfrak{k}_h$ is a semisimple ideal of $\mathfrak{z}_\mathfrak{g}(h)$. Let $z_1 \in \mathfrak{c}$, $z_2 \in \mathfrak{k}_h$, and set $z = z_1 + z_2 \in \mathfrak{z}_\mathfrak{g}(h)$. We see from $\mathfrak{c} \subset \mathfrak{h}$ that $z_1$ is a semisimple element of $\mathfrak{g}$. Suppose that the Jordan decomposition of $z_2$ in $\mathfrak{k}_h$ is given by $z_2 = (z_2)_s + (z_2)_n$. Then by applying Theorem 10.2.1 (ii) to the representation $\mathrm{ad} : \mathfrak{k}_h \to \mathfrak{gl}(\mathfrak{g})$ of $\mathfrak{k}_h$ we see that the Jordan decomposition of $z$ in $\mathfrak{g}$ is given by $z_s = z_1 + (z_2)_s$, $z_n = (z_2)_n$. It follows that $\mathcal{N}_h$ coincides with the set of nilpotent elements in $\mathfrak{k}_h$. Since $Z_G(h)$ is connected, $\mathrm{Ad}(Z_G(h))\big|_{\mathfrak{k}_h}$ coincides with the adjoint group of $\mathfrak{k}_h$. This completes the proof. □

By Theorems 10.2.2 and 10.2.3, in order to classify conjugacy classes, it suffices to classify nilpotent conjugacy classes. Namely, our problem is to classify $G$-orbits in the *nilpotent cone*

$$\mathcal{N} = \{\text{nilpotent elements in } \mathfrak{g}\} \subset \mathfrak{g} \tag{10.2.2}$$

for an arbitrary simple Lie algebra $\mathfrak{g}$.

Let us show that $\mathcal{N}$ is an irreducible closed algebraic subvariety of $\mathfrak{g}$. Define $f_i \in \mathbb{C}[\mathfrak{g}]$ by $\det(t1 - \mathrm{ad}(x)) = t^n + \sum_{i=0}^{n-1} f_i(x)t^i$. Then $\mathcal{N}$ is the common zero set of $f_1, \ldots, f_{n-1}$. In particular, it is a closed subvariety of $\mathfrak{g}$. Since any point $x \in \mathcal{N}$ is contained in a Borel subalgebra, it follows from Theorem 9.9.1 that $\mathcal{N} = \mathrm{Ad}(G)(\mathfrak{b} \cap \mathcal{N})$. Hence the irreducibility of $\mathcal{N}$ follows from $\mathfrak{b} \cap \mathcal{N} = \mathfrak{n}$.

To classify nilpotent conjugacy classes we need case-by-case arguments for each simple Lie algebra. For simple Lie algebras of classical type, we can perform the

classification by using linear algebra (essentially by the theory of Jordan normal forms). To treat exceptional types, some further preparation from general theories (the Jacobson–Morozov theorem) is required. The result of the classification can be found in Dynkin [Dy]. Here we only state the following general fact (It follows from the Dynkin–Kostant theory. The proof below is due to R. Richardson [Ri]).

**Theorem 10.2.4.** *The number of $G$-orbits in $\mathcal{N}$ is finite.*

*Proof.* Set $\widetilde{G} = GL(\mathfrak{g})$ and $\widetilde{\mathfrak{g}} = \mathfrak{gl}(\mathfrak{g})$. We regard $\mathfrak{g}$ as a subalgebra of $\widetilde{\mathfrak{g}}$ by the embedding $\mathrm{ad} : \mathfrak{g} \hookrightarrow \widetilde{\mathfrak{g}}$. Define an action of $\widetilde{G}$ on $\widetilde{\mathfrak{g}}$ by

$$\widetilde{g} \cdot \widetilde{x} = \widetilde{g}\widetilde{x}\widetilde{g}^{-1} \quad (\widetilde{g} \in \widetilde{G}, \widetilde{x} \in \widetilde{\mathfrak{g}}). \tag{10.2.3}$$

Then for $g \in G$, we see that the adjoint action of $g$ on $\mathfrak{g}$ coincides with the restriction of the action (10.2.3) of $\mathrm{Ad}(g) \in \widetilde{G}$ on $\widetilde{\mathfrak{g}}$. Denote the set of nilpotent linear endomorphisms of $\widetilde{\mathfrak{g}}$ by $\widetilde{\mathcal{N}}$. Then by the definition of $\mathcal{N}$ we have $\mathcal{N} = \widetilde{\mathcal{N}} \cap \mathfrak{g}$. On the other hand, it follows from the theory of Jordan normal forms that the number of $\widetilde{G}$-orbits in $\widetilde{\mathcal{N}}$ is finite (see Example 10.2.6). Therefore, it suffices to show that for a $\widetilde{G}$-orbit $\widetilde{O}$ in $\widetilde{\mathfrak{g}}$ the intersection $Z = \widetilde{O} \cap \mathfrak{g}$ with $\mathfrak{g}$ consists of finitely many $\mathrm{Ad}(G)$-orbits.

Let us consider a decreasing sequence of closed subvarieties of $Z$

$$Z = Z_0 \supset Z_1 \supset Z_2 \supset \cdots \tag{10.2.4}$$

defined inductively by $Z_i = (Z_{i-1})_{\mathrm{sing}} =$ (the set of singular points in $Z_{i-1}$). Then every $Z_i \backslash Z_{i+1}$ is a $G$-invariant set, and we have to show that $Z_i \backslash Z_{i+1}$ is a union of finitely many $\mathrm{Ad}(G)$-orbits. For this, it suffices to prove that every $\mathrm{Ad}(G)$-orbit $O = O_G(x)$ (through $x \in Z_i \backslash Z_{i+1}$) in $Z_i \backslash Z_{i+1}$ is open. Since $Z_i \backslash Z_{i+1}$ is smooth, it is enough to show the equality $T_x O = T_x(Z_i \backslash Z_{i+1})$ of tangent spaces. Let us concentrate on proving the inclusion $T_x O \supset T_x(Z_i \backslash Z_{i+1})$, because the converse $T_x O \subset T_x(Z_i \backslash Z_{i+1})$ is trivial. Under the identification of $T_x \widetilde{\mathfrak{g}}$ with $\widetilde{\mathfrak{g}}$, we have natural isomorphisms $T_x O = [\mathfrak{g}, x]$ and $T_x(Z_i \backslash Z_{i+1}) \subset T_x(\widetilde{O}) \cap \mathfrak{g} = [\widetilde{\mathfrak{g}}, x] \cap \mathfrak{g}$. Consequently it remains to show the following inclusion:

$$[\widetilde{\mathfrak{g}}, x] \cap \mathfrak{g} \subset [\mathfrak{g}, x]. \tag{10.2.5}$$

Let us define a symmetric bilinear form on $\widetilde{\mathfrak{g}}$ by

$$(x_1, x_2) = \mathrm{Tr}(x_1 x_2) \qquad (x_1, x_2 \in \widetilde{\mathfrak{g}}). \tag{10.2.6}$$

It is easily checked that this bilinear form is non-degenerate. Furthermore, its restriction to $\mathfrak{g}$ is the Killing form, and hence it is also non-degenerate. Therefore, we have the orthogonal decomposition $\widetilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{g}^{\perp}$, where $\mathfrak{g}^{\perp} = \{y \in \widetilde{\mathfrak{g}} \mid (y, \mathfrak{g}) = 0\}$. Moreover, we have $[\mathfrak{g}, \mathfrak{g}^{\perp}] \subset \mathfrak{g}^{\perp}$. Indeed, for $x_1, x_2 \in \mathfrak{g}$, $y \in \mathfrak{g}^{\perp}$ we have

$$\begin{aligned}(x_1, [x_2, y]) &= \mathrm{Tr}(x_1 x_2 y) - \mathrm{Tr}(x_1 y x_2) = \mathrm{Tr}(x_1 x_2 y) - \mathrm{Tr}(x_2 x_1 y) \\ &= ([x_1, x_2], y) = 0.\end{aligned}$$

Hence we get $[\widetilde{\mathfrak{g}}, x] \cap \mathfrak{g} = ([\mathfrak{g}, x] + [\mathfrak{g}^{\perp}, x]) \cap \mathfrak{g} \subset [\mathfrak{g}, x] + ([\mathfrak{g}^{\perp}, \mathfrak{g}] \cap \mathfrak{g}) = [\mathfrak{g}, x]$. This completes the proof. □

The irreducibility of the nilpotent cone $\mathcal{N}$ implies the following result.

**Corollary 10.2.5.** *There exists an open dense orbit in* $\mathcal{N}$.

**Example 10.2.6.** Let us consider the case of $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ and $G = SL_n(\mathbb{C})$. Since the adjoint action of $G$ on $\mathfrak{g}$ is given by $\mathrm{Ad}(g)x = gxg^{-1}$, there exists a bijection between the set conjugacy classes of $\mathfrak{g}$ and that of Jordan normal forms. Moreover, the Jordan decomposition of a Jordan normal form is given by writing it into a sum of a diagonal matrix and a strictly upper triangular matrix. As a representative of each nilpotent conjugacy class, we can choose a Jordan normal form whose eigenvalues are all zero. Consequently, the number of nilpotent conjugacy classes is equal to the number of the ways of writing $n$ as a sum of natural numbers (i.e., the partition number $P(n)$ of $n$).

## 10.3 Geometry of conjugacy classes

By Theorem 10.1.1 we have $\mathbb{C}[\mathfrak{h}]^W \simeq \mathbb{C}[\mathfrak{g}]^G \subset \mathbb{C}[\mathfrak{g}]$. On the other hand the ring $\mathbb{C}[\mathfrak{h}]^W$ is isomorphic to the polynomial ring of $l$ $(= \dim \mathfrak{h})$-variables by Theorem 10.1.2. Therefore, we get a natural morphism

$$\chi : \mathfrak{g} \to W\backslash\mathfrak{h} \cong \mathbb{A}^l \tag{10.3.1}$$

of algebraic varieties. Set-theoretically this morphism $\chi$ is described as follows.

**Proposition 10.3.1.** *Let* $x \in \mathfrak{g}$. *Then the image* $\chi(x)$ *is the point of* $W\backslash\mathfrak{h}$ *which corresponds to the* $W$*-orbit* $O_G(x_s) \cap \mathfrak{h}$ (*see Theorem* 10.2.2).

*Proof.* Since any fiber of $\chi$ is $G$-invariant, we may assume that $x_s = h \in \mathfrak{h}$. Under the notation in Section 10.2, it is enough to show that the set $\chi(h + \mathcal{N}_h)$ is just a one-point set. By Theorem 10.2.3 and Corollary 10.2.5 there is a dense $Z_G(h)$-orbit in $\mathcal{N}_h$. Hence there is also a dense $Z_G(h)$-orbit in $h + \mathcal{N}_h$. Since $\chi$ is constant on each $G$-orbit, our assertion is clear. □

The next corollary follows immediately from the proof of this proposition and Corollary 10.2.5.

**Corollary 10.3.2.** *Every fiber of* $\chi$ *is irreducible and consists of finitely many conjugacy classes. In particular, there exists an open dense conjugacy class in each fiber.*

In the rest of this section we study geometric properties of the morphism $\chi$ and its fibers. Since $\mathfrak{b}$ is a $B$-module by the adjoint action, we can associate to it a vector bundle $\widetilde{\mathfrak{g}} := \Lambda(\mathfrak{b}) \to X$ on the flag variety $X = G/B$ (see Section 9.11). This vector bundle $\widetilde{\mathfrak{g}}$ is isomorphic to the quotient $B\backslash(G \times \mathfrak{b})$ of the product bundle $G \times \mathfrak{b}$ by the locally free $B$-action on $G \times \mathfrak{b}$ given by

$$b \cdot (g, x) = (gb^{-1}, \mathrm{Ad}(b)x) \qquad (b \in B, (g, x) \in G \times \mathfrak{b}). \tag{10.3.2}$$

We define morphisms $\rho : \widetilde{\mathfrak{g}} \to \mathfrak{g}$ and $\theta : \widetilde{\mathfrak{g}} \to \mathfrak{h}$ by

$$\rho(\overline{(g,x)}) = \mathrm{Ad}(g)x, \qquad \theta(\overline{(g,x)}) = p(x) \quad (x \in \mathfrak{b},\ g \in G), \tag{10.3.3}$$

respectively, where $p : \mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n} \to \mathfrak{h}$ is the projection. Note that $\rho$ is a composite of the closed embedding $\widetilde{\mathfrak{g}} \hookrightarrow \mathfrak{g} \times X$ ($\overline{(g,x)} \mapsto (\mathrm{Ad}(g)x, gB)$) and the projection $\mathfrak{g} \times X \to \mathfrak{g}$. Therefore, $\rho$ is a proper morphism (note that $X$ is a projective variety). Set $d = \dim X = |\Delta^+|$.

**Proposition 10.3.3.** *$\chi$ is a flat morphism, and for any $\overline{h} \in W\backslash\mathfrak{h}$ the dimension of the fiber $\chi^{-1}(\overline{h})$ is equal to $2d$.*

*Proof.* Let us consider the fiber $\chi^{-1}(\overline{h})$ for $h \in \mathfrak{h}$. By Proposition 10.3.1 we have $\chi^{-1}(\overline{h}) = \mathrm{Ad}(G)(h + \mathfrak{n}) = \rho(B\backslash(G \times (h + \mathfrak{n})))$. Hence by $\dim(B\backslash(G \times (h + \mathfrak{n}))) = 2d$ we obtain $\dim \chi^{-1}(\overline{h}) \leqq 2d$. On the other hand, if we take a point $h \in \mathfrak{h}$ such that $\alpha(h) \neq 0$ for any $\alpha \in \Delta$, we have $\mathfrak{z}_{\mathfrak{g}}(h) = \mathfrak{h}$, $Z_G(h) = H$, $\mathcal{N}_h = \{0\}$ and $\chi^{-1}(\overline{h}) = O_G(h) \cong G/H$. This implies that the general fibers of $\chi$ are $2d$ $(= \dim G/H)$-dimensional. Hence, by the upper semicontinuity of the dimensions of fibers, the dimension of each fiber of $\chi$ should be $2d$. The flatness of $\chi$ follows from the fact that $\chi$ is an affine morphism such that all fibers have the same dimension. □

**Corollary 10.3.4.** *The dimension of a conjugacy class is not greater than $2d$.*

We call a $2d$-dimensional conjugacy class in $\mathfrak{g}$ a *regular conjugacy class*, and an element of a regular conjugacy class is called a *regular element*. Each fiber of $\chi$ contains a unique regular conjugacy class, and it is open dense in the fiber.

For each $x \in \mathfrak{g}$ we define an antisymmetric bilinear form $\beta_x$ on $\mathfrak{g}$ by

$$\beta_x(y,z) = (x,[y,z]) \quad (y,z \in \mathfrak{g}), \tag{10.3.4}$$

where $(\bullet,\bullet)$ is the Killing form of $\mathfrak{g}$:

$$(y,z) = \mathrm{Tr}(\mathrm{ad}(y)\,\mathrm{ad}(z)) \quad (y,z \in \mathfrak{g}). \tag{10.3.5}$$

By the relation $(x,[y,z]) = ([x,y],z)$ and the non-degeneracy of the Killing form we get

$$\beta_x(y,\mathfrak{g}) = 0 \Longleftrightarrow y \in \mathfrak{z}_{\mathfrak{g}}(x). \tag{10.3.6}$$

Hence $\beta_x$ induces a non-degenerate antisymmetric bilinear form on $\mathfrak{g}/\mathfrak{z}_{\mathfrak{g}}(x)$. In particular, $\dim \mathfrak{g}/\mathfrak{z}_{\mathfrak{g}}(x)$ is even. By $\mathfrak{g}/\mathfrak{z}_{\mathfrak{g}}(x) \cong T_x(O_G(x))$ we obtain the following.

**Proposition 10.3.5.** *Any conjugacy class is even-dimensional.*

The bilinear forms $\beta_x$ $(x \in \mathfrak{g})$ gives a global 2-form $\beta$ on $\mathfrak{g}$. Define a $2d$-form $\omega$ by $\omega = \beta^d$. Then by (10.3.6) we see that

$$x \text{ is a regular element} \Longleftrightarrow \omega_x \neq 0. \tag{10.3.7}$$

Now take a nowhere vanishing $\dim \mathfrak{g}$-form $v_{\mathfrak{g}}$ on $\mathfrak{g}$ and consider the Hodge star operator

$$* : \bigwedge^{2d} \mathfrak{g}^* \to \bigwedge^{l} \mathfrak{g}^*$$

defined by

$$\psi \wedge \varphi = ({}^*\psi, \varphi) v_{\mathfrak{g}} \quad \left( \psi \in \bigwedge^{2d} \mathfrak{g}^*, \ \varphi \in \bigwedge^{l} \mathfrak{g}^* \right).$$

Here $(\bullet, \bullet)$ stands for the non-degenerate symmetric bilinear form on $\bigwedge^l \mathfrak{g}^*$ induced by the Killing form.

**Lemma 10.3.6.** *Let $\chi_1, \ldots, \chi_l$ be homogeneous algebraically independent generators of $\mathbb{C}[\mathfrak{g}]^G$. Then the $l$-form ${}^*\omega$ coincides with $d\chi_1 \wedge \cdots \wedge d\chi_l$ up to a non-zero constant multiple.*

*Proof.* Set $\mathfrak{h}' = \{h \in \mathfrak{h} \mid \alpha(h) \neq 0 \ (\alpha \in \Delta)\}$. Since $\mathrm{Ad}(G)\mathfrak{h}' = \chi^{-1}(W \backslash \mathfrak{h}')$ is dense in $\mathfrak{g}$, so is $\mathrm{Ad}(G)\mathfrak{h}$. Note that ${}^*\omega$ and $d\chi_1 \wedge \cdots \wedge d\chi_l$ are $G$-invariant $l$-forms on $\mathfrak{g}$. Therefore, it suffices to show that they coincide on $\mathfrak{h}$. Take a nowhere vanishing $l$-form $v_{\mathfrak{h}}$ on $\mathfrak{h}$. We define polynomials $\pi_1, \pi_2$ on $\mathfrak{h}$ by

$$({}^*\omega)_h = \pi_1(h) v_{\mathfrak{h}}, \ (d\chi_1 \wedge \cdots \wedge d\chi_l)_h = \pi_2(h) v_{\mathfrak{h}} \quad (h \in \mathfrak{h}). \tag{10.3.8}$$

Then we have

$$\begin{cases} \text{both } \pi_1 \text{ and } \pi_2 \text{ are skew-invariants of } W, \\ \text{namely, for any } \alpha \in \Delta, \text{ we have } s_\alpha \pi_i = -\pi_i. \end{cases} \tag{10.3.9}$$

$$\text{both } \pi_1 \text{ and } \pi_2 \text{ are homogeneous polynomials of degree } d. \tag{10.3.10}$$

The assertion (10.3.9) follows from the $W$-invariance of ${}^*\omega|_{\mathfrak{h}}$, $(d\chi_1 \wedge \cdots \wedge d\chi_l)|_{\mathfrak{h}}$ and the $W$-skew-invariance of $v_{\mathfrak{h}}$. Since the coefficients of $\beta$ are homogeneous of degree 1 on $\mathfrak{g}$, those of $\omega$, ${}^*\omega$ are homogeneous of degree $d$. Therefore, $\pi_1$ is a homogeneous polynomial of degree $d$. If the degree of $\chi_i$ is $m_i + 1$, the degree of the coefficients of $d\chi_i$ is $m_i$. This means that the homogeneous degree of $\pi_2$ is $\sum_{i=1}^{l} m_i$, which is equal to $d$ by (10.1.12). Hence the assertion (10.3.10) is also proved.

On the other hand it is known that the space of $W$-skew-invariant polynomials on $\mathfrak{h}$ coincides with $\mathbb{C}[\mathfrak{h}]^W (\prod_{\alpha \in \Delta^+} \alpha)$ (Bourbaki [Bou]). Hence both $\pi_1$ and $\pi_2$ coincide with $\prod_{\alpha \in \Delta^+} \alpha$ up to non-zero constant multiples. □

**Theorem 10.3.7 (Kostant).**

(i) *$\chi$ is smooth at $x \in \mathfrak{g}$ if and only if $x$ is a regular element of $\mathfrak{g}$.*

(ii) *Every fiber of $\chi$ is a reduced normal algebraic variety.*

*Proof.* Note that $\chi : \mathfrak{g} \to W \backslash \mathfrak{h} \cong \mathbb{A}^l$ is explicitly given by

$$\chi(x) = (\chi_1(x), \ldots, \chi_l(x)).$$

Therefore, the part (i) follows immediately from Lemma 10.3.6 and (10.3.7). By Proposition 10.3.3 any fiber of $\chi$ is complete intersection. Hence any fiber is reduced since it contains a smooth point. Moreover, the codimension of the singular set of each fiber is at least two by Proposition 10.3.5, from which the normality of fibers follows. □

Theorem 10.3.7 is due to Kostant [Kos]. The proof presented here, which uses Lemma 10.3.6, is due to W. Rossman.

**Theorem 10.3.8.** *Let $h \in \mathfrak{h}$. Then the morphism*

$$\rho_h : \theta^{-1}(h) \to \chi^{-1}(\overline{h})$$

*obtained by restricting $\rho$ to $\theta^{-1}(h)$ (the morphisms $\rho$ and $\theta$ were defined in* (10.3.3)*) gives a* **resolution of singularities** *of $\chi^{-1}(\overline{h})$.*

*Proof.* Since $\rho$ is a proper morphism, its restriction $\rho_h$ is also proper. Note that $\theta^{-1}(h)$ is an affine bundle on $X$ whose fibers are isomorphic to $h + \mathfrak{n}$. This in particular implies that $\theta^{-1}(h)$ is a non-singular variety. Hence it is enough to prove the following statement by Theorem 10.3.7 (i):

$$\text{If } x \in \chi^{-1}(\overline{h}) \text{ is a regular element of } \mathfrak{g}, \text{ then } \rho_h^{-1}(x) \text{ is a single point.} \quad (10.3.11)$$

We only prove it in the case where $h = 0$. The proof for the general case is reduced to that of this special case by the technique reducing the study of conjugacy classes to that of nilpotent conjugacy classes in smaller semisimple Lie algebras (see Theorem 10.2.3); however, the details are omitted. By $\chi^{-1}(\overline{0}) = \mathcal{N}$ our problem is to show that $\rho_0^{-1}(x)$ is a single point for any regular nilpotent element $x$ in $\mathfrak{g}$. Since $\rho_0$ is $G$-equivariant, it suffices to show it for just some regular nilpotent element $x$. Let $\alpha_1, \ldots, \alpha_l$ be the set of simple roots and set

$$\mathfrak{n}_1 = \{y \in \mathfrak{n} \mid y \text{ is a regular nilpotent element}\},$$

$$\mathfrak{n}_2 = \left\{ \sum_{\alpha \in \Delta^+} y_\alpha \;\middle|\; y_\alpha \in \mathfrak{g}_\alpha, y_{\alpha_i} \neq 0 \right\}.$$

Then these are open subsets of $\mathfrak{n}$ and hence we have $\mathfrak{n}_1 \cap \mathfrak{n}_2 \neq \emptyset$ (it is known that $\mathfrak{n}_1 = \mathfrak{n}_2$, but we do not need it here). Hence we take $x \in \mathfrak{n}_1 \cap \mathfrak{n}_2$. Since $\rho_0^{-1}(x) \cong \{gB \in X \mid \mathrm{Ad}(g^{-1})x \in \mathfrak{n}\}$, it is enough to show that $g \in B$ whenever $g \in G$ satisfies $\mathrm{Ad}(g)x \in \mathfrak{n}$. Using the Bruhat decomposition, let us rewrite this $g \in G$ by $g = b_1 \dot{w} b_2$ ($b_1, b_2 \in B$, $\dot{w} \in N_G(H)$ is a representative of $w \in W = N_G(H)/H$). Then it follows from $\mathrm{Ad}(g)x \in \mathfrak{n}$ that $\mathrm{Ad}(\dot{w})\,\mathrm{Ad}(b_2)x \in \mathfrak{n}$, and by the $B$-invariance of $\mathfrak{n}_2$ we obtain

$$\mathrm{Ad}(\dot{w})\,\mathrm{Ad}(b_2)x = \sum_{\alpha \in \Delta^+} y_{w(\alpha)} \in \mathfrak{n}, \quad y_{w(\alpha)} \in \mathfrak{g}_{w(\alpha)}, \quad y_{w(\alpha_i)} \neq 0.$$

This implies $w(\alpha_i) \in \Delta^+ (i = 1, \ldots, l)$ and we get $w = 1$ and $g \in B$. □

Since the subspace of $\mathfrak{g}$ consisting of elements orthogonal to $\mathfrak{b}$ (with respect to the Killing form) coincides with $\mathfrak{n}$, we obtain an isomorphism $\mathfrak{n} \cong (\mathfrak{g}/\mathfrak{b})^*$, from which we see that $\theta^{-1}(0) \cong T^*X$. On the other hand we have $\chi^{-1}(\bar{0}) = \mathcal{N}$. Hence we obtain a resolution of singularities $\rho_0 : T^*X \to \mathcal{N}$ of the *nilpotent cone* $\mathcal{N}$. It is called the *Springer resolution* of $\mathcal{N}$. Now identify $\mathfrak{g}$ with $\mathfrak{g}^*$ by the Killing form of $\mathfrak{g}$ and consider the composite of the morphisms $\rho_0$ and $\mathcal{N} \hookrightarrow \mathfrak{g} \cong \mathfrak{g}^*$. Then the morphism

$$\gamma : T^*X \to \mathfrak{g}^* \tag{10.3.12}$$

coincides with the *moment map* obtained by the $G$-action on the symplectic variety $T^*X$.

# 11

# Representations of Lie Algebras and $D$-Modules

In this chapter we give a proof of the Beilinson–Bernstein correspondence [BB] between representations of semisimple Lie algebras and $D$-modules on flag varieties.

## 11.1 Universal enveloping algebras and differential operators

Let $Y$ be a smooth algebraic variety. For a locally free $\mathcal{O}_Y$-module $\mathcal{V}$ of finite rank we define a sheaf $D_Y^{\mathcal{V}} \subset \mathcal{E}nd_{\mathbb{C}_Y}(\mathcal{V})$of differential operators acting on $\mathcal{V}$ by

$$F_p(D_Y^{\mathcal{V}}) = \{0\} \quad (p < 0), \tag{11.1.1}$$

$$F_p(D_Y^{\mathcal{V}}) = \{P \in \mathcal{E}nd_{\mathbb{C}_Y}(\mathcal{V}) \mid Pf - fP \in F_{p-1}(D_Y^{\mathcal{V}}) \ \ (f \in \mathcal{O}_Y)\} \quad (p \geqq 0), \tag{11.1.2}$$

$$D_Y^{\mathcal{V}} = \bigcup_{p=0}^{\infty} F_p(D_Y^{\mathcal{V}}). \tag{11.1.3}$$

The sheaf $D_Y^{\mathcal{V}}$ thus obtained is a sheaf of rings, and it contains $\mathcal{O}_Y$ as a subring. The ordinary sheaf $D_Y$ of differential operators on $Y$ is $D_Y^{\mathcal{O}_Y}$. In general we have an isomorphism

$$\mathcal{V} \otimes_{\mathcal{O}_Y} D_Y \otimes_{\mathcal{O}_Y} \mathcal{V}^* \simeq D_Y^{\mathcal{V}} \tag{11.1.4}$$

of sheaves of rings given by

$$\big(s \otimes P \otimes s^*\big)(t) = P(\langle s^*, t\rangle)s \quad (s, t \in \mathcal{V},\ s^* \in \mathcal{V}^*,\ P \in D_Y). \tag{11.1.5}$$

Now assume that a linear algebraic group $K$ acts on $Y$ and that $\mathcal{V}$ is a $K$-equivariant vector bundle. Denote by $U(\mathfrak{k})$ the universal enveloping algebra of the Lie algebra $\mathfrak{k}$ of $K$. Then we can construct a ring homomorphism

$$U(\mathfrak{k}) \to \Gamma(Y, D_Y^{\mathcal{V}}) \qquad (a \mapsto \partial_a) \tag{11.1.6}$$

from $U(\mathfrak{k})$ to $\Gamma(Y, D_Y^{\mathcal{V}})$ as follows. Let $\varphi : p_2^*\mathcal{V} \xrightarrow{\sim} \sigma^*\mathcal{V}$ be the isomorphism giving the $K$-equivariant structure of $\mathcal{V}$, where $p_2 : K \times Y \to Y$ and $\sigma : K \times Y \to Y$ are the second projection and the action of $K$ on $Y$, respectively. Then the section $\partial_a$ is uniquely determined by the condition

$$\varphi((a \otimes 1) \cdot \varphi^{-1}(\sigma^* s)) = \sigma^*(\partial_a s) \quad (s \in \mathcal{V},\ a \in \mathfrak{k}), \tag{11.1.7}$$

where $a \in \mathfrak{k}$ in the left-hand side is regarded as a right-invariant vector field on $K$. Note that if we identify $U(\mathfrak{k})$ with the ring of right-invariant differential operators on $K$, the formula (11.1.7) holds also for any $a \in U(\mathfrak{k})$.

**Remark 11.1.1.** In the complex analytic category, by regarding $K$ as a complex Lie group we have

$$(\partial_a s)(y) = \frac{d}{dt}(\exp(ta)s((\exp ta)^{-1}y))\Big|_{t=0} \quad (a \in \mathfrak{k},\ s \in \mathcal{V},\ y \in Y).$$

Since $\Gamma(Y, \mathcal{V})$ is a $K$-module, it is naturally a $U(\mathfrak{k})$-module. The corresponding homomorphism $U(\mathfrak{k}) \to \mathrm{End}(\Gamma(Y, \mathcal{V}))$ coincides with the one obtained by composing (11.1.6) with the homomorphism $\Gamma(Y, D_Y^{\mathcal{V}}) \to \mathrm{End}(\Gamma(Y, \mathcal{V}))$ induced by the inclusion $D_Y^{\mathcal{V}} \subset \mathcal{E}nd_{\mathbb{C}_Y}(\mathcal{V})$.

## 11.2 Rings of twisted differential operators on flag varieties

Recall that for each $\lambda \in P$ we have a $G$-equivariant line bundle $\mathcal{L}(\lambda)$ on the flag variety $X = G/B$ (see Section 9.11). We set

$$D_\lambda = D_X^{\mathcal{L}(\lambda+\rho)} \qquad (\lambda \in P) \tag{11.2.1}$$

(see (9.4.8) for the definition of the Weyl vector $\rho$). Namely, $D_\lambda$ is the sheaf of differential operators acting on $\mathcal{L}(\lambda + \rho)$. Since $\mathcal{L}(\lambda + \rho)$ is $G$-equivariant, we have a homomorphism

$$\Phi_\lambda : U(\mathfrak{g}) \to \Gamma(X, D_\lambda) \quad (a \mapsto \partial_a) \tag{11.2.2}$$

of associative algebras (see Section 11.1).

We denote by $\mathrm{Mod}_{qc}(D_\lambda)$ the abelian category of $D_\lambda$-modules which are quasi-coherent over $\mathcal{O}_X$, and by $\mathrm{Mod}_c(D_\lambda)$ its full subcategory consisting of coherent $D_\lambda$-modules. We also denote by $\mathrm{Mod}(\mathfrak{g})$ the category of $U(\mathfrak{g})$-modules. By (11.2.2) we have the additive functors

$$\mathrm{Mod}_{qc}(D_\lambda) \to \mathrm{Mod}(\mathfrak{g}) \qquad (\mathcal{M} \mapsto \Gamma(X, \mathcal{M})), \tag{11.2.3}$$

$$\mathrm{Mod}(\mathfrak{g}) \to \mathrm{Mod}_{qc}(D_\lambda) \qquad \big(M \mapsto D_\lambda \otimes_{U(\mathfrak{g})} M\big). \tag{11.2.4}$$

We easily see that the functor $D_\lambda \otimes_{U(\mathfrak{g})} (\bullet)$ is the left adjoint functor of $\Gamma(X, \bullet)$. Namely, we have

$$\mathrm{Hom}_{D_\lambda}(D_\lambda \otimes_{U(\mathfrak{g})} M, \mathcal{N}) \cong \mathrm{Hom}_{U(\mathfrak{g})}(M, \Gamma(X, \mathcal{N})) \tag{11.2.5}$$

for any $M \in \mathrm{Mod}(\mathfrak{g})$ and $\mathcal{N} \in \mathrm{Mod}_{qc}(D_\lambda)$.

**Example 11.2.1.** Let $G = SL_2(\mathbb{C})$. We follow the notation in Example 9.11.1. The restrictions of the sheaf $D_{n\rho}$ on $X = \mathbb{P}^1$ to the open subsets $U_1$ and $U_2$ are isomorphic to the ordinary sheaves of differential operators, i.e., we have $D_{n\rho}|_{U_1} \cong D_{U_1}$ and $D_{n\rho}|_{U_2} \cong D_{U_2}$. Using the coordinates $(z)$, $(x)$ of $U_1$, $U_2$, respectively, the gluing rule of these sheaves is given by

$$\begin{aligned} x &= \frac{1}{z}, & z &= \frac{1}{x} \\ \frac{d}{dx} &= -z^2\frac{d}{dz} - (n+1)z, & \frac{d}{dz} &= -x^2\frac{d}{dx} - (n+1)x. \end{aligned} \tag{11.2.6}$$

Moreover, the homomorphism $\Phi_{n\rho} : U(\mathfrak{g}) \to \Gamma(X, D_{n\rho})$ is given by

$$\begin{aligned} h &\mapsto -2z\frac{d}{dz} - (n+1) = 2x\frac{d}{dx} + (n+1), \\ e &\mapsto -\frac{d}{dz} = x^2\frac{d}{dx} + (n+1)x, \\ f &\mapsto z^2\frac{d}{dz} + (n+1)z = -\frac{d}{dx}. \end{aligned} \tag{11.2.7}$$

The aim of this chapter is to establish the following fundamental theorems due to *Beilinson–Bernstein* [BB].

**Theorem 11.2.2.** *Let* $\lambda \in P$.

(i) *The homomorphism* $\Phi_\lambda : U(\mathfrak{g}) \to \Gamma(X, D_\lambda)$ *is surjective.*
(ii) *For any* $z \in \mathfrak{z}$ *we have* $\Phi_\lambda(z) = \chi_\lambda(z)\mathrm{id}$. *Moreover, we have* $\mathrm{Ker}\,\Phi_\lambda = U(\mathfrak{g})(\mathrm{Ker}\,\chi_\lambda)$.

*Here,* $\chi_\lambda : \mathfrak{z} \to \mathbb{C}$ *denotes the central character associated to* $\lambda$ (*see Section* 9.4).

**Theorem 11.2.3.** *Suppose that* $\lambda \in P$ *satisfies the condition*

$$\langle \lambda, \alpha^\vee \rangle \notin \mathbb{N}^+ = \{1, 2, 3, \dots\} \text{ for any } \alpha \in \Delta^+. \tag{11.2.8}$$

*Then for any* $\mathcal{M} \in \mathrm{Mod}_{qc}(D_\lambda)$ *we have*

$$H^k(X, \mathcal{M}) = 0 \quad (k \neq 0). \tag{11.2.9}$$

**Theorem 11.2.4.** *Assume that* $\lambda \in P$ *satisfies the condition*

$$\langle \lambda, \alpha^\vee \rangle \notin \mathbb{N} = \{0, 1, 2, \dots\} \text{ for any } \alpha \in \Delta^+. \tag{11.2.10}$$

*Then for any* $\mathcal{M} \in \mathrm{Mod}_{qc}(D_\lambda)$, *the morphism*

$$D_\lambda \otimes_{U(\mathfrak{g})} \Gamma(X, \mathcal{M}) \to \mathcal{M} \tag{11.2.11}$$

*is surjective.*

The proof of these theorems will be given in subsequent sections.

**Definition 11.2.5.** Let $\chi$ be a central character. We denote by $\mathrm{Mod}(\mathfrak{g}, \chi)$ the abelian category of $U(\mathfrak{g})$-modules with central character $\chi$. Namely, $\mathrm{Mod}(\mathfrak{g}, \chi)$ is the full subcategory of $\mathrm{Mod}(\mathfrak{g})$ consisting of $U(\mathfrak{g})$-modules $M$ satisfying the condition

$$zm = \chi(z)m \qquad (z \in \mathfrak{z},\ m \in M). \tag{11.2.12}$$

We also denote by $\mathrm{Mod}_f(\mathfrak{g}, \chi)$ the full abelian subcategory of $\mathrm{Mod}(\mathfrak{g}, \chi)$ consisting of finitely generated $U(\mathfrak{g})$-modules.

For $\lambda \in P$ the category $\mathrm{Mod}(\mathfrak{g}, \chi_\lambda)$ is naturally equivalent to that of $\Gamma(X, D_\lambda)$-modules by Theorem 11.2.2.

For $\lambda \in P$ satisfying the condition (11.2.8) we denote by $\mathrm{Mod}^e_{qc}(D_\lambda)$ the full subcategory of $\mathrm{Mod}_{qc}(D_\lambda)$ consisting of objects $\mathcal{M} \in \mathrm{Mod}_{qc}(D_\lambda)$ such that we have the following:

(a) The canonical morphism $D_\lambda \otimes_{U(\mathfrak{g})} \Gamma(X, \mathcal{M}) \to \mathcal{M}$ is surjective.
(b) For any non-zero subobject $\mathcal{N}$ of $\mathcal{M}$ in $\mathrm{Mod}_{qc}(D_\lambda)$ we have $\Gamma(X, \mathcal{N}) \neq 0$.

Set $\mathrm{Mod}^e_c(D_\lambda) = \mathrm{Mod}^e_{qc}(D_\lambda) \cap \mathrm{Mod}_c(D_\lambda)$. By Theorem 11.2.4 we have $\mathrm{Mod}^e_{qc}(D_\lambda) = \mathrm{Mod}_{qc}(D_\lambda)$ for $\lambda \in P$ satisfying the condition (11.2.10). Indeed, if $\mathcal{N}$ is a non-zero object of $\mathrm{Mod}_{qc}(D_\lambda)$, the surjectivity of $D_\lambda \otimes_{U(\mathfrak{g})} \Gamma(X, \mathcal{N}) \to \mathcal{N}$ implies $\Gamma(X, \mathcal{N}) \neq 0$.

**Corollary 11.2.6.**

(i) *Assume that $\lambda \in P$ satisfies the condition* (11.2.8). *Then the functor $\Gamma(X, \bullet)$ induces equivalences*

$$\mathrm{Mod}^e_{qc}(D_\lambda) \cong \mathrm{Mod}(\mathfrak{g}, \chi_\lambda), \qquad \mathrm{Mod}^e_c(D_\lambda) \cong \mathrm{Mod}_f(\mathfrak{g}, \chi_\lambda)$$

*of categories.*

(ii) *Assume that $\lambda \in P$ satisfies the condition* (11.2.10). *Then the functor $\Gamma(X, \bullet)$ induces equivalences*

$$\mathrm{Mod}_{qc}(D_\lambda) \cong \mathrm{Mod}(\mathfrak{g}, \chi_\lambda), \qquad \mathrm{Mod}_c(D_\lambda) \cong \mathrm{Mod}_f(\mathfrak{g}, \chi_\lambda)$$

*of abelian categories. The inverse functor is given by $D_\lambda \otimes_{U(\mathfrak{g})} (\bullet)$.*

*Proof.* (i) We first show that the canonical homomorphism

$$M \to \Gamma(X, D_\lambda \otimes_{U(\mathfrak{g})} M) \tag{11.2.13}$$

is an isomorphism for any $M \in \mathrm{Mod}(\mathfrak{g}, \chi_\lambda)$. Choose an exact sequence

$$\Gamma(X, D_\lambda)^{\oplus I} \longrightarrow \Gamma(X, D_\lambda)^{\oplus J} \longrightarrow M \longrightarrow 0.$$

Note that the functor $\Gamma(X, \bullet) : \mathrm{Mod}_{qc}(D_\lambda) \to \mathrm{Mod}(\mathfrak{g}, \chi_\lambda)$ is exact by Theorem 11.2.3. Applying the right exact functor $\Gamma(X, D_\lambda \otimes_{U(\mathfrak{g})} (\bullet))$ to the above exact sequence we obtain a commutative diagram

$$\begin{array}{ccccccc}
\Gamma(X, D_\lambda)^{\oplus I} & \longrightarrow & \Gamma(X, D_\lambda)^{\oplus J} & \longrightarrow & M & \longrightarrow & 0 \\
\| & & \| & & \downarrow & & \\
\Gamma(X, D_\lambda)^{\oplus I} & \longrightarrow & \Gamma(X, D_\lambda)^{\oplus J} & \longrightarrow & \Gamma(X, D_\lambda \otimes_{U(\mathfrak{g})} M) & \longrightarrow & 0
\end{array}$$

whose rows are exact (note that $D_\lambda \otimes_{U(\mathfrak{g})} \Gamma(X, D_\lambda) = D_\lambda \otimes_{\Gamma(X, D_\lambda)} \Gamma(X, D_\lambda) = D_\lambda$ by Theorem 11.2.2). Hence (11.2.13) is an isomorphism.

Let us show that the functor $\Gamma(X, \bullet) : \mathrm{Mod}^e_{qc}(D_\lambda) \to \mathrm{Mod}(\mathfrak{g}, \chi_\lambda)$ is fully faithful. Let $\mathcal{M}_1, \mathcal{M}_2 \in \mathrm{Mod}^e_{qc}(D_\lambda)$. The homomorphism

$$\Gamma : \mathrm{Hom}_{D_\lambda}(\mathcal{M}_1, \mathcal{M}_2) \to \mathrm{Hom}_{U(\mathfrak{g})}(\Gamma(X, \mathcal{M}_1), \Gamma(X, \mathcal{M}_2))$$

is injective since $D_\lambda \otimes_{U(\mathfrak{g})} \Gamma(X, \mathcal{M}_1) \to \mathcal{M}_1$ is surjective. We now let $\varphi \in \mathrm{Hom}_{U(\mathfrak{g})}(\Gamma(X, \mathcal{M}_1), \Gamma(X, \mathcal{M}_2))$. Denote the kernel of $D_\lambda \otimes_{U(\mathfrak{g})} \Gamma(X, \mathcal{M}_1) \to \mathcal{M}_1$ by $\mathcal{K}_1$. Applying the exact functor $\Gamma(X, \bullet)$ to the exact sequence

$$0 \longrightarrow \mathcal{K}_1 \longrightarrow D_\lambda \otimes_{U(\mathfrak{g})} \Gamma(X, \mathcal{M}_1) \longrightarrow \mathcal{M}_1 \longrightarrow 0,$$

we obtain an exact sequence

$$0 \longrightarrow \Gamma(X, \mathcal{K}_1) \longrightarrow \Gamma(X, \mathcal{M}_1) \longrightarrow \Gamma(X, \mathcal{M}_1) \longrightarrow 0.$$

Here we have used the fact that (11.2.13) is an isomorphism. Hence we have $\Gamma(X, \mathcal{K}_1) = 0$. Let $\mathcal{K}_2$ be the image of the composite of

$$\mathcal{K}_1 \to D_\lambda \otimes_{U(\mathfrak{g})} \Gamma(X, \mathcal{M}_1) \to D_\lambda \otimes_{U(\mathfrak{g})} \Gamma(X, \mathcal{M}_2) \to \mathcal{M}_2.$$

Then by $\Gamma(X, \mathcal{K}_1) = 0$ and the exactness of $\Gamma(X, \bullet)$ we have $\Gamma(X, \mathcal{K}_2) = 0$. Since $\mathcal{M}_2$ is an object of $\mathrm{Mod}^e_{qc}(D_\lambda)$, we have $\mathcal{K}_2 = 0$. Hence we obtain a homomorphism $\psi : \mathcal{M}_1 (\cong D_\lambda \otimes_{U(\mathfrak{g})} \Gamma(X, \mathcal{M}_1)/\mathcal{K}_1) \to \mathcal{M}_2$ satisfying $\Gamma(\psi) = \varphi$.

We next show that for any $M \in \mathrm{Mod}(\mathfrak{g}, \chi_\lambda)$ there exists $\mathcal{M} \in \mathrm{Mod}^e_{qc}(D_\lambda)$ satisfying $\Gamma(X, \mathcal{M}) \cong M$. The set of subobjects $\mathcal{K}$ of $D_\lambda \otimes_{U(\mathfrak{g})} M$ satisfying $\Gamma(X, \mathcal{K}) = 0$ contains a unique largest element $\mathcal{L}$. Set $\mathcal{M} = D_\lambda \otimes_{U(\mathfrak{g})} M/\mathcal{L}$. Then by the definition of $\mathcal{M}$ we have $\mathcal{M} \in \mathrm{Mod}^e_{qc}(D_\lambda)$. By applying the exact functor $\Gamma(X, \bullet)$ to the exact sequence

$$0 \longrightarrow \mathcal{L} \longrightarrow D_\lambda \otimes_{U(\mathfrak{g})} M \longrightarrow \mathcal{M} \longrightarrow 0,$$

we see that $M \cong \Gamma(X, \mathcal{M})$.

It remains to show that $\mathrm{Mod}^e_c(D_\lambda)$ and $\mathrm{Mod}_f(\mathfrak{g}, \chi_\lambda)$ correspond to each other under this equivalence of categories $\mathrm{Mod}^e_{qc}(D_\lambda) \cong \mathrm{Mod}(\mathfrak{g}, \chi_\lambda)$. Let $M \in \mathrm{Mod}_f(\mathfrak{g}, \chi_\lambda)$. Since $\Gamma(X, D_\lambda)$ is a quotient of $U(\mathfrak{g})$, it is a left noetherian ring. Hence there exists an exact sequence

$$\Gamma(X, D_\lambda)^{\oplus I} \longrightarrow \Gamma(X, D_\lambda)^{\oplus J} \longrightarrow M \longrightarrow 0,$$

for finite sets $I$ and $J$. If we apply the right exact functor $D_\lambda \otimes_{U(\mathfrak{g})} (\bullet)$, we obtain an exact sequence

$$D_\lambda^{\oplus I} \longrightarrow D_\lambda^{\oplus J} \longrightarrow D_\lambda \otimes_{U(\mathfrak{g})} M \longrightarrow 0,$$

and hence $D_\lambda \otimes_{U(\mathfrak{g})} M$ is coherent. Since the object of $\mathrm{Mod}^e_{qc}(D_\lambda)$ corresponding to $M$ is a quotient of $D_\lambda \otimes_{U(\mathfrak{g})} M$, it is also coherent. Let $\mathcal{M} \in \mathrm{Mod}^e_c(D_\lambda)$. Since $\mathcal{M}$ is coherent, it is locally generated by finitely many sections. By the surjectivity of the morphism $D_\lambda \otimes_{U(\mathfrak{g})} \Gamma(X, \mathcal{M}) \to \mathcal{M}$ we can take the local finite generators from $\Gamma(X, \mathcal{M})$. Since $X$ is quasi-compact, we see that $\mathcal{M}$ is globally generated by finitely many elements of $\Gamma(X, \mathcal{M})$. This means that we have a surjective morphism $D_\lambda^{\oplus I} \to \mathcal{M}$ for a finite set $I$. From this we obtain a surjective homomorphism $\Gamma(X, D_\lambda)^{\oplus I} \to \Gamma(X, \mathcal{M})$, and hence $\Gamma(X, \mathcal{M})$ is a finitely generated $U(\mathfrak{g})$-module.

Finally, the assertion (ii) follows from Theorem 11.2.4 and (the proof of) (i). □

For any $\lambda \in P$ there always exists $w \in W$ such that $w(\lambda) \in P$ satisfies the condition (11.2.8),

$$\langle w(\lambda),\, \alpha^\vee \rangle \notin \mathbb{N}^+ \quad \text{for any } \alpha \in \Delta^+.$$

By $\chi_\lambda = \chi_{w(\lambda)}$ (see Section 9.4) we have $\mathrm{Mod}(\mathfrak{g}, \chi_\lambda) = \mathrm{Mod}(\mathfrak{g}, \chi_{w(\lambda)}) \cong \mathrm{Mod}^e_{qc}(D_{w(\lambda)})$. This means that we can translate various problems in $\mathrm{Mod}(\mathfrak{g}, \chi_\lambda)$ into those of $\mathrm{Mod}^e_{qc}(D_{w(\lambda)})$.

It is also necessary in representation theory to consider problems in $\mathrm{Mod}(\mathfrak{g}, \chi_\lambda)$ for a general element $\lambda \in \mathfrak{h}^* = \mathbb{C} \otimes_{\mathbb{Z}} P$. In fact, the results in this section can be formulated in this more general situation and are known to be true. For example, let us consider the case when $G = SL_2(\mathbb{C})$. In Example 11.2.1 we assumed that $n$ is an integer. However, even if $n$ is a general complex number, we can define the sheaf $D_{n\rho}$ by gluing the ordinary sheaves of differential operators on $U_1$ and $U_2$ by the rule (11.2.6). Furthermore, it is also possible to define a homomorphism $U(\mathfrak{g}) \to \Gamma(X, D_{n\rho})$ by (11.2.7). Namely, we can also perform the constructions in (11.2.6) and (11.2.7) for general $n \in \mathbb{C}$ (the condition $n \in \mathbb{Z}$ was necessary only for the existence of line bundles). The situation is similar for general semisimple algebraic groups $G$. Although there is no corresponding line bundle for a generic $\lambda \in \mathfrak{h}^* = \mathbb{C} \otimes_{\mathbb{Z}} P$, we can construct a sheaf $D_\lambda$ of twisted differential operators and a homomorphism $\Phi_\lambda : U(\mathfrak{g}) \to \Gamma(X, D_\lambda)$ of algebras, and then Theorems 11.2.2, 11.2.3, 11.2.4 hold without any modification. In this book we only treat the case where $\lambda \in P$ for the sake of simplicity, and refer to Beilinson–Bernstein [BB] and Kashiwara [Kas14] for details about the general case.

## 11.3 Proof of Theorem 11.2.2

As in the case of $D_X$ we introduce a natural filtration $\{F_p(U(\mathfrak{g}))\}_{p\in\mathbb{Z}}$ on $U(\mathfrak{g})$ by

$$F_p(U(\mathfrak{g})) = \begin{cases} 0 & (p < 0) \\ \{\text{the subspace spanned by} & \\ \ \text{products of at most } p \text{ elements in } \mathfrak{g}\} & (p \geqq 0). \end{cases} \tag{11.3.1}$$

Then we have

$$F_{p_1}(U(\mathfrak{g}))F_{p_2}(U(\mathfrak{g})) \subset F_{p_1+p_2}(U(\mathfrak{g})), \qquad (11.3.2)$$
$$\big[F_{p_1}(U(\mathfrak{g})),\ F_{p_2}(U(\mathfrak{g}))\big] \subset F_{p_1+p_2-1}(U(\mathfrak{g})).$$

Hence if we set

$$\mathrm{gr}_p\, U(\mathfrak{g}) = F_p(U(\mathfrak{g}))/F_{p-1}(U(\mathfrak{g})) \quad \text{and} \quad \mathrm{gr}\, U(\mathfrak{g}) = \bigoplus_p \mathrm{gr}_p\, U(\mathfrak{g}), \qquad (11.3.3)$$

then $\mathrm{gr}\, U(\mathfrak{g})$ is a commutative $\mathbb{C}$-algebra. By the PBW theorem, this algebra is isomorphic to the symmetric algebra $S(\mathfrak{g})$ over $\mathfrak{g}$. Also for $D_\lambda$, set

$$\mathrm{gr}_p\, D_\lambda = F_p(D_\lambda)/F_{p-1}(D_\lambda),\ \mathrm{gr}\, D_\lambda = \bigoplus_p \mathrm{gr}_p\, D_\lambda. \qquad (11.3.4)$$

Then $\mathrm{gr}\, D_\lambda$ is a sheaf of commutative $\mathcal{O}_X$-algebras. If we denote by $\pi : T^*X \to X$ the cotangent bundle of $X$, then we have a natural isomorphism $\mathrm{gr}\, D_\lambda \simeq \pi_*\mathcal{O}_{T^*X}$.

We first prove the commutative version of Theorem 11.2.2 where $U(\mathfrak{g})$, $D_\lambda$ are replaced by $\mathrm{gr}\, D_\lambda$, $\mathrm{gr}\, U(\mathfrak{g})$, respectively. By applying the left exact functor $\Gamma(X, \bullet)$ to the exact sequence

$$0 \longrightarrow F_{p-1}(D_\lambda) \longrightarrow F_p(D_\lambda) \longrightarrow \mathrm{gr}_p\, D_\lambda \longrightarrow 0, \qquad (11.3.5)$$

we obtain the exact sequence

$$0 \longrightarrow \Gamma(X, F_{p-1}(D_\lambda)) \longrightarrow \Gamma(X, F_p(D_\lambda)) \longrightarrow \Gamma(X, \mathrm{gr}_p\, D_\lambda). \qquad (11.3.6)$$

Hence we have $\Gamma(X, F_p(D_\lambda))/\Gamma(X, F_{p-1}(D_\lambda)) \subset \Gamma(X, \mathrm{gr}_p\, D_\lambda)$. Therefore, by $\Phi_\lambda(F_p(U(\mathfrak{g}))) \subset \Gamma(X, F_p(D_\lambda))$ we get a homomorphism

$$\mathrm{gr}\, \Phi_\lambda : S(\mathfrak{g}) = \mathrm{gr}\, U(\mathfrak{g}) \to \Gamma(X, \mathrm{gr}\, D_\lambda) \qquad (11.3.7)$$

of $\mathbb{C}$-algebras. Denote by $S(\mathfrak{g})^G$ the set of $G$-invariant elements in $S(\mathfrak{g})$ and set $S(\mathfrak{g})^G_+ = S(\mathfrak{g})^G \cap (\bigoplus_{p>0} S(\mathfrak{g})_p)$, where $S(\mathfrak{g})_p$ denotes the subspace of $S(\mathfrak{g})$ consisting of homogeneous elements of degree $p \geqq 0$.

**Proposition 11.3.1.** *The homomorphism* $\mathrm{gr}\, \Phi_\lambda$ *is surjective and its kernel* $\mathrm{Ker}\, \mathrm{gr}\, \Phi_\lambda$ *is the ideal generated by* $S(\mathfrak{g})^G_+$.

*Proof.* By the identifications $S(\mathfrak{g}) \cong \mathbb{C}[\mathfrak{g}^*] = \Gamma(\mathfrak{g}^*, \mathcal{O}_{\mathfrak{g}^*})$ and $\Gamma(X, \mathrm{gr}\, D_\lambda) \cong \Gamma(X, p_*\mathcal{O}_{T^*X}) \cong \Gamma(T^*X, \mathcal{O}_{T^*X})$, $\mathrm{gr}\, \Phi_\lambda$ gives a homomorphism

$$\Gamma(\mathfrak{g}^*, \mathcal{O}_{\mathfrak{g}^*}) \to \Gamma(T^*X, \mathcal{O}_{T^*X}) \qquad (11.3.8)$$

of $\mathbb{C}$-algebras. Moreover, we see by a simple calculation that this homomorphism (11.3.8) coincides with the pull-back $\gamma^*$ of the moment map (see Section 10.3):

$$\gamma : T^*X \to \mathfrak{g}^*. \qquad (11.3.9)$$

Recall that the image of $\gamma$ is the set $\mathcal{N}$ of nilpotent elements in $\mathfrak{g}$ (we identify $\mathfrak{g}^*$ with $\mathfrak{g}$ by the Killing form of $\mathfrak{g}$ as in Section 10.3). Let us factorize the morphism $\gamma$ as

$$T^*X \xrightarrow{\gamma'} \!\!\!\rightarrow \mathcal{N} \xhookrightarrow{\gamma''} \mathfrak{g}^*.$$

Since $\gamma'$ is a resolution of singularities of $\mathcal{N}$ and $\mathcal{N}$ is a normal variety, the induced map $\gamma'^* : \Gamma(\mathcal{N}, \mathcal{O}_{\mathcal{N}}) \to \Gamma(T^*X, \mathcal{O}_{T^*X})$ is an isomorphism. Furthermore, the algebra homomorphism $\gamma''^* : \Gamma(\mathfrak{g}^*, \mathcal{O}_{\mathfrak{g}^*}) \to \Gamma(\mathcal{N}, \mathcal{O}_{\mathcal{N}})$ is surjective because the nilpotent cone $\mathcal{N}$ is a closed subvariety of $\mathfrak{g}^*$. Hence their composite $\gamma^* = \gamma'^* \circ \gamma''^*$ is also surjective, and its kernel is the defining ideal of $\mathcal{N}$ in $\mathfrak{g}^*$. To finish the proof, it suffices to note that this defining ideal is generated by $S(\mathfrak{g})^G_+$ (see the proof of Theorem 10.3.7). □

We next study how the center $\mathfrak{z}$ of $U(\mathfrak{g})$ acts on $\mathcal{L}(\lambda + \rho)$.

**Proposition 11.3.2.** *For any $z \in \mathfrak{z}$, we have $\Phi_\lambda(z) = \chi_\lambda(z)\mathrm{id}$.*

*Proof.* Note that for any $z \in \mathfrak{z} = U(\mathfrak{g})^G$ we have $\Phi_\lambda(z) \in \Gamma(X, D_\lambda)^G$. We first prove

$$\Gamma(X, D_\lambda)^G = \mathbb{C}\mathrm{id}. \tag{11.3.10}$$

Recall the notation in the proof of Proposition 11.3.1. Since there exists an open dense $G$-orbit in $\mathcal{N}$ (Corollary 10.2.5), we have $\Gamma(\mathcal{N}, \mathcal{O}_{\mathcal{N}})^G = \mathbb{C}$. So it follows from the isomorphism $\gamma'^*$ that $\Gamma(X, \mathrm{gr}\, D_\lambda)^G = \Gamma(T^*X, \mathcal{O}_{T^*X})^G = \mathbb{C}$. Namely, we have

$$\Gamma(X, \mathrm{gr}_p D_\lambda)^G = \begin{cases} \mathbb{C} & (p = 0) \\ 0 & (p > 0). \end{cases} \tag{11.3.11}$$

By taking the $G$-invariant part of (11.3.6), we get an exact sequence

$$0 \longrightarrow \Gamma(X, F_{p-1}(D_\lambda))^G \longrightarrow \Gamma(X, F_p(D_\lambda))^G \longrightarrow \Gamma(X, \mathrm{gr}_p D_\lambda)^G.$$

If $p = 0$, it implies $\Gamma(X, F_0(D_\lambda))^G = \Gamma(X, \mathcal{O}_X)^G = \mathbb{C}$ (this follows also from the fact that $X$ is projective). Furthermore, by induction on $p$ we can show that $\Gamma(X, F_p(D_\lambda))^G = \mathbb{C}$ for any $p \geqq 0$. Hence the assertion (11.3.10) follows from $D_\lambda = \bigcup_p F_p(D_\lambda)$.

It remains to show that $\Phi_\lambda(z)s = \chi_\lambda(z)s$ for a non-zero local section $s$ of $\mathcal{L}(\lambda+\rho)$. Take a non-zero element $v_0$ in the fiber $\Lambda(\lambda + \rho)_{eB}$ of $\mathcal{L}(\lambda + \rho)$ at $eB \in X$, and define a section $s$ of $\mathcal{L}(\lambda + \rho)$ on the open subset $N^-B/B$ of $X$ by

$$s(uB) = uv_0 \qquad (u \in N^-). \tag{11.3.12}$$

Since $\Lambda(\lambda + \rho)_{eB}$ is a $B$-module associated to $\lambda + \rho$, we have

$$\Phi_\lambda(h)s = (\lambda + \rho)(h)s \qquad (h \in \mathfrak{h}). \tag{11.3.13}$$

It also follows from the definition of $s$ that

$$\Phi_\lambda(a)s = 0 \qquad (a \in \mathfrak{n}^-). \tag{11.3.14}$$

Write $z \in \mathfrak{z}$ as

$$z = u_1 + u_2 \quad (u_1 \in U(\mathfrak{h}),\ u_2 \in \mathfrak{n}U(\mathfrak{g}) \cap U(\mathfrak{g})\mathfrak{n}^-)$$

(see Lemma 9.4.4). Then by (11.3.13), (11.3.14) we obtain

$$\Phi_\lambda(z)s = \Phi_\lambda(u_1)s = \langle u_1, \lambda + \rho\rangle s.$$

Finally, under the notation in Theorem 9.4.3, we see that

$$\langle u_1,\ \lambda + \rho\rangle = \langle p'(z),\ \lambda + \rho\rangle = \langle f' \circ p'(z),\ \lambda\rangle = \chi_\lambda(z).$$

This completes the proof. □

*Proof of Theorem* 11.2.2. For $p \geqq 0$ let us set

$$\begin{aligned} I_p &= \operatorname{Ker} \chi_\lambda \cap F_p(U(\mathfrak{g})), \\ J_p &= \sum_{k+l=p} F_k(U(\mathfrak{g}))I_l, \\ K_p &= S(\mathfrak{g})S(\mathfrak{g})_+^G \cap S(\mathfrak{g})_p = \bigoplus_{\substack{k+l=p \\ l>0}} S(\mathfrak{g})_k S(\mathfrak{g})_l^G \subset S(\mathfrak{g})_p. \end{aligned} \tag{11.3.15}$$

It suffices to prove that

$$J_p \longrightarrow F_p(U(\mathfrak{g})) \longrightarrow \Gamma(X, F_p(D_\lambda)) \longrightarrow 0 \tag{11.3.16}$$

is an exact sequence for any $p \geqq 0$. We will prove this assertion by induction on $p$. Assume that $p = 0$. Then we have obviously $F_0(U(\mathfrak{g})) = \mathbb{C}$. Moreover, since $X$ is projective, we have $\Gamma(X, F_0(D_\lambda)) = \Gamma(X, \mathcal{O}_X) = \mathbb{C}$. Finally, we have $I_0 = 0$ and hence $J_0 = 0$. The assertion is verified for $p = 0$. Assume that $p > 0$. Let us consider the following commutative diagram:

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \downarrow & & \\
J_{p-1} & \longrightarrow & F_{p-1}(U(\mathfrak{g})) & \longrightarrow & \Gamma(X, F_{p-1}(D_\lambda)) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
J_p & \longrightarrow & F_p(U(\mathfrak{g})) & \longrightarrow & \Gamma(X, F_p(D_\lambda)) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
K_p & \longrightarrow & S(\mathfrak{g})_p & \longrightarrow & \Gamma(X, \operatorname{gr}_p D_\lambda) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & & & \\
0 & & 0. & & & &
\end{array}$$

The commutativity is clear. It is also easy to see that all rows and all columns are complexes. The exactness of the rightmost column follows from the fact that the functor $\Gamma(X, \bullet)$ is left exact. Furthermore, the exactness of the middle column is trivial, and that of the first (resp. third) row is our hypothesis of (resp. follows from Proposition 11.3.1). Hence to prove the exactness of the middle row it is sufficient to show that the leftmost column is exact. Note that this exactness is clear except for the surjectivity of $J_p \to K_p$. Hence it suffices to prove the surjectivity of $I_l \to S(\mathfrak{g})_l^G$ for any $l > 0$. Since $F_l(U(\mathfrak{g})) \to S(\mathfrak{g})_l$ is a surjective homomorphism of $G$-modules and since all finite-dimensional $G$-modules are completely reducible, its $G$-invariant part

$$F_l(U(\mathfrak{g}))^G = \mathfrak{z} \cap F_l(U(\mathfrak{g})) \to S(\mathfrak{g})_l^G$$

is also surjective. Therefore, for any $a \in S(\mathfrak{g})_l^G$ there exists $z' \in \mathfrak{z} \cap F_l(U(\mathfrak{g}))$ such that $\sigma_l(z') = a$ (here $\sigma_l : F_l(U(\mathfrak{g})) \to S(\mathfrak{g})_l$ is the natural map). Now let us set $z = z' - \chi_\lambda(z')1$. Then we have $z \in I_l$ and $\sigma_l(z) = a$. The surjectivity of $J_p \to K_p$ is verified. □

## 11.4 Proof of Theorems 11.2.3 and 11.2.4

For $\nu \in -P^+$ we have a surjective morphism

$$p_\nu : \mathcal{O}_X \otimes_{\mathbb{C}} L^-(\nu) \twoheadrightarrow \mathcal{L}(\nu) \tag{11.4.1}$$

of $\mathcal{O}_X$-modules by the Borel–Weil theorem. By taking its dual we get an injective morphism

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}(\nu), \mathcal{O}_X) \hookrightarrow \mathcal{O}_X \otimes_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}}(L^-(\nu), \mathbb{C}).$$

By $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}(\nu), \mathcal{O}_X) \cong \mathcal{L}(-\nu)$ and $\mathrm{Hom}_{\mathbb{C}}(L^-(\nu), \mathbb{C}) \cong L^+(-\nu)$ we can rewrite it as

$$\mathcal{L}(-\nu) \hookrightarrow \mathcal{O}_X \otimes_{\mathbb{C}} L^+(-\nu).$$

Applying the functor $\mathcal{L}(\nu) \otimes_{\mathcal{O}_X}$, we obtain an injective morphism

$$i_\nu : \mathcal{O}_X \hookrightarrow \mathcal{L}(\nu) \otimes_{\mathbb{C}} L^+(-\nu) \tag{11.4.2}$$

of $\mathcal{O}_X$-modules.

Now let $\lambda \in P$ and let $\mathcal{M}$ be a $D_\lambda$-module. If we apply the functor $\mathcal{M} \otimes_{\mathcal{O}_X}$ to (11.4.1) and (11.4.2), then we get morphisms

$$\overline{p_\nu} : \mathcal{M} \otimes_{\mathbb{C}} L^-(\nu) \to \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{L}(\nu), \tag{11.4.3}$$

$$\overline{i_\nu} : \quad \mathcal{M} \to \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{L}(\nu) \otimes_{\mathbb{C}} L^+(-\nu) \tag{11.4.4}$$

of $\mathcal{O}_X$-modules. Since $\mathrm{Ker}(p_\nu)$ and $\mathrm{Im}(i_\nu)$ are locally direct summands of the $\mathcal{O}_X$-modules $\mathcal{O}_X \otimes_{\mathbb{C}} L^-(\nu)$ and $\mathcal{L}(\nu) \otimes_{\mathbb{C}} L^+(-\nu)$, respectively, the morphism $\overline{p_\nu}$ (resp. $\overline{i_\nu}$) is surjective (resp. injective). The following results will be essential in our arguments in this section.

**Proposition 11.4.1.**

(i) *Assume that* $\lambda \in P$ *satisfies the condition* (11.2.10). *Then* $\mathrm{Ker}(\overline{p_\nu})$ *is a direct summand of* $\mathcal{M} \otimes_{\mathbb{C}} L^-(\nu)$ (*globally on* $X$) *as a sheaf of abelian groups.*

(ii) *Assume that* $\lambda \in P$ *satisfies the condition* (11.2.8). *Then* $\mathrm{Im}(\overline{i_\nu})$ *is a direct summand of* $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{L}(\nu) \otimes_{\mathbb{C}} L^+(-\nu)$ (*globally on* $X$) *as a sheaf of abelian groups.*

*Proof.* In general, if $M_1, \ldots, M_k$ are $\mathfrak{g}$-modules, then their tensor product $M_1 \otimes \cdots \otimes M_k$ is also a $\mathfrak{g}$-module by

$$x \cdot (m_1 \otimes \cdots \otimes m_k) = \sum_{i=1}^{k} m_1 \otimes \cdots \otimes x \cdot m_i \otimes \cdots \otimes m_k \quad (x \in \mathfrak{g} \text{ and } m_i \in M_i).$$

Since $\mathcal{M}$, $\mathcal{L}(\nu)$, $L^{\pm}(\mp\nu)$ are $\mathfrak{g}$-modules, $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{L}(\nu) \otimes_{\mathbb{C}} L^+(-\nu)$ are also $\mathfrak{g}$-modules, hence are $U(\mathfrak{g})$-modules. Moreover, we easily see that (11.4.3) and (11.4.4) are homomorphisms of $U(\mathfrak{g})$-modules. Decomposing these $U(\mathfrak{g})$-modules by the action of the center $\mathfrak{z}$ of $U(\mathfrak{g})$, we shall show that $\mathrm{Ker}(\overline{p_\nu})$ and $\mathrm{Im}(\overline{i_\nu})$ are direct summands.

Note that $L^-(\nu)$ has a filtration

$$L^-(\nu) = L^1 \supset L^2 \supset \cdots \supset L^{r-1} \supset L^r = \{0\} \tag{11.4.5}$$

by $B$-submodules $L^i$ satisfying the following conditions (a), (b):

(a) $L^i/L^{i+1}$ is the irreducible $B$-module which corresponds to $\mu_i \in P$. In particular, we have $\dim L^i/L^{i+1} = 1$.

(b) $\{\mu_1, \ldots, \mu_{r-1}\}$ are the weights of $L^-(\nu)$, and we have $\mu_i < \mu_j \Rightarrow i < j$ and $\mu_i = \nu \Longleftrightarrow i = 1$.

Now consider the trivial vector bundle $X \times L^-(\nu)$ which corresponds to the sheaf $\mathcal{O}_X \otimes_{\mathbb{C}} L^-(\nu)$. We define a filtration

$$X \times L^-(\nu) = U^1 \supset U^2 \supset \cdots \supset U^{r-1} \supset U^r \tag{11.4.6}$$

of $X \times L^-(\nu)$ by

$$U^i = \{(gB, l) \in X \times L^-(\nu) \mid l \in g(L^i)\}. \tag{11.4.7}$$

Denote by $\mathcal{V}^i$ the $\mathcal{O}_X$-submodule of $\mathcal{O}_X \otimes_{\mathbb{C}} L^-(\nu)$ consisting of sections of $U^i$. Then the filtration

$$\mathcal{O}_X \otimes_{\mathbb{C}} L^-(\nu) = \mathcal{V}^1 \supset \mathcal{V}^2 \supset \cdots\cdots \supset \mathcal{V}^r = 0 \tag{11.4.8}$$

satisfies the condition $\mathcal{V}^i/\mathcal{V}^{i+1} \cong \mathcal{L}(\mu_i)$. Therefore, $\mathcal{M} \otimes_{\mathbb{C}} L^-(\nu)$ has a filtration

$$\mathcal{M} \otimes_{\mathbb{C}} L^-(\nu) = \overline{\mathcal{V}^1} \supset \overline{\mathcal{V}^2} \supset \cdots \supset \overline{\mathcal{V}^r} = 0 \tag{11.4.9}$$

satisfying $\overline{\mathcal{V}^i}/\overline{\mathcal{V}^{i+1}} \cong \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{L}(\mu_i)$. Similarly, we can define a filtration of $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{L}(\nu) \otimes_{\mathbb{C}} L^+(-\nu)$

$$\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{L}(\nu) \otimes_{\mathbb{C}} L^+(-\nu) = \overline{\mathcal{W}^r} \supset \overline{\mathcal{W}^{r-1}} \supset \cdots \supset \overline{\mathcal{W}^1} = 0 \tag{11.4.10}$$

so that we have $\overline{\mathcal{W}^{i+1}}/\overline{\mathcal{W}^i} \cong \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{L}(\nu - \mu_i)$. Note that the preceding morphisms $\overline{p_\nu}$ and $\overline{i_\nu}$ coincide with the natural morphisms $\overline{\mathcal{V}^1} \to \overline{\mathcal{V}^1}/\overline{\mathcal{V}^2}$ and $\overline{\mathcal{W}^2} \to \overline{\mathcal{W}^r}$, respectively. Now let us consider the action of $\mathfrak{z}$. Since $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{L}(\mu)$ is a $D_{\lambda+\mu}$-module, for any $z \in \mathfrak{z}$ we have $(z - \chi_{\lambda+\mu}(z))(\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{L}(\mu)) = 0$ by Proposition 11.3.2. So, it follows from (11.4.9) and (11.4.10) that

$$\prod_{i=1}^{r-1} (z - \chi_{\lambda+\mu_i}(z))(\mathcal{M} \otimes_{\mathbb{C}} L^-(\nu)) = 0, \tag{11.4.11}$$

$$\prod_{i=1}^{r-1} (z - \chi_{\lambda+\nu-\mu_i}(z))(\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{L}(\nu) \otimes_{\mathbb{C}} L^+(-\nu)) = 0, \tag{11.4.12}$$

for any $z \in \mathfrak{z}$. In particular, the actions of $\mathfrak{z}$ on $\mathcal{M} \otimes_{\mathbb{C}} L^-(\nu)$ and $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{L}(\nu) \otimes_{\mathbb{C}} L^+(-\nu)$ are locally finite. In general let $N$ be a vector space equipped with a locally finite $\mathfrak{z}$-action. For a central character $\chi$, we set

$$N^\chi = \{n \in N \mid {}^\forall z \in \mathfrak{z},\ {}^\exists p \in \mathbb{N};\ (z - \chi(z))^p n = 0\}.$$

Then, it follows from the commutativity of $\mathfrak{z}$ that we get a direct sum decomposition $N = \bigoplus_\chi N^\chi$. Applying this general result to our situation, we obtain

$$\mathcal{M} \otimes_{\mathbb{C}} L^-(\nu) = \bigoplus_\chi (\mathcal{M} \otimes_{\mathbb{C}} L^-(\nu))^\chi, \tag{11.4.13}$$

$$\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{L}(\nu) \otimes_{\mathbb{C}} L^+(-\nu) = \bigoplus_\chi (\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{L}(\nu) \otimes_{\mathbb{C}} L^+(-\nu))^\chi. \tag{11.4.14}$$

Therefore, it remains for us to prove the following two assertions:

$$\text{Under the condition (11.2.10), } \chi_{\lambda+\mu_i} = \chi_{\lambda+\nu} \text{ implies } i = 1. \tag{11.4.15}$$

$$\text{Under the condition (11.2.8), } \chi_{\lambda+\nu-\mu_i} = \chi_\lambda \text{ implies } i = 1. \tag{11.4.16}$$

First we prove (11.4.15). If $\chi_{\lambda+\mu_i} = \chi_{\lambda+\nu}$, then there exists an element $w \in W$ such that $w(\lambda + \mu_i) = \lambda + \nu$, that is, $(w(\lambda) - \lambda) + (w(\mu_i) - \nu) = 0$. By (11.2.10) we have $w(\lambda) - \lambda \geqq 0$. Since $w(\mu_i)$ is also a weight of $L^-(\nu)$, we have $w(\mu_i) - \nu \geqq 0$. Consequently we obtain $w(\lambda) - \lambda = w(\mu_i) - \nu = 0$. By (11.2.10) and $w(\lambda) = \lambda$ we get $w = 1$ and hence $\mu_i = \nu$. This implies $i = 1$.

Next we prove (11.4.16). By $\chi_{\lambda+\nu-\mu_i} = \chi_\lambda$ there exists an element $w \in W$ such that $w(\lambda) = \lambda + \nu - \mu_i$. By $(\mu_i - \nu) + (w(\lambda) - \lambda) = 0$, $\mu_i - \nu \geqq 0$ and $w(\lambda) - \lambda \geqq 0$, we have $\mu_i = \nu$. This implies $i = 1$. □

*Proof of Theorem* 11.2.3. Note that $H^k(X, \mathcal{M}) = \varinjlim_{\mathcal{N}} H^k(X, \mathcal{N})$, where $\mathcal{N}$ ranges through the family of coherent $\mathcal{O}_X$-submodules of $\mathcal{M}$. Hence it suffices to show that

$H^k(X, \mathcal{N}) \to H^k(X, \mathcal{M})$ $(k \neq 0)$ is the zero map for any coherent $\mathcal{O}_X$-submodule $\mathcal{N}$. According to Theorem 9.11.2 (ii) there exists $\nu \in -P^+$ such that $H^k(X, \mathcal{N} \otimes \mathcal{L}(\nu)) = 0$. For such $\nu \in -P^+$ let us consider the following commutative diagram:

$$\begin{array}{ccc} H^k(X, \mathcal{N}) & \longrightarrow & H^k(X, \mathcal{M}) \\ \downarrow & & \downarrow (\bar{i}_\nu)_* \\ H^k(X, \mathcal{N} \otimes_{\mathcal{O}_X} \mathcal{L}(\nu) \otimes L^+(-\nu)) & \longrightarrow & H^k(X, \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{L}(\nu) \otimes L^+(-\nu)). \end{array}$$

We have $H^k(X, \mathcal{N} \otimes_{\mathcal{O}_X} \mathcal{L}(\nu) \otimes_{\mathbb{C}} L^+(-\nu)) = H^k(X, \mathcal{N} \otimes_{\mathcal{O}_X} \mathcal{L}(\nu)) \otimes_{\mathbb{C}} L^+(-\nu) = 0$. On the other hand, the map $(\bar{i}_\nu)_*$ is injective by Proposition 11.4.1 (ii) . Hence $H^k(X, \mathcal{N}) \to H^k(X, \mathcal{M})$ is the zero map. □

*Proof of Theorem* 11.2.4. Denote the image of the morphism $D_\lambda \otimes_{U(\mathfrak{g})} \Gamma(X, \mathcal{M}) \to \mathcal{M}$ by $\mathcal{M}'$ and set $\mathcal{M}'' = \mathcal{M}/\mathcal{M}'$. We have to show that $\mathcal{M}'' = 0$. Assume $\mathcal{M}'' \neq 0$. If we take a coherent $\mathcal{O}_X$-submodule $\mathcal{N} \neq 0$ of $\mathcal{M}''$, then there exists $\nu \in -P^+$ so that $\Gamma(X, \mathcal{N} \otimes_{\mathcal{O}_X} \mathcal{L}(\nu)) \neq 0$ by Theorem 9.11.2 (ii). In particular, we have $\Gamma(X, \mathcal{M}'' \otimes_{\mathcal{O}_X} \mathcal{L}(\nu)) \neq 0$. On the other hand, the map $\Gamma(X, \mathcal{M}'' \otimes_{\mathbb{C}} L^-(\nu)) = \Gamma(X, \mathcal{M}'') \otimes_{\mathbb{C}} L^-(\nu) \to \Gamma(X, \mathcal{M}'' \otimes_{\mathcal{O}_X} \mathcal{L}(\nu))$ is surjective by Proposition 11.4.1 (i). Therefore, we obtain $\Gamma(X, \mathcal{M}'') \neq 0$. Now let us consider the exact sequence

$$0 \longrightarrow \Gamma(X, \mathcal{M}') \longrightarrow \Gamma(X, \mathcal{M}) \longrightarrow \Gamma(X, \mathcal{M}'') \longrightarrow 0,$$

(the exactness follows from Theorem 11.2.3). Note that there exists an isomorphism $\Gamma(X, \mathcal{M}') \xrightarrow{\sim} \Gamma(X, \mathcal{M})$ by the definition of $\mathcal{M}'$. Hence we get $\Gamma(X, \mathcal{M}'') = 0$. This is a contradiction. Thus we must have $\mathcal{M}'' = 0$. □

## 11.5 Equivariant representations and equivariant $D$-modules

The $\mathfrak{g}$-modules which can be lifted to representations of a certain large subgroup $K$ of $G$ are especially important in representation theory. Such $\mathfrak{g}$-modules are called $K$-equivariant $\mathfrak{g}$-modules, or simply $(\mathfrak{g}, K)$-*modules*. As a large subgroup $K$ of $G$, we mainly consider the following two cases:

$$K = B \text{ (a Borel subgroup).} \tag{11.5.1}$$

$$\begin{cases} K = G^\theta = \{g \in G \mid \theta(g) = g\}, \\ \text{where } \theta \text{ is an involution of } G. \end{cases} \tag{11.5.2}$$

We will treat the case (11.5.1) in Chapter 12. The case (11.5.2) is closely related to the study of admissible representations of real semisimple Lie groups. The precise definition of $K$-equivariant $\mathfrak{g}$-modules is as follows.

**Definition 11.5.1.** Let $K$ be a closed subgroup of $G$ and set $\mathfrak{k} = \mathrm{Lie}(K)$. We say that a (not necessarily finite-dimensional) vector space $M$ over $\mathbb{C}$ is a $K$-*equivariant* $\mathfrak{g}$-*module*, if it has both a $\mathfrak{g}$-module structure and a $K$-module structure (see Section 9.6) satisfying the following conditions:

$$\begin{cases} \text{The action of } \mathfrak{k} \text{ on } M \text{ obtained by differentiating that of } K \\ \text{coincides with the restriction of the } \mathfrak{g}\text{-action.} \end{cases} \tag{11.5.3}$$

$$k \cdot (a \cdot m) = ((\mathrm{Ad}(k))(a)) \cdot (k \cdot m) \quad (k \in K, a \in \mathfrak{g}, m \in M). \tag{11.5.4}$$

We denote the category of $K$-equivariant $\mathfrak{g}$-modules by $\mathrm{Mod}(\mathfrak{g}, K)$. Moreover, $\mathrm{Mod}(\mathfrak{g}, \chi, K)$ (resp. $\mathrm{Mod}_f(\mathfrak{g}, \chi, K)$) stands for its full subcategory consisting of objects of which also belong to $\mathrm{Mod}(\mathfrak{g}, \chi)$ (resp. $\mathrm{Mod}_f(\mathfrak{g}, \chi)$) as $\mathfrak{g}$-modules.

From now on, we introduce $D$-modules which correspond to $K$-equivariant $\mathfrak{g}$-modules. Assume that an algebraic group $K$ is acting on a smooth algebraic variety $Y$. We define morphisms $p_2 : K \times Y \to Y$, $\sigma : K \times Y \to Y$, $m : K \times K \to K$ by $p_2(k, y) = y, \sigma(k, y) = ky, m(k_1, k_2) = k_1 k_2$, respectively. Just by imitating the definition of $K$-equivariant locally free $\mathcal{O}_Y$-modules in Chapter 9, we can define the notion of $K$-equivariant $D$-modules as follows.

**Definition 11.5.2.** Let $\mathcal{M}$ be a $D_Y$-module. Suppose that we are given an isomorphism

$$\varphi : p_2^* \mathcal{M} \xrightarrow{\sim} \sigma^* \mathcal{M} \tag{11.5.5}$$

of $D_{K \times Y}$-modules satisfying the cocycle condition. Then we we say that $\mathcal{M}$ is a *$K$-equivariant $D_Y$*-module. Here the cocycle condition is the commutativity of the diagram obtained by replacing $\mathcal{V}$ with $\mathcal{M}$ in (9.10.10). We denote the abelian category of $\mathcal{O}_Y$-quasi-coherent (resp. $D_Y$-coherent) and $K$-equivariant $D_Y$-modules by $\mathrm{Mod}_{qc}(D_Y, K)$ (resp. $\mathrm{Mod}_c(D_Y, K)$).

By $D_{-\rho} = D_X$ we have $\mathrm{Mod}(\mathfrak{g}, \chi_{-\rho}) \cong \mathrm{Mod}_{qc}(D_X)$ and $\mathrm{Mod}_f(\mathfrak{g}, \chi_{-\rho}) \cong \mathrm{Mod}_c(D_X)$ by Corollary 11.2.6.

**Theorem 11.5.3.** *For any closed subgroup $K$ of $G$, we have* $\mathrm{Mod}(\mathfrak{g}, \chi_{-\rho}, K) \cong \mathrm{Mod}_{qc}(D_X, K)$ *and* $\mathrm{Mod}_f(\mathfrak{g}, \chi_{-\rho}, K) \cong \mathrm{Mod}_c(D_X, K)$.

*Proof.* Since $K$ is an affine variety, it is $D$-affine in the sense that the category $\mathrm{Mod}_{qc}(D_K)$ is equivalent to that of $\Gamma(K, D_K)$-modules (see Proposition 1.4.3). Moreover, we have also proved that the flag variety $X$ of $G$ is $D$-affine (see Corollary 11.2.6). The arguments used to prove the $D$-affinity of $X$ can also be applied to $K \times X$ as well and we conclude that $K \times X$ is $D$-affine. Namely, the category $\mathrm{Mod}_{qc}(D_{K \times X})$ is equivalent to the category of $\Gamma(K \times X, D_{K \times X})$-modules. As a result, giving an isomorphism $\varphi : p_2^* \mathcal{M} \to \sigma^* \mathcal{M}$ of $D_{K \times X}$-modules for $\mathcal{M} \in \mathrm{Mod}_{qc}(D_X)$ is equivalent to giving an isomorphism

$$\widetilde{\varphi} : \Gamma(K \times X, p_2^* \mathcal{M}) \to \Gamma(K \times X, \sigma^* \mathcal{M}) \tag{11.5.6}$$

of $\Gamma(K \times X, D_{K \times X})$-modules. Note that we have

$$\begin{aligned} \Gamma(K \times X, p_2^* \mathcal{M}) &\cong \Gamma(K \times X, \mathcal{O}_K \boxtimes \mathcal{M}) \\ &\cong \Gamma(K, \mathcal{O}_K) \otimes \Gamma(X, \mathcal{M}). \end{aligned} \tag{11.5.7}$$

On the other hand, if we define morphisms $\varepsilon_i : K \times X \to K \times X$ $(i = 1, 2)$ by $\varepsilon_1(k, x) = (k, kx)$, $\varepsilon_2(k, x) = (k, k^{-1}x)$, then it follows from $\varepsilon_1 = \varepsilon_2^{-1}$, $p_2 \circ \varepsilon_1 = \sigma$ that

$$\begin{aligned}\Gamma(K \times X, \sigma^*\mathcal{M}) &\cong \Gamma(K \times X, \varepsilon_1^* p_2^* \mathcal{M}) \cong \Gamma(K \times X, (\varepsilon_2)_* p_2^* \mathcal{M}) \\ &\cong \Gamma(K \times X, p_2^*\mathcal{M}) \cong \Gamma(K, \mathcal{O}_K) \otimes \Gamma(X, \mathcal{M}).\end{aligned} \tag{11.5.8}$$

Therefore, both $\Gamma(K \times X, p_2^*\mathcal{M})$ and $\Gamma(K \times X, \sigma^*\mathcal{M})$ are isomorphic to $\Gamma(K, \mathcal{O}_K) \otimes \Gamma(X, \mathcal{M})$ as vector spaces. To distinguish the two actions of $\Gamma(K \times X, D_{K\times X})$ on $\Gamma(K, \mathcal{O}_K) \otimes \Gamma(X, \mathcal{M})$ defined by (11.5.7) and (11.5.8), we denote the action through the isomorphism (11.5.7) by $(\xi, n) \mapsto \xi \bullet n$, and the one through the isomorphism (11.5.8) by $(\xi, n) \mapsto \xi \blacksquare n$ (where $\xi \in \Gamma(K \times X, D_{K\times X})$, $n \in \Gamma(K, \mathcal{O}_K) \otimes \Gamma(X, \mathcal{M})$). We give descriptions of these actions in the following.

Note that we have

$$\Gamma(K \times X, D_{K\times X}) \cong \Gamma(K \times X, D_K \boxtimes D_X) = \Gamma(K, D_K) \otimes \Gamma(X, D_X)$$

and

$$\Gamma(K, D_K) \cong \Gamma(K, \mathcal{O}_K) \otimes_{\mathbb{C}} U(\mathfrak{k}),$$

where $\mathfrak{k} = \mathrm{Lie}(K)$ is identified with the set of left invariant vector fields on $K$. In particular, $\Gamma(K, D_X)$ is generated by $\Gamma(K, \mathcal{O}_K)$ and $\mathfrak{k}$. Moreover, it follows from Theorem 11.2.2 that $\Gamma(X, D_X) \cong U(\mathfrak{g})/U(\mathfrak{g})\,\mathrm{Ker}\,\chi_{-\rho}$. In particular, $\Gamma(X, D_X)$ is generated by the vector fields $\overline{p} = \Phi_{-\rho}(p)$ on $X$ corresponding to $p \in \mathfrak{g}$. Now note that an element $h \otimes m$ in $\Gamma(K, \mathcal{O}_K) \otimes \Gamma(X, \mathcal{M})$ corresponds to the section $h \circ p_1 \otimes p_2^{-1}m$ of $p_2^*\mathcal{M} = \mathcal{O}_{K\times X} \otimes_{p_2^{-1}\mathcal{O}_X} p_2^{-1}\mathcal{M}$ (resp. the section $h \circ p_1 \otimes \sigma^{-1}m$ of $\sigma^*\mathcal{M} = \mathcal{O}_{K\times X} \otimes_{\sigma^{-1}\mathcal{O}_X} \sigma^{-1}\mathcal{M}$) through the isomorphism (11.5.7) (resp. (11.5.8)), where $p_1 : K \times X \to K$ is the first projection. Then it follows from these observations and simple computations that

$$\begin{cases} (f \otimes 1) \bullet (h \otimes m) = fh \otimes m & (f \in \Gamma(K, \mathcal{O}_K)), \\ (a \otimes 1) \bullet (h \otimes m) = a \cdot h \otimes m & (a \in \mathfrak{k}), \\ (1 \otimes \overline{p}) \bullet (h \otimes m) = h \otimes \overline{p} \cdot m & (p \in \mathfrak{g}), \end{cases} \tag{11.5.9}$$

$$\begin{cases} (f \otimes 1) \blacksquare (h \otimes m) = fh \otimes m & (f \in \Gamma(K, \mathcal{O}_K)), \\ (a \otimes 1) \blacksquare (h \otimes m) = a \cdot h \otimes m - (1 \otimes \overline{a}) \blacksquare (h \otimes m) & (a \in \mathfrak{k}), \\ (1 \otimes \overline{p}) \blacksquare (h \otimes m) = \sum_i hh_i \otimes \overline{p}_i \cdot m & (p \in \mathfrak{g}), \end{cases} \tag{11.5.10}$$

where we set $\mathrm{Ad}(k)p = \sum_i h_i(k)p_i$ $(k \in K, h_i \in \Gamma(K, \mathcal{O}_K), p_i \in \mathfrak{g})$. Since the two actions of $\Gamma(K, \mathcal{O}_K) \otimes 1$ are the same as we see in the formula above, the map $\widetilde{\varphi}$ is uniquely determined by its restriction

$$\widehat{\varphi} : \Gamma(X, \mathcal{M}) \to \Gamma(K, \mathcal{O}_K) \otimes \Gamma(X, \mathcal{M}) \tag{11.5.11}$$

to $\Gamma(X, \mathcal{M}) = 1 \otimes \Gamma(X, \mathcal{M})$. From this map we obtain a morphism

$$\tau : K \times \Gamma(X, \mathcal{M}) \to \Gamma(X, \mathcal{M}) \tag{11.5.12}$$

defined by

$$\widehat{\varphi}(m) = \sum_i h_i \otimes m_i \Longrightarrow \tau(k, m) = \sum_i h_i(k) m_i. \tag{11.5.13}$$

Hence giving a homomorphism (11.5.6) of $\Gamma(K, \mathcal{O}_K)$-modules is equivalent to giving an algebraic morphism (11.5.12) satisfying the condition

$$\tau(k, m) \text{ is linear with respect to } m \in \Gamma(X, \mathcal{M}). \tag{11.5.14}$$

Furthermore, the cocycle condition for $\varphi$ is equivalent to the condition

$$\tau(k_1, \tau(k_2, m)) = \tau(k_1 k_2, m) \quad (k_1, k_2 \in K,\ m \in \Gamma(X, \mathcal{M})). \tag{11.5.15}$$

Therefore, giving an isomorphism (11.5.6) of $\Gamma(K, \mathcal{O}_K)$-modules such that the corresponding morphism $\varphi : p_2^*\mathcal{M} \to \sigma^*\mathcal{M}$ satisfies the cocycle condition is equivalent to giving a $K$-module structure on $\Gamma(X, \mathcal{M})$ ($k \cdot m = \tau(k, m)$). Finally, by using (11.5.9), (11.5.10) we can easily check that the conditions for $\widetilde{\varphi}$ to preserve the actions of $\mathfrak{k} \otimes 1$, $1 \otimes \overline{\mathfrak{g}}$ correspond, respectively, to (11.5.3), (11.5.4). This completes the proof. □

**Remark 11.5.4.** We can also define for general $\lambda$ the notion of $K$-equivariant $D_\lambda$-modules. Moreover, if $\lambda$ satisfies the condition (11.2.10) we have an equivalence of categories $\mathrm{Mod}(\mathfrak{g}, \chi_\lambda, K) \cong \mathrm{Mod}_{qc}(D_\lambda, K)$ (see [Kas14]). However, in order to present such general results we need to define an action of $K$ on $D_\lambda$, by which the arguments become more complicated. In order to avoid it we only treat in this book the special case where $\lambda = -\rho$.

## 11.6 Classification of equivariant $D$-modules

When a subgroup $K$ of $G$ is either (11.5.1) or (11.5.2), there exist only finitely many $K$-orbits on $X$ (in the case (11.5.1) it is a consequence of Theorem 9.9.4, and in the case (11.5.2) this is a result of T. Matsuki [Mat]). This is one way of saying that $K$ is sufficiently "large." In such cases the following remarkable results hold.

**Theorem 11.6.1.** *Let $Y$ be a smooth algebraic variety and $K$ an algebraic group acting on $Y$. Suppose that there exist only finitely many $K$-orbits on $Y$. Denote the category of $K$-equivariant regular holonomic $D_Y$-modules by* $\mathrm{Mod}_{rh}(D_Y, K)$. *We denote also by $\Upsilon(K, Y)$ the set of pairs $(O, L)$ of a $K$-orbit $O \subset Y$ and an irreducible $K$-equivariant local system $L$ on $O^{\mathrm{an}}$ (the notion of $K$-equivariant local systems can be defined in the same way as in the case of equivariant $D$-modules. We call an irreducible object in the category of $K$-equivariant local systems an irreducible $K$-equivariant local system). Then we have the following:*

(i) $\mathrm{Mod}_c(D_Y, K) = \mathrm{Mod}_{rh}(D_Y, K)$. *That is, the regular holonomicity of a coherent $D_Y$-module follows from the $K$-equivariance.*

(ii) *The irreducible objects in* $\mathrm{Mod}_c(D_Y, K)$ *are parameterized by the set* $\Upsilon(K, Y)$.

*Proof.* (i) We first consider the case when $Y$ consists of a single $K$-orbit, i.e., the case when $Y = K/K'$ for a subgroup $K'$ of $K$. Denoting the one-point algebraic variety by $\mathrm{pt} = \{x_0\}$ we define morphisms

$$\sigma : K \times Y \to Y, \qquad p_2 : K \times Y \to Y, \qquad i : K \to K \times Y,$$
$$\pi : K \to Y, \qquad j : \mathrm{pt} \to Y, \qquad l : K \to \mathrm{pt}$$

by $\sigma(k_1, k_2K') = k_1k_2K'$, $p_2(k_1, k_2K') = k_2K'$, $i(k) = (k^{-1}, kK')$, $\pi(k) = kK'$, $j(x_0) = K'$, $l(k) = x_0$, respectively. Then for any $\mathcal{M} \in \mathrm{Mod}_c(D_Y, K)$ we have

$$\pi^*\mathcal{M} = (p_2 \circ i)^*\mathcal{M} = i^*p_2^*\mathcal{M} \cong i^*\sigma^*\mathcal{M} = (\sigma \circ i)^*\mathcal{M}$$
$$= (j \circ l)^*\mathcal{M} = l^*j^*\mathcal{M} = \mathcal{O}_K \otimes_{\mathbb{C}} (j^*\mathcal{M})_{x_0},$$

and it follows from the coherence of $\pi^*\mathcal{M}$ that $\dim(j^*\mathcal{M})_{x_0} < \infty$. This implies that $\pi^*\mathcal{M} \cong \mathcal{O}_K \otimes_{\mathbb{C}} (j^*\mathcal{M})_{x_0}$ is a regular holonomic $D$-module. Since $\pi$ is smooth, $\mathcal{M}$ itself is also regular holonomic.

The general case can be proved by induction on the number of $K$-orbits. Let $\mathcal{M} \in \mathrm{Mod}_c(D_Y, K)$. Take a closed $K$-orbit $O$ in $Y$, set $Y' = Y \setminus O$ and consider the injections $i : O \hookrightarrow Y$, $j : Y' \hookrightarrow Y$. Then we have a distinguished triangle

$$\int_i i^\dagger\mathcal{M} \longrightarrow \mathcal{M} \longrightarrow \int_j j^\dagger\mathcal{M} \xrightarrow{+1} .$$

By our inductive assumption, (the cohomology sheaves of) $j^\dagger\mathcal{M}$ and $i^\dagger\mathcal{M}$ are regular holonomic. Hence (the cohomology sheaves of) $\int_j j^\dagger\mathcal{M}$ and $\int_i i^\dagger\mathcal{M}$ are as well. This implies that $\mathcal{M}$ is also regular holonomic.

(ii) By the Riemann–Hilbert correspondence the category $\mathrm{Mod}_{rh}(D_Y, K)$ is equivalent to the category $\mathrm{Perv}(\mathbb{C}_Y, K)$ of $K$-equivariant perverse sheaves on $Y^{\mathrm{an}}$. The irreducible objects of $\mathrm{Perv}(\mathbb{C}_Y, K)$ are parameterized by the set $\Upsilon(K, Y)$, where a pair $(O, L) \in \Upsilon(K, Y)$ corresponds to the intersection cohomology complex $\mathrm{IC}_Y(L)$. □

**Remark 11.6.2.** Let $O = K/K'$ be a $K$-orbit. Then the category of $K$-equivariant local systems on $O^{\mathrm{an}}$ is equivalent to that of finite-dimensional representations of $K'/(K')^0$ (here $(K')^0$ is the identity component subgroup of $K'$). In particular, there exists a one-to-one correspondence between irreducible $K$-equivariant local systems on $O^{\mathrm{an}}$ and irreducible representations of the finite group $K'/(K')^0$.

By Theorem 11.5.3 and 11.6.1 we obtain the following result.

**Corollary 11.6.3.** *Let $K$ be a closed subgroup of $G$ for which there exist only finitely many $K$-orbits on the flag variety $X$ of $G$. Then the irreducible $K$-equivariant $\mathfrak{g}$-modules with the central character $\chi_{-\rho}$ are parameterized by the set $\Upsilon(K, X)$.*

Although we stated our results only in the case where the central character is $\chi_{-\rho}$, we can argue similarly also in the general case to get a geometric classification of irreducible $K$-equivariant $\mathfrak{g}$-modules. When $K$ is of type (11.5.2), we thus obtain a classification of irreducible admissible representations of real semisimple Lie groups. This gives a new approach to the Langlands classification.

# 12

# Character Formula of Highest Weight Modules

In this chapter we will give an account of the famous character formula for irreducible highest weight modules over semisimple Lie algebras (the Kazhdan–Lusztig conjecture, a theorem due to Brylinski–Kashiwara and Beilinson–Bernstein). It became a starting point of various applications of $D$-module theory to representation theory.

## 12.1 Highest weight modules

Let $M$ be a (not necessarily finite-dimensional) $\mathfrak{h}$-module. For each $\mu \in \mathfrak{h}^*$ we set

$$M_\mu = \{m \in M \mid hm = \mu(h)m \quad (h \in \mathfrak{h})\} \tag{12.1.1}$$

and call it the weight space of $M$ with weight $\mu$. When $M_\mu \neq \{0\}$, we say that $\mu$ is a *weight* of $M$ and the elements of $M_\mu$ are called weight vectors with weight $\mu$. If an $\mathfrak{h}$-module $M$ satisfies the conditions

$$M = \bigoplus_{\mu \in \mathfrak{h}^*} M_\mu, \tag{12.1.2}$$

$$\dim M_\mu < \infty \quad (\mu \in \mathfrak{h}^*), \tag{12.1.3}$$

then we call $M$ a *weight module* and define its *character* by the formal infinite sum

$$\mathrm{ch}(M) = \sum_{\mu \in \mathfrak{h}^*} (\dim M_\mu) e^\mu. \tag{12.1.4}$$

**Definition 12.1.1.** Let $\lambda \in \mathfrak{h}^*$, and let $M$ be a $\mathfrak{g}$-module. If there is an element $m \neq 0$ of $M$ such that

$$m \in M_\lambda, \quad \mathfrak{n}m = \{0\}, \quad U(\mathfrak{g})m = M, \tag{12.1.5}$$

we say that $M$ is a *highest weight module* with *highest weight* $\lambda$. In this case $m$ is called a *highest weight vector* of $M$.

**Lemma 12.1.2.** *Let $M$ be a highest weight module with highest weight $\lambda$, and let $m$ be its highest weight vector.*

(i) $M = U(\mathfrak{n}^-)m$. *Namely, $M$ is generated by $m$ as an $\mathfrak{n}^-$-module.*
(ii) $M = \bigoplus_{\mu \leqq \lambda} M_\mu$ *and* $M_\lambda = \mathbb{C}m$. *In particular, $M$ is a weight module as an $\mathfrak{h}$-module and the highest weight vector of $M$ is uniquely determined up to constant multiples. (Here, the partial ordering of $\mathfrak{h}^*$ is the one defined in* (9.3.17)*).*

*Proof.* (i) By $m \in M_\lambda$ we have $U(\mathfrak{h})m = \mathbb{C}m$. It also follows from $\mathfrak{n}m = 0$ that $U(\mathfrak{n})m = \mathbb{C}m$. By $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$ and PBW we have $U(\mathfrak{g}) = U(\mathfrak{n}^-)U(\mathfrak{h})U(\mathfrak{n})$. Therefore,

$$M = U(\mathfrak{g})m = U(\mathfrak{n}^-)m.$$

(ii) Note that $\mathfrak{n}^-$ is an $\mathfrak{h}$-module by the adjoint action. We can extend it to the adjoint action of $\mathfrak{h}$ on the whole $U(\mathfrak{n}^-)$ by

$$\mathrm{ad}(h)u = hu - uh \quad (h \in \mathfrak{h}, u \in U(\mathfrak{n}^-)).$$

By $\mathfrak{n}^- = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_{-\alpha}$ and $U(\mathfrak{n}^-)_\mu U(\mathfrak{n}^-)_\nu \subset U(\mathfrak{n}^-)_{\mu+\nu}$ we have a direct sum decomposition $U(\mathfrak{n}^-) = \bigoplus_{\mu \leqq 0} U(\mathfrak{n}^-)_\mu$. Hence by $U(\mathfrak{n}^-)_\mu M_\nu \subset M_{\mu+\nu}$ we obtain

$$M = U(\mathfrak{n}^-)m = \sum_{\mu \leqq \lambda} M_\mu = \bigoplus_{\mu \leqq \lambda} M_\mu \quad (M_\mu = U(\mathfrak{n}^-)_{\mu-\lambda}m).$$

The assertion $M_\lambda = \mathbb{C}m$ follows from $U(\mathfrak{n}^-)_0 = \mathbb{C}1$. □

Now let us introduce the notion of Verma modules which plays a crucial role in subsequent arguments. For each $\lambda \in \mathfrak{h}^*$ we set

$$M(\lambda) = U(\mathfrak{g})/(U(\mathfrak{g})\mathfrak{n} + \sum_{h \in \mathfrak{h}} U(\mathfrak{g})(h - \lambda(h)1)). \tag{12.1.6}$$

Since $U(\mathfrak{g})$ is a left $U(\mathfrak{g})$-module by its left multiplication and $M(\lambda)$ is a quotient of $U(\mathfrak{g})$ by a left $U(\mathfrak{g})$-submodule, $M(\lambda)$ is naturally a left $U(\mathfrak{g})$-module. We set

$$m_\lambda = \overline{1} \in M(\lambda). \tag{12.1.7}$$

**Lemma 12.1.3.**

(i) *The natural homomorphism $U(\mathfrak{n}^-) \to M(\lambda)$ $(u \mapsto um_\lambda)$ is an isomorphism. In other words, $M(\lambda)$ is a free $U(\mathfrak{n}^-)$-module of rank one with a free generator $m_\lambda$.*
(ii) *The $\mathfrak{g}$-module $M(\lambda)$ is a highest weight module with highest weight $\lambda$.*
(iii) *Assume that $M$ is a highest weight module with highest weight $\lambda$. Let $m \in M$ be a highest weight vector of $M$. Then there exists a unique surjective homomorphism $f : M(\lambda) \to M$ of $U(\mathfrak{g})$-modules such that $f(m_\lambda) = m$.*

*Proof.* We first show (i). Set $I = U(\mathfrak{g})\mathfrak{n} + \sum_{h \in \mathfrak{h}} U(\mathfrak{g})(h - \lambda(h)1)$. Then it is enough to prove that $U(\mathfrak{g}) = U(\mathfrak{n}^-) \oplus I$. By PBW there exists an isomorphism

$$U(\mathfrak{n}^-) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n}) \xrightarrow{\sim} U(\mathfrak{g}) \quad (u_1 \otimes u_2 \otimes u_3 \mapsto u_1 u_2 u_3) \tag{12.1.8}$$

of vector spaces. Hence we have

$$\begin{aligned}\sum_{h\in\mathfrak{h}} U(\mathfrak{g})(h-\lambda(h)1) &= \sum_{h\in\mathfrak{h}} U(\mathfrak{n}^-)U(\mathfrak{h})U(\mathfrak{n})(h-\lambda(h)1)\\ &= \sum_{h\in\mathfrak{h}} U(\mathfrak{n}^-)U(\mathfrak{h})(\mathbb{C}\oplus U(\mathfrak{n})\mathfrak{n})(h-\lambda(h)1)\\ &\subset U(\mathfrak{n}^-)\Bigg(\sum_{h\in\mathfrak{h}} U(\mathfrak{h})(h-\lambda(h)1)\Bigg) + U(\mathfrak{g})\mathfrak{n}.\end{aligned}$$

Therefore,

$$I = U(\mathfrak{g})\mathfrak{n} + U(\mathfrak{n}^-)\sum_{h\in\mathfrak{h}} U(\mathfrak{h})(h-\lambda(h)1).$$

Finally, by (12.1.8) we have the following chain of isomorphisms:

$$\begin{aligned}U(\mathfrak{g}) &= U(\mathfrak{n}^-)U(\mathfrak{h})U(\mathfrak{n})\\ &= U(\mathfrak{n}^-)U(\mathfrak{h})(\mathbb{C}\oplus U(\mathfrak{n})\mathfrak{n})\\ &= U(\mathfrak{n}^-)U(\mathfrak{h}) \oplus U(\mathfrak{g})\mathfrak{n}\\ &= U(\mathfrak{n}^-)\Bigg(\mathbb{C}\oplus\sum_{h\in\mathfrak{h}} U(\mathfrak{h})(h-\lambda(h)1)\Bigg) \oplus U(\mathfrak{g})\mathfrak{n}\\ &= U(\mathfrak{n}^-) \oplus I.\end{aligned}$$

This completes the proof of (i). By (i) we get $M(\lambda) \neq 0$, and (ii) follows from this. The assertion (iii) also follows directly from the definition of $M(\lambda)$. □

We call $M(\lambda)$ the *Verma module* with highest weight $\lambda$. The characters of Verma modules can be computed easily as follows. By Lemma 12.1.3 (i) we have $M(\lambda)_\mu \cong U(\mathfrak{n}^-)_{\mu-\lambda}$, and hence

$$\begin{aligned}\mathrm{ch}(M(\lambda)) &= \sum_\mu \dim M(\lambda)_\mu e^\mu = \sum_{\beta\leqq 0} \dim U(\mathfrak{n}^-)_\beta e^{\lambda+\beta}\\ &= e^\lambda \sum_{\beta\leqq 0} \dim U(\mathfrak{n}^-)_\beta e^\beta.\end{aligned}$$

Moreover, by PBW we obtain

$$\sum_{\beta\leqq 0} \dim U(\mathfrak{n}^-)_\beta e^\beta = \prod_{\beta\in\Delta^+} (1 + e^{-\beta} + e^{-2\beta} + \cdots\cdots) = \frac{1}{\prod_{\beta\in\Delta^+}(1-e^{-\beta})},$$

and hence we get the character formula

$$\mathrm{ch}(M(\lambda)) = \frac{e^{\lambda}}{\prod_{\beta\in\Delta^+}(1-e^{-\beta})}. \tag{12.1.9}$$

According to Lemma 12.1.3 (iii), any highest weight module is a quotient of a Verma module. Namely, Verma modules are the "largest" highest weight modules. The existence of the "smallest" highest weight modules is also guaranteed by the next lemma.

**Lemma 12.1.4.** *Let $\lambda \in \mathfrak{h}^*$. Then there exists a unique maximal proper $\mathfrak{g}$-submodule of $M(\lambda)$.*

*Proof.* Let $N$ be a proper $\mathfrak{g}$-submodule of $M(\lambda)$. Since submodules of a weight module are also weight modules, we have a direct sum decomposition

$$N = \bigoplus_{\mu\leqq\lambda} \big(M(\lambda)_\mu \cap N\big).$$

Moreover, by $\dim M(\lambda)_\lambda = 1$ and $M(\lambda) = U(\mathfrak{g})M(\lambda)_\lambda$ we have $M(\lambda)_\lambda \cap N = 0$. In particular, we obtain $N \subset \bigoplus_{\mu<\lambda} M(\lambda)_\mu$. Therefore, the sum of all proper submodules is also contained in $\bigoplus_{\mu<\lambda} M(\lambda)_\mu$. This is the largest proper submodule of $M(\lambda)$. □

Denote by $K(\lambda)$ the largest proper submodule of $M(\lambda)$ and set

$$L(\lambda) = M(\lambda)/K(\lambda). \tag{12.1.10}$$

It is clear from the definition that $L(\lambda)$ is a highest weight module with highest weight $\lambda$, which is irreducible as a $\mathfrak{g}$-module. If $M$ is a highest weight module with highest weight $\lambda$, then we have two surjective homomorphisms

$$M(\lambda) \xrightarrow{\varphi} M, \quad M \xrightarrow{\psi} L(\lambda)$$

of $\mathfrak{g}$-modules. These homomorphisms are uniquely determined up to constant multiples. In this sense, $M(\lambda)$ (resp. $L(\lambda)$) is the "largest" (resp. "smallest") highest weight module with highest weight $\lambda$. It is natural to ask the following problem.

**Basic Problem 12.1.5.** Compute the character $\mathrm{ch}\, L(\lambda)$ of the irreducible highest weight module $L(\lambda)$.

Although the character of the Verma module $M(\lambda)$ was fairly easily computed as (12.1.9), the computation of the character of $L(\lambda)$ is much more difficult. This is because $L(\lambda)$ is a quotient of the Verma module $M(\lambda)$ by a submodule $K(\lambda)$ which admits no explicit description in general. However, in the case of dominant integral highest weights $\lambda \in P^+$, we have an isomorphism $L^+(\lambda) \simeq L(\lambda)$ (recall that $L^+(\lambda)$ is the finite-dimensional irreducible $\mathfrak{g}$-module with highest weight $\lambda$ introduced in Section 9.5). In this case we have

$$\begin{aligned}\mathrm{ch}(L(\lambda)) &= \frac{\sum_{w\in W}(-1)^{l(w)}e^{w(\lambda+\rho)-\rho}}{\prod_{\beta\in\Delta^+}(1-e^{-\beta})} \\ &= \sum_{w\in W}(-1)^{l(w)}\,\mathrm{ch}(M(w(\lambda+\rho)-\rho))\end{aligned} \tag{12.1.11}$$

by Weyl's character formula. Therefore, our problem is to generalize the celebrated Weyl character formula to general highest weights.

**Lemma 12.1.6.** *Let $M$ be a highest weight module with highest weight $\lambda$. Then for any $z \in \mathfrak{z} =$ (the center of $U(\mathfrak{g})$) and $m \in M$ we have*

$$zm = \chi_{\lambda+\rho}(z)m.$$

*Proof.* Let $m_0$ be a highest weight vector of $M$. By $M = U(\mathfrak{g})m_0$ it is enough to show that $zm_0 = \chi_{\lambda+\rho}(z)m_0$ for any $z \in \mathfrak{z}$. Let us write

$$z = u + v \quad (u \in U(\mathfrak{h}) \cong \mathbb{C}[\mathfrak{h}^*],\ v \in \mathfrak{n}^- U(\mathfrak{g}) \cap U(\mathfrak{g})\mathfrak{n})$$

(see Lemma 9.4.4). By $\mathfrak{n}m_0 = 0$ we have $zm_0 = um_0 = u(\lambda)m_0$. Moreover, by the definition of the Harish-Chandra homomorphism we obtain $u(\lambda) = (\gamma(z))(\lambda+\rho) = \chi_{\lambda+\rho}(z)$. □

**Proposition 12.1.7.** *Let $M$ be a highest weight $\mathfrak{g}$-module with highest weight $\lambda$. Then $M$ has a composition series of finite length, and each composition factor in it is isomorphic to an irreducible $\mathfrak{g}$-module $L(\mu)$ associated to $\mu \in \mathfrak{h}^*$ satisfying the condition*

$$\mu \leqq \lambda, \quad \mu+\rho \in W(\lambda+\rho). \tag{12.1.12}$$

*Proof.* If $M$ is irreducible, then our assertion is obvious. So suppose that $M$ is not irreducible and take a proper submodule $N \neq 0$ of $M$. Then $N$ is a weight module, and any weight $\nu$ of $N$ is a weight of $M$. Since $N$ is a proper submodule, this weight $\nu$ must satisfy $\nu < \lambda$. We can choose a weight $\mu$ ($\mu < \lambda$) of $N$ which is maximal in the set of weights of $N$ because the number of $\mu$'s satisfying $\nu \leqq \mu < \lambda$ is finite. If $m$ is a non-zero vector of $N_\mu$, then by $\mathfrak{g}_\alpha N_\mu \subset N_{\mu+\alpha}$ and the maximality of $\mu$ implies $\mathfrak{n}m = 0$. This implies that $U(\mathfrak{g})m$ is a highest weight module with highest weight $\mu$. Hence we may assume from the beginning that $N$ is a highest weight module with highest weight $\mu$. In this situation, we can prove

$$\mu < \lambda,\ \mu+\rho \in W(\lambda+\rho). \tag{12.1.13}$$

Indeed, by Lemma 12.1.6 for any $z \in \mathfrak{z}$ we have $z|_M = \chi_{\lambda+\rho}(z)\mathrm{id}$ and $z|_N = \chi_{\mu+\rho}(z)\mathrm{id}$. Consequently we get $\chi_{\lambda+\rho} = \chi_{\mu+\rho}$. Hence it follows from Proposition 9.4.5 that $\mu+\rho \in W(\lambda+\rho)$. If $N$ is not irreducible, applying the same argument to $N$ itself we can find a proper highest weight submodule $N' \neq 0$ of $N$ whose highest weight $\mu'$ satisfies

$$\mu' < \mu < \lambda, \quad \mu'+\rho \in W(\mu+\rho) = W(\lambda+\rho).$$

Since $W(\lambda+\rho)$ is a finite set, repeating this procedure we obtain a proper irreducible submodule $N_1 \neq 0$ of $M$ having a highest weight $\mu$ satisfying the condition (12.1.13). Now $M' = M/N_1$ is a highest weight module with highest weight $\lambda$. If $M'$ is irreducible, we are done. Otherwise, we apply the above arguments to $M'$ instead of $M$. By repeating this procedure, we finally get an increasing sequence

$$0 = N_0 \subset N_1 \subset N_2 \subset \cdots \subset M \tag{12.1.14}$$

of proper submodules of $M$ such that $N_i/N_{i-1} \cong L(\mu_i)$, $\mu_i < \lambda$, $\mu_i+\rho \in W(\lambda+\rho)$. The number of $i$'s such that $\mu_i = \mu$ for any given $\mu$ satisfying the condition (12.1.13) does not exceed $\dim M_\mu$, because $\mathrm{ch}(N_j) = \sum_{i=1}^{j} \mathrm{ch}(N_i/N_{i-1})$. Since $W(\lambda+\rho)$ is a finite set, the increasing sequence (12.1.14) must terminate after a finite number of steps. □

Let us denote by $[M(\lambda) : L(\mu)]$ the multiplicity of $L(\mu)$ appearing in a composition series of the Verma module $M(\lambda)$ (by Artin–Schreier theorem, this number does not depend on the choice of composition series). Then we have

$$\mathrm{ch}(M(\lambda)) = \sum_{\mu} [M(\lambda) : L(\mu)]\, \mathrm{ch}(L(\mu)). \tag{12.1.15}$$

Now we consider the following problem.

**Basic Problem 12.1.8.** Compute the multiplicity $[M(\lambda) : L(\mu)]$.

Let us consider the equivalence relation on $\mathfrak{h}^*$ defined by

$$\lambda \sim \mu \Longleftrightarrow \lambda - \mu \in Q,\ \mu+\rho \in W(\lambda+\rho). \tag{12.1.16}$$

Then by Proposition 12.1.7 we have $[M(\lambda) : L(\mu)] \neq 0$ only if $\lambda \sim \mu$. Hence it is enough to study Basic Problem 12.1.8 within each equivalence class. Now fix an equivalence class $\Lambda \subset \mathfrak{g}^*$ and for any pair $\lambda, \mu \in \Lambda$ set $a_{\mu\lambda} = [M(\lambda) : L(\mu)]$. Then we have

$$a_{\mu\lambda} \in \mathbb{N}, \quad a_{\lambda\lambda} = 1; \quad \text{if } \lambda \not\geqslant \mu, \text{ then } a_{\mu\lambda} = 0. \tag{12.1.17}$$

Therefore, the inverse matrix $(b_{\mu\lambda})_{\mu,\lambda\in\Lambda}$ of $(a_{\mu\lambda})_{\mu,\lambda\in\Lambda}$ satisfies the conditions

$$b_{\mu\lambda} \in \mathbb{Z}, \quad b_{\lambda\lambda} = 1; \quad \text{if } \lambda \not\geqslant \mu, \text{ then } b_{\mu\lambda} = 0. \tag{12.1.18}$$

Using this notation we have

$$\mathrm{ch}(M(\lambda)) = \sum_{\mu\in\Lambda} a_{\mu\lambda}\, \mathrm{ch}(L(\mu)) \quad (\lambda \in \Lambda), \tag{12.1.19}$$

$$\mathrm{ch}(L(\lambda)) = \sum_{\mu\in\Lambda} b_{\mu\lambda}\, \mathrm{ch}(M(\mu)) \quad (\lambda \in \Lambda), \tag{12.1.20}$$

which shows that Basic Problem 12.1.8 is equivalent to Basic Problem 12.1.5. This problem was initiated by Verma in the late 1960s and was intensively studied by

many mathematicians (Bernstein–Gelfand–Gelfand, Jantzen) in the 1970s. The final answer was given by Beilinson–Bernstein and Brylinski–Kashiwara around 1980 (the settlement of the Kazhdan–Lusztig conjecture). This spectacular application of $D$-module theory was astonishing to the researchers of representation theory, who had been studying this problem by purely algebraic methods.

**Example 12.1.9.** Consider the case of $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$. Let us take the basis $h, e, f$ of $\mathfrak{g}$ in Example 9.5.3. Then we have $\mathfrak{h} = \mathbb{C}h$, $\mathfrak{h}^* = \mathbb{C}\rho$, $\rho(h) = 1$. For each $k \in \mathbb{C}$ the Verma module $M(k\rho)$ is given by

$$M(k\rho) = \bigoplus_{n=0}^{\infty} \mathbb{C}v_n \qquad (v_{-1} = 0),$$
$$hv_n = (k - 2n)v_n, \quad fv_n = v_{n+1}, \quad ev_n = n(k + 1 - n)v_{n-1}.$$

Hence for $k \notin \mathbb{N}$ the Verma module $M(k\rho)$ is irreducible and $M(k\rho) = L(k\rho)$. For $k \in \mathbb{N}$ the largest proper submodule of $M(k\rho)$ is given by $\bigoplus_{n=k+1}^{\infty} \mathbb{C}v_n$, and it is isomorphic to $M(-(k + 2)\rho) = L(-(k + 2)\rho)$. Therefore, the answer to our basic problems in this case is given by

$$M(k\rho) = L(k\rho) \qquad (k \notin \mathbb{N}), \tag{12.1.21}$$
$$\mathrm{ch}(M(k\rho)) = \mathrm{ch}(L(k\rho)) + \mathrm{ch}(L(-(k + 2)\rho)) \qquad (k \in \mathbb{N}), \tag{12.1.22}$$
$$\mathrm{ch}(L(k\rho)) = \mathrm{ch}(M(k\rho)) - \mathrm{ch}(M(-(k + 2)\rho)) \qquad (k \in \mathbb{N}). \tag{12.1.23}$$

## 12.2 Kazhdan–Lusztig conjecture

We will give the answer to Basic Problems 12.1.5, 12.1.8 for the equivalence class $\Lambda$ containing $-2\rho$ with respect to the equivalence relation (12.1.16) on $\mathfrak{h}^*$. In this case we have $\Lambda = \{-w\rho - \rho \mid w \in W\}$, and hence $\Lambda$ is parameterized by the Weyl group $W$. If we denote the longest element of $W$ by $w_0$, we have $-w_0\rho - \rho = 0$ and hence $0 \in \Lambda$. See Remark 12.2.8 below for other equivalence classes.

Note that the highest weight $\mathfrak{g}$-modules $M$ such as $M(-w\rho - \rho)$, $L(-w\rho - \rho)$ that we will treat are locally finite as $\mathfrak{b}$-modules (that is, for any $m \in M$ we have $\dim U(\mathfrak{b})m < \infty$). Moreover, $M$ is a weight module as an $\mathfrak{h}$-module, and their weights belong to $P$. Therefore, the action of $\mathfrak{b}$ on $M$ can be lifted to an algebraic $B$-action, and $M$ is a $B$-equivariant $\mathfrak{g}$-modules in the sense of Section 11.5. Furthermore, by Lemma 12.1.6 such a $\mathfrak{g}$-module $M$ has the central character $\chi_{-\rho}$. Namely, we have

$$M(-w\rho - \rho),\ L(-w\rho - \rho) \in \mathrm{Mod}_f(\mathfrak{g}, \chi_{-\rho}, B) \quad (w \in W). \tag{12.2.1}$$

By the arguments similar to those in Proposition 12.1.7, we have the following.

**Proposition 12.2.1.** *Let $M \in \mathrm{Mod}_f(\mathfrak{g}, \chi_{-\rho}, B)$. Then $M$ has a composition series of finite length, and each composition factor is isomorphic to an irreducible $\mathfrak{g}$-module $L(-w\rho - \rho)$ for some $w \in W$.*

We denote by $K(\mathrm{Mod}_f(\mathfrak{g}, \chi_{-\rho}, B))$ the *Grothendieck group* of the abelian category $\mathrm{Mod}_f(\mathfrak{g}, \chi_{-\rho}, B)$. It is a free $\mathbb{Z}$-module with a basis $\{[L(-w\rho - \rho)]\}_{w\in W}$. By the arguments in Section 12.1 $\{[M(-w\rho - \rho)]\}_{w\in W}$ is also a basis of $K(\mathrm{Mod}_f(\mathfrak{g}, \chi_{-\rho}, B))$. Hence the basic problem in Section 12.1 is to determine the transfer matrices between these two bases of the free $\mathbb{Z}$-module $K(\mathrm{Mod}_f(\mathfrak{g}, \chi_{-\rho}, B))$. Namely, if we have

$$\begin{aligned} \big[L(-w\rho - \rho)\big] &= \sum_y b_{yw}\big[M(-y\rho - \rho)\big], \\ \big[M(-w\rho - \rho)\big] &= \sum_y a_{yw}\big[L(-y\rho - \rho)\big] \end{aligned} \tag{12.2.2}$$

in $K(\mathrm{Mod}_f(\mathfrak{g}, \chi_{-\rho}, B))$, then we have

$$\begin{aligned} \mathrm{ch}(L(-w\rho - \rho)) &= \sum_y b_{yw}\,\mathrm{ch}(M(-y\rho - \rho)), \\ \mathrm{ch}(M(-w\rho - \rho)) &= \sum_y a_{yw}\,\mathrm{ch}(L(-y\rho - \rho)). \end{aligned} \tag{12.2.3}$$

and

$$(a_{y,w})_{y,w\in W} = \Big((b_{y,w})_{y,w\in W}\Big)^{-1}. \tag{12.2.4}$$

**Example 12.2.2.** In the case of $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ we have $W = \{e, s\}$. By Example 12.1.9 we get

$$\begin{aligned} &a_{ee} = a_{ss} = a_{es} = 1, \quad a_{se} = 0, \\ &b_{ee} = b_{ss} = 1, \qquad\quad b_{es} = -1, \quad b_{se} = 0 \end{aligned}$$

in this case.

To state the solution to our basic problems we need some results on the Hecke algebra $H(W)$ of the Weyl group $W$. If we set $S = \{s_\alpha \mid \alpha \in \Pi\} \subset W$, then the pair $(W, S)$ is a Coxeter group. In general the Hecke algebra $H(W)$ is defined for any Coxeter group $(W, S)$ as follows. Let us consider a free $\mathbb{Z}[q, q^{-1}]$-module $H(W)$ with the basis $\{T_w\}_{w\in W}$. We can define a $\mathbb{Z}[q, q^{-1}]$-algebra structure on $H(W)$ by

$$T_{w_1}T_{w_2} = T_{w_1,w_2} \quad (l(w_1) + l(w_2) = l(w_1w_2)) \tag{12.2.5}$$

$$(T_s + 1)(T_s - q) = 0 \qquad (s \in S). \tag{12.2.6}$$

Note that $T_e = 1$ by (12.2.5), where $e$ is the identity element of $W$. We call this $\mathbb{Z}[q, q^{-1}]$-algebra the *Hecke algebra* of $(W, S)$. The Hecke algebras originate from the study by N. Iwahori on reductive groups over finite fields, and they are sometimes called Iwahori–Hecke algebras.

**Proposition 12.2.3.** *There exists a unique family* $\{P_{y,w}(q)\}_{y,w\in W}$ *of polynomials* $P_{y,w}(q) \in \mathbb{Z}[q]$ $(y, w \in W)$ *satisfying the following conditions:*

$$P_{y,w}(q) = 0 \qquad (y \not\leqq w), \tag{12.2.7}$$

$$P_{w,w}(q) = 1 \qquad (w \in W), \tag{12.2.8}$$

$$\deg P_{y,w}(q) \leqq (l(w) - l(y) - 1)/2 \qquad (y < w), \tag{12.2.9}$$

$$\sum_{y \leqq w} P_{y,w}(q) T_y = q^{l(w)} \sum_{y \leqq w} P_{y,w}(q^{-1}) T_{y-1}^{-1}. \tag{12.2.10}$$

We call $P_{y,w}(q)$ the *Kazhdan–Lusztig polynomial*.

**Example 12.2.4.** When $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ we have

$$P_{ee} = P_{ss} = P_{es} = 1, \ P_{se} = 0.$$

More generally, in the case where $|S| \leqq 2$ we have

$$P_{y,w}(q) = \begin{cases} 1 & (y \leqq w), \\ 0 & (y \not\leqq w). \end{cases} \tag{12.2.11}$$

If $\mathfrak{g} = \mathfrak{sl}(4, \mathbb{C})$ and $S = \{s_1, s_2, s_3\}$ with the relation $s_1 s_3 = s_3 s_1$, then (12.2.11) holds except for the cases $(y, w) = (s_2, \ s_2 s_1 s_3 s_2), (s_1 s_3, \ s_1 s_3 s_2 s_3 s_1)$ for which we have $P_{y,w}(q) = 1 + q$.

In [KL1] Kazhdan–Lusztig introduced the above-mentioned Kazhdan–Lusztig polynomials and conjectured that

$$b_{y,w} = (-1)^{l(w)-l(y)} P_{y,w}(1) \tag{12.2.12}$$

(the *Kazhdan–Lusztig conjecture*). They also predicted in loc. cit. that these Kazhdan–Lusztig polynomials should be closely related to the geometry of Schubert varieties, which was more precisely formulated in a subsequent paper [KL2] as follows.

For each $w \in W$ denote the intersection cohomology complex of the Schubert variety $\overline{X}_w$ by $\mathrm{IC}(\mathbb{C}_{X_w})$ and set

$${}^{\pi}\mathbb{C}_{X_w} = \mathrm{IC}(\mathbb{C}_{X_w})[-\dim X_w]. \tag{12.2.13}$$

**Theorem 12.2.5 ([KL2]).** *For $y, w \in W$ consider the stalk $H^i({}^{\pi}\mathbb{C}_{X_w})_{yB}$ of the $i$th cohomology sheaf $H^i({}^{\pi}\mathbb{C}_{X_w})$ of ${}^{\pi}\mathbb{C}_{X_w}$ at $yB \in X$. Then we have*

$$\sum_i (\dim H^i({}^{\pi}\mathbb{C}_{X_w})_{yB}) q^{\frac{i}{2}} = P_{y,w}(q). \tag{12.2.14}$$

*In particular, for an odd number $i$ we have $H^i({}^{\pi}\mathbb{C}_{X_w})_{yB} = 0$, and*

$$\sum_j (-1)^j \dim H^j({}^{\pi}\mathbb{C}_{X_w})_{yB} = P_{y,w}(1). \tag{12.2.15}$$

The proof of this theorem will be postponed until the end of Chapter 13.

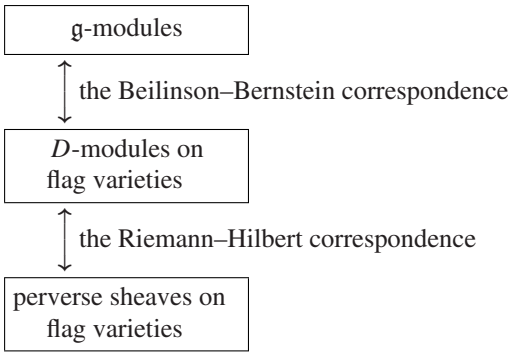
**Example 12.2.6.** Recall that for $G = SL_2(\mathbb{C})$ we have $X = \mathbb{P}^1$. Under the notation in Example 9.11.1, we have $X_s = U_1 \cong \mathbb{A}^1$, $X_e = \{eB\} \cong \mathbb{A}^0$. Hence in this case we get ${}^{\pi}\mathbb{C}_{X_s} = \mathbb{C}_X$, ${}^{\pi}\mathbb{C}_{X_e} = \mathbb{C}_{X_e}$.

As a conclusion, one should find a certain link between highest weight modules and intersection cohomology complexes of Schubert varieties in order to prove the conjecture (12.2.12). Beilinson–Bernstein [BB] and Brylinski–Kashiwara [BK] could make such a link via $D$-modules on flag varieties, and succeeded in proving the conjecture (12.2.12). Namely, we have the following.

**Theorem 12.2.7.** *In the Grothendieck group* $K(\mathrm{Mod}_f(\mathfrak{g}, B, \chi_{-\rho}))$ *we have*

$$[L(-w\rho - \rho)] = \sum_{y \leqq w} (-1)^{l(w)-l(y)} P_{y,w}(1)[M(-y\rho - \rho)].$$

The strategy of the proof of Theorem 12.2.7 can be illustrated in the following diagram:




We have already given accounts of the *Beilinson–Bernstein correspondence* and the Riemann–Hilbert correspondence in Chapter 11 and Chapter 7, respectively. In view of these correspondences it remains to determine which perverse sheaves correspond to the $\mathfrak{g}$-modules $M(-w\rho - \rho)$, $L(-w\rho - \rho)$. This problem will be studied in the next section. We note that Brylinski–Kashiwara gave a direct proof of the Beilinson–Bernstein correspondence for the special case of highest weight modules. In this book we employ the general results due to Beilinson–Bernstein.

**Remark 12.2.8.** We have described the answer to Basic Problem 12.1.5 only in the case $\lambda = -w\rho - \rho$ for some $w \in W$. The answer in the case $\lambda \in P$ can be obtained from this special case using the translation principle, which is a standard technique in representation theory. More generally, for general $\lambda \in \mathfrak{h}^*$ we can perform the similar arguments indicated in the above diagram using twisted differential operators, and the problem turns out to be the computation of certain twisted intersection cohomology groups. The result corresponding to Theorem 12.2.5 is shown for $\lambda \in \mathbb{Q} \otimes_{\mathbb{Z}} P \subset \mathbb{C} \otimes_{\mathbb{Z}} P = \mathfrak{h}^*$ (this rationality condition is necessary in order to apply the theory of Weil sheaves or Hodge modules). The general case is reduced to this rational case

by using Jantzen's deformation argument [J]. The final answer is given in terms of Kazhdan–Lusztig polynomials of the subgroup

$$W(\lambda) = \{w \in W \mid w\lambda - \lambda \in Q\}$$

of $W$. For details we refer to [KT7]. We finally note that Kashiwara's conjecture on semisimple holonomic system (see Kashiwara [Kas17]), which would in particular give the result corresponding to Theorem 12.2.5 for all $\lambda$, seems to have been established by a recent remarkable progress (see Drinfeld [Dr], Gaitsgory [Gai], and Mochizuki [Mo] and Sabbah [Sab2]).

## 12.3 $D$-modules associated to highest weight modules

In Chapter 11 we proved the equivalence of categories

$$\mathrm{Mod}_f(\mathfrak{g}, B, \chi_{-\rho}) \xrightarrow{\sim} \mathrm{Mod}_c(D_X, B) = \mathrm{Mod}_{rh}(D_X, B). \tag{12.3.1}$$

Let us denote by $\mathcal{M}_w$, $\mathcal{L}_w$ the objects in $\mathrm{Mod}_{rh}(D_X, B)$ which correspond to $M(-w\rho - \rho)$, $L(-w\rho - \rho) \in \mathrm{Mod}_f(\mathfrak{g}, B, \chi_{-\rho})$, respectively. Namely, we set

$$\mathcal{M}_w = D_X \otimes_{U(\mathfrak{g})} M_w, \quad \mathcal{L}_w = D_X \otimes_{U(\mathfrak{g})} L_w \quad (w \in W). \tag{12.3.2}$$

In view of (12.3.1) our problem of determining the transfer matrix between two bases $\{[M(-w\rho - \rho)]\}_{w\in W}$, $\{[L(-w\rho - \rho)]\}_{w\in W}$ of the Grothendieck group $K(\mathrm{Mod}_f(\mathfrak{g}, B, \chi_{-\rho}))$ can be reduced to determining the transfer matrix between two bases $\{[\mathcal{M}_w]\}_{w\in W}$, $\{[\mathcal{L}_w]\}_{w\in W}$ of $K(\mathrm{Mod}_{rh}(D_X, B))$. For this purpose, we need more concrete descriptions of $\mathcal{M}_w$ and $\mathcal{L}_w$.

For $w \in W$ let $i_w : X_w \hookrightarrow X$ be the embedding. Then $\overline{X_w}\backslash X_w$ is a divisor of $\overline{X_w}$ by Theorem 9.9.5, and hence $i_w$ is an affine morphism. Let us set

$$\mathcal{N}_w = \int_{i_w} \mathcal{O}_{X_w} = i_{w*}(D_{X\leftarrow X_w} \otimes_{D_{X_w}} \mathcal{O}_{X_w}) \quad (w \in W). \tag{12.3.3}$$

Then we have $\mathcal{N}_w \in \mathrm{Mod}_{rh}(D_X, B)$ because $X_w$ is a $B$-orbit.

**Lemma 12.3.1.** *Let $w \in W$. Then*

(i) *We have* $\mathrm{ch}(\Gamma(X, \mathcal{N}_w)) = \mathrm{ch}(M(-w\rho - \rho))$. *In particular, in the Grothendieck group $K(\mathrm{Mod}_{rh}(D_X, B))$ we obtain $[\mathcal{M}_w] = [\mathcal{N}_w]$.*

(ii) *The only $D_X$-submodule of $\mathcal{N}_w$ whose support is contained in $\overline{X_w}\backslash X_w$ is zero.*

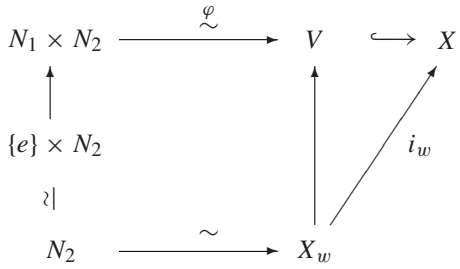
*Proof.* (i) We define two subalgebras of $\mathfrak{g}$ by

$$\mathfrak{n}_1 = \bigoplus_{\alpha\in\Delta^+\cap w(\Delta^+)} \mathfrak{g}_{-\alpha}, \quad \mathfrak{n}_2 = \bigoplus_{\alpha\in\Delta^+\cap(-w(\Delta^+))} \mathfrak{g}_{\alpha}.$$

Then the corresponding unipotent subgroups $N_1$, $N_2$ of $G$ with the properties $\mathrm{Lie}(N_1) = \mathfrak{n}_1$, $\mathrm{Lie}(N_2) = \mathfrak{n}_2$ are determined. For each $w \in W = N_G(H)/H$ we fix a representative $\dot{w} \in N_G(H)$ of it and define a morphism $\varphi : N_1 \times N_2 \to X$ by

$$\varphi(n_1, n_2) = n_1 n_2 \dot{w} B \quad (n_1 \in N_1,\ n_2 \in N_2).$$

Then $\varphi$ is an open embedding satisfying $\varphi(\{e\} \times N_2) = X_w$. Namely, setting $V = \mathrm{Im}(\varphi)$ we obtain the following commutative diagram:




Therefore, we have

$$\begin{aligned}
\Gamma(X, \mathcal{N}_w) &= \Gamma\Big(X\ ,\ i_{w*}\Big(D_{X \leftarrow X_w} \otimes_{D_{X_w}} \mathcal{O}_{X_w}\Big)\Big) \\
&= \Gamma\Big(X_w\ ,\ D_{X \leftarrow X_w} \otimes_{D_{X_w}} \mathcal{O}_{X_w}\Big) \\
&= \Gamma\Big(X_w\ ,\ D_{V \leftarrow X_w} \otimes_{D_{X_w}} \mathcal{O}_{X_w}\Big) \\
&\cong \Gamma\Big(N_2\ ,\ D_{N_1 \times N_2 \leftarrow N_2} \otimes_{D_{N_2}} \mathcal{O}_{N_2}\Big).
\end{aligned}$$

Moreover, by

$$D_{N_1 \times N_2 \leftarrow N_2} = \Big(D_{N_1,e} \otimes_{\mathcal{O}_{N_1,e}} \mathbb{C}\Big) \otimes_{\mathbb{C}} \Big(\Omega_{N_1,e}^{\otimes -1} \otimes_{\mathcal{O}_{N_1,e}} \mathbb{C}\Big) \otimes_{\mathbb{C}} D_{N_2},$$

we obtain

$$\Gamma(X, \mathcal{N}_w) \cong \Big(D_{N_1,e} \otimes_{\mathcal{O}_{N_1,e}} \mathbb{C}\Big) \otimes_{\mathbb{C}} \Big(\Omega_{N_1,e}^{\otimes -1} \otimes_{\mathcal{O}_{N_1,e}} \mathbb{C}\Big) \otimes_{\mathbb{C}} \Gamma(N_2, \mathcal{O}_{N_2}).$$

Identifying $\mathfrak{n}_1$ with the set of right-invariant vector fields on $N_1$, we get $D_{N_1} \cong U(\mathfrak{n}_1) \otimes_{\mathbb{C}} \mathcal{O}_{N_1}$ and hence $D_{N_1,e} \otimes_{\mathcal{O}_{N_1,e}} \mathbb{C} \cong U(\mathfrak{n}_1)$. Furthermore, we have obviously $\Omega_{N_1,e}^{\otimes -1} \otimes_{\mathcal{O}_{N_1,e}} \mathbb{C} \cong \bigwedge^p \mathfrak{n}_1$ ($p = \dim \mathfrak{n}_1$). Since $N_2$ is a unipotent algebraic group, the exponential map $\exp : \mathfrak{n}_2 \to N_2$ is an isomorphism of algebraic varieties, and hence we obtain $\Gamma(N_2, \mathcal{O}_{N_2}) \cong \Gamma(\mathfrak{n}_2, \mathcal{O}_{\mathfrak{n}_2}) = S(\mathfrak{n}_2^*)$. Therefore, we have

$$\Gamma(X, \mathcal{N}_w) \cong U(\mathfrak{n}_1) \otimes_{\mathbb{C}} \bigwedge^p \mathfrak{n}_1 \otimes_{\mathbb{C}} S(\mathfrak{n}_2^*). \tag{12.3.4}$$

Recall that our problem was to study the action of $\mathfrak{h}$ on $\Gamma(X, \mathcal{N}_w)$. Let us define an $H$-action on $N_1 \times N_2$ by

$$t \cdot (n_1, n_2) = (tn_1t^{-1}, tn_2t^{-1}) \quad (t \in H, n_1 \in N_1, n_2 \in N_2).$$

Then $\varphi$ is $H$-equivariant, from which we see that (12.3.4) is an isomorphism of $\mathfrak{h}$-modules. Therefore, we obtain $\mathrm{ch}(\Gamma(X, \mathcal{N}_w)) = \mathrm{ch}(U(\mathfrak{n}_1))\,\mathrm{ch}(\bigwedge^p \mathfrak{n}_1)\,\mathrm{ch}(S(\mathfrak{n}_2^*))$. We easily see that

$$\mathrm{ch}(U(\mathfrak{n}_1)) = \frac{1}{\prod_{\alpha \in \Delta^+ \cap w(\Delta^+)}(1 - e^{-\alpha})}, \tag{12.3.5}$$

$$\mathrm{ch}(S(\mathfrak{n}_2^*)) = \frac{1}{\prod_{\alpha \in \Delta^+ \cap (-w(\Delta^+))}(1 - e^{-\alpha})}. \tag{12.3.6}$$

Moreover, by $\rho = \frac{1}{2}\sum_{\alpha \in \Delta^+} \alpha$ we have $\sum_{\alpha \in \Delta^- \cap w(\Delta^-)} \alpha = -w\rho - \rho$, and hence

$$\mathrm{ch}\Big(\bigwedge^p \mathfrak{n}_1\Big) = e^{-w\rho-\rho}. \tag{12.3.7}$$

Therefore, we finally get

$$\mathrm{ch}(\Gamma(X, \mathcal{N}_w)) = \frac{e^{-w\rho-\rho}}{\prod_{\alpha \in \Delta^+}(1 - e^{-\alpha})} = \mathrm{ch}(M(-w\rho - \rho)).$$

(ii) Set $Z = X \backslash V$ and let $j : V \hookrightarrow X$ be the open embedding. Then we have a distinguished triangle

$$R\Gamma_Z(\mathcal{N}_w) \longrightarrow \mathcal{N}_w \longrightarrow j_*(\mathcal{N}_w|_V) \xrightarrow{+1} .$$

It follows from the definition of $\mathcal{N}_w$ that $\mathcal{N}_w \to j_*(\mathcal{N}_w|_V)$ is an isomorphism, and hence we get $R\Gamma_Z(\mathcal{N}_w) = 0$. In particular, we have $\Gamma_Z(\mathcal{N}_w) = 0$. Thus the only $D_X$-submodule of $\mathcal{N}_w$ whose support is contained in $Z$ is zero. By $\overline{X_w} \backslash X_w \subset Z$ the assertion (ii) is now clear. □

We denote by $\mathcal{L}(X_w, \mathcal{O}_{X_w})$ the minimal extension (see Section 3.4) of the regular holonomic $D_{X_w}$-module $\mathcal{O}_{X_w}$. We have obviously $\mathcal{L}(X_w, \mathcal{O}_{X_w}) \in \mathrm{Mod}_{rh}(D_X, B)$.

**Proposition 12.3.2.** *Let $w \in W$. Then we have*

(i) $\mathcal{L}_w = \mathcal{L}(X_w, \mathcal{O}_{X_w})$,
(ii) $\mathcal{M}_w = \mathbb{D}\mathcal{N}_w$ (*here $\mathbb{D} = \mathbb{D}_X$ is the dualizing functor introduced in Section* 2.6).

*Proof.* (i) By the results in Section 11.6 and the fact that $X_w^{\mathrm{an}}$ is simply connected, the set $\mathcal{P}$ of isomorphism classes of irreducible objects in $\mathrm{Mod}_{rh}(D_X, B)$ is given by $\mathcal{P} = \{\mathcal{L}(X_w, \mathcal{O}_{X_w})\}_{w \in W}$. On the other hand, the set of isomorphism classes of irreducible objects in $\mathrm{Mod}_f(\mathfrak{g}, B, \chi_{-\rho})$ is $\{L(-w\rho - \rho)\}_{w \in W}$. So by the definition of $\mathcal{L}_w$ we get $\mathcal{P} = \{\mathcal{L}_w\}_{w \in W}$. This means that for any $w \in W$ there exists a unique $y \in W$ such that $\mathcal{L}_w = \mathcal{L}(X_y, \mathcal{O}_{X_y})$. By the definition of $\mathcal{M}_w$ and $\mathcal{L}_w$ we see that $\mathcal{L}_w = \mathcal{L}(X_y, \mathcal{O}_{X_y})$ is a composition factor of $\mathcal{M}_w$ (note that $L(-w\rho - \rho)$ is a composition factor of $M(-w\rho - \rho)$). Recall that by Lemma 12.3.1 (i) the composition factors of $\mathcal{M}_w$ and those of $\mathcal{N}_w$ are the same. Hence $\mathcal{L}_w = \mathcal{L}(X_y, \mathcal{O}_{X_y})$

is a composition factor of $\mathcal{N}_w$. By the results in Section 3.4, any composition factor of $\mathcal{N}_w$ should be written as $\mathcal{L}(X_x, \mathcal{O}_{X_x})$ for some $x \leqq w$, which shows that $y \leqq w$. Hence we obtain $y = w$ by an induction on $w$ with respect to the Bruhat order.

(ii) Since the irreducible objects $\mathcal{L}(X_w, \mathcal{O}_{X_w})$ of $\mathrm{Mod}_{rh}(D_X, B)$ are self-dual, the composition factors of $\mathcal{M} \in \mathrm{Mod}_{rh}(D_X, B)$ and those of $\mathbb{D}\mathcal{M}$ coincide. Hence we have $\mathrm{ch}(\Gamma(X, \mathcal{M})) = \mathrm{ch}(\Gamma(X, \mathbb{D}\mathcal{M}))$. In particular, we get $\mathrm{ch}(\Gamma(X, \mathbb{D}\mathcal{N}_w)) = \mathrm{ch}(\Gamma(X, \mathcal{N}_w)) = \mathrm{ch}(M(-w\rho - \rho))$. We see from this that $U(\mathfrak{g}) \cdot \Gamma(X, \mathbb{D}\mathcal{N}_w)_{-w\rho-\rho}$ is a highest weight module with highest weight $-w\rho - \rho$. Hence there exists a non-trivial homomorphism $f_1 : M(-w\rho - \rho) \to \Gamma(X, \mathbb{D}\mathcal{N}_w)$ of $\mathfrak{g}$-modules. Then $L(-w\rho - \rho)$ is not a composition factor of $N = \mathrm{Coker}(f_1)$. From the exact sequence

$$M(-w\rho - \rho) \xrightarrow{f_1} \Gamma(X, \mathbb{D}\mathcal{N}_w) \longrightarrow N \longrightarrow 0 \tag{12.3.8}$$

of $\mathfrak{g}$-module, we obtain an exact sequence

$$\mathcal{M}_w \xrightarrow{f_2} \mathbb{D}\mathcal{N}_w \longrightarrow \mathcal{N} \longrightarrow 0 \tag{12.3.9}$$

of $D_X$-modules, where $\mathcal{N} = D_X \otimes_{U(\mathfrak{g})} N$. Taking its dual we obtain also the exact sequence

$$0 \longrightarrow \mathbb{D}\mathcal{N} \longrightarrow \mathcal{N}_w \xrightarrow{f_3} \mathbb{D}\mathcal{M}_w. \tag{12.3.10}$$

Since $\mathcal{L}_w$ is not contained in the set of composition factors of $\mathcal{N}$ and $\mathbb{D}\mathcal{N}$, the support of $\mathbb{D}\mathcal{N}$ is contained in $\overline{X_w} \backslash X_w$. Hence we get $\mathbb{D}\mathcal{N} = 0$ by Lemma 12.3.1 (ii). It follows that $\mathcal{N} = 0$ and hence $N = 0$. In other words $f_1$ is surjective. We conclude from this that $f_1$ is an isomorphism by $\mathrm{ch}(\Gamma(X, \mathbb{D}\mathcal{N}_w)) = \mathrm{ch}(M(-w\rho - \rho))$. Therefore, $f_2$ is also an isomorphism. □

By the Riemann–Hilbert correspondence we get the following.

**Corollary 12.3.3.**
(i) $\mathrm{DR}_X(\mathcal{M}_w) = \mathbb{C}_{X_w}[\dim X_w]$ $(w \in W)$.
(ii) $\mathrm{DR}_X(\mathcal{L}_w) = {}^{\pi}\mathbb{C}_{X_w}[\dim X_w]$ $(w \in W)$.

*Proof of Theorem* 12.2.7. By the definition of $\mathcal{M}_w$ and $\mathcal{L}_w$ it is enough to show that

$$[\mathcal{L}_w] = \sum_{y \leqq w} (-1)^{l(w)-l(y)} P_{y,w}(1)[\mathcal{M}_y] \quad (w \in W) \tag{12.3.11}$$

in the Grothendieck group $K(\mathrm{Mod}_{rh}(D_X, B))$. Let us define a homomorphism $\varphi : K(\mathrm{Mod}_{rh}(D_X, B)) \to \mathbb{Z}[W]$ of $\mathbb{Z}$-modules by

$$\varphi([\mathcal{M}]) = \sum_{y \in w} \left( \sum_i (-1)^i \dim H^i(\mathrm{DR}_X(\mathcal{M}))_{yB} \right) y. \tag{12.3.12}$$

Note that $\varphi$ is well defined since we are taking an alternating sum of the dimensions of cohomology groups. Then by Corollary 12.3.3 (i) we have $\varphi([\mathcal{M}_w]) = (-1)^{l(w)} w$

and $\varphi$ is an isomorphism of $\mathbb{Z}$-modules. Furthermore, by Corollary 12.3.3 (ii) and Theorem 12.2.5 we obtain

$$\begin{aligned}\varphi([\mathcal{L}_w]) &= (-1)^{l(w)} \sum_y \left( \sum_i (-1)^i \dim H^i({}^{\pi}\mathbb{C}_{X_w})_{yB} \right) y \\ &= (-1)^{l(w)} \sum_{y \leqq w} P_{y,w}(1)\, y \\ &= \sum_{y \leqq w} (-1)^{l(w)-l(y)} P_{y,w}(1) \varphi([\mathcal{M}_y]),\end{aligned}$$

from which our claim (12.3.11) is clear. □

# 13

# Hecke Algebras and Hodge Modules

In this chapter we give geometric realization of group algebras of Weyl groups (resp. Hecke algebras) via $D$-modules (resp. via Hodge modules). We also include a proof of Kazhdan–Lusztig's theorem (Theorem 12.2.5, i.e., the calculation of intersection cohomology complexes of Schubert varieties) using the theory of Hodge modules.

## 13.1 Weyl groups and $D$-modules

Let $\Delta G = \{(g, g) \mid g \in G\} \simeq G$ be the diagonal subgroup of $G \times G$ and consider the diagonal action of $\Delta G$ on $X \times X$. Let us give the orbit decomposition of $X \times X$ for this $\Delta G$-action. Since $G$ acts transitively on $X$, we can take as a representative of each $\Delta G$-orbit in $X \times X$ an element of the form $(eB, gB)$. Furthermore, two elements $(eB, g_1B)$ and $(eB, g_2B)$ lie in the same $\Delta G$-orbit if and only if $g_1B$ and $g_2B$ lie in the same $B$-orbit. Namely, there exists a bijection between the set of $\Delta G$-orbits in $X \times X$ and that of $B$-orbits in $X$. Under this bijection, the $B$-orbit $X_w$ ($w \in W$) in $X$ corresponds to the $\Delta G$-orbit

$$Z_w = \Delta G(eB, wB) \quad (w \in W) \tag{13.1.1}$$

in $X \times X$. Hence we obtain the orbit decomposition

$$X \times X = \coprod_{w \in W} Z_w. \tag{13.1.2}$$

We can also describe geometric properties of $\Delta G$-orbits in $X \times X$ from the ones of $B$-orbits in $X$ as follows. Define morphisms $p_k : X \times X \to X$, $i_k : X \to X \times X$ ($k = 1, 2$) by

$$p_1(a, b) = a, \quad p_2(a, b) = b, \quad i_1(a) = (eB, a), \quad i_2(a) = (a, eB). \tag{13.1.3}$$

Then we can regard $X \times X$ via the projection $p_1$ as a $G$-equivariant fiber bundle on $X = G/B$ whose fiber $i_1(X)$ over $eB \in X$ is isomorphic to $X$. Moreover, for each

$w \in W$, $Z_w$ is a $G$-equivariant subbundle of $X \times X$ whose fiber over $eB \in X$ is isomorphic to $i_1^{-1}(Z_w) = X_w$. Hence we have

$$X \times X = G \overset{B}{\times} X, \quad Z_w = G \overset{B}{\times} X_w \quad (w \in W) \tag{13.1.4}$$

under the standard notation of associated fiber bundles (we can also obtain similar descriptions using $p_2$ and $i_2$ in which $Z_{w^{-1}}$ corresponds to $X_w$).

Namely, the $\Delta G$-orbit decomposition of $X \times X$ corresponds to that of the $B$-orbit decomposition of $X$ including geometric natures such as singularities of the closures. In particular, we have the following by Theorem 9.9.5.

**Proposition 13.1.1.**
(i) $\overline{Z}_w = \coprod_{y \leqq w} Z_y$.
(ii) $\dim Z_w = \dim X + l(w)$.

In this chapter we will deal with $\Delta G$-equivariant $D$-modules on $X \times X$ instead of $B$-equivariant $D$-modules on $X$ considered in Chapter 12. Indeed, we have the following correspondence between them.

**Proposition 13.1.2.** *The inverse image functors* $Li_1^* = i_1^\dagger[\dim X]$ *and* $Li_2^* = i_2^\dagger[\dim X]$ *of D-modules induce the following equivalences of categories:*

$$i_1^* : \mathrm{Mod}_c(D_{X\times X}, \Delta G) \xrightarrow{\sim} \mathrm{Mod}_c(D_X, B),$$
$$i_2^* : \mathrm{Mod}_c(D_{X\times X}, \Delta G) \xrightarrow{\sim} \mathrm{Mod}_c(D_X, B).$$

We omit the proof because it follows easily from the above geometric observations.

Note that

$$H^p Li_k^*(\mathcal{M}) = 0 \quad (p \neq 0,\ k = 1, 2)$$

holds for any $\mathcal{M} \in \mathrm{Mod}_c(D_{X\times X}, \Delta G)$. Note also that

$$\mathrm{Mod}_c(D_{X\times X}, \Delta G) = \mathrm{Mod}_{rh}(D_{X\times X}, \Delta G) \tag{13.1.5}$$

by Theorem 11.6.1. For $w \in W$ consider the embedding $j_w : Z_w \hookrightarrow X \times X$ and set

$$\widetilde{\mathcal{N}}_w = \int_{j_w} \mathcal{O}_{Z_w}, \quad \widetilde{\mathcal{M}}_w = \mathbb{D}\widetilde{\mathcal{N}}_w, \quad \widetilde{\mathcal{L}}_w = \mathcal{L}(Z_w, \mathcal{O}_{Z_w}). \tag{13.1.6}$$

They are objects in $\mathrm{Mod}_c(D_{X\times X}, \Delta G)$. Moreover, we have

$$\begin{aligned} &i_1^*(\widetilde{\mathcal{N}}_w) = \mathcal{N}_w, && i_1^*(\widetilde{\mathcal{M}}_w) = \mathcal{M}_w, && i_1^*(\widetilde{\mathcal{L}}_w) = \mathcal{L}_w, \\ &i_2^*(\widetilde{\mathcal{N}}_w) = \mathcal{N}_{w^{-1}}, && i_2^*(\widetilde{\mathcal{M}}_w) = \mathcal{M}_{w^{-1}}, && i_2^*(\widetilde{\mathcal{L}}_w) = \mathcal{L}_{w^{-1}} \end{aligned} \tag{13.1.7}$$

under the notation of Chapter 12. Therefore, we get the following proposition.

**Proposition 13.1.3.**
(i) $K(\mathrm{Mod}_c(D_{X\times X}, \Delta G)) = \bigoplus_{w\in W} \mathbb{Z}[\widetilde{\mathcal{M}}_w] = \bigoplus_{w\in W} \mathbb{Z}[\widetilde{\mathcal{L}}_w]$.
(ii) *For any* $w \in W$ *we have* $[\widetilde{\mathcal{M}}_w] = [\widetilde{\mathcal{N}}_w]$.

Let us define morphisms $p_{13} : X \times X \times X \to X \times X$ and $r : X \times X \times X \to X \times X \times X \times X$ by

$$p_{13}(a, b, c) = (a, c) \quad r(a, b, c) = (a, b, b, c). \tag{13.1.8}$$

**Remark 13.1.4.** For any $\widetilde{\mathcal{M}}, \widetilde{\mathcal{N}} \in \mathrm{Mod}_c(D_{X\times X}, \Delta G)$ and $p \neq 0$, we can prove that $H^p Lr^*(\widetilde{\mathcal{M}} \boxtimes \widetilde{\mathcal{N}}) = 0$ (see the proof of Proposition 13.2.7 (i) below).

**Proposition 13.1.5.**
(i) *The Grothendieck group* $K(\mathrm{Mod}_c(D_{X\times X}, \Delta G))$ *has a ring structure defined by*

$$[\widetilde{\mathcal{M}}] \cdot [\widetilde{\mathcal{N}}] = \sum_k (-1)^k \left[ H^k \int_{p_{13}} r^*(\widetilde{\mathcal{M}} \boxtimes \widetilde{\mathcal{N}}) \right].$$

(ii) *The ring* $K(\mathrm{Mod}_c(D_{X\times X}, \Delta G))$ *is isomorphic to the group ring* $\mathbb{Z}[W]$ *of the Weyl group* $W$ *by the correspondence* $[\widetilde{\mathcal{M}}_w] \leftrightarrow (-1)^{l(w)} w$.

*Proof.* (i) The proof is a simple application of the base change theorem for $D$-modules. By

$$\left([\widetilde{\mathcal{M}}] \cdot [\widetilde{\mathcal{N}}]\right) \cdot [\widetilde{\mathcal{L}}] = \sum_k (-1)^k \left[ H^k \int_{p_{13}} r^* \left( \left( \int_{p_{13}} r^*(\widetilde{\mathcal{M}} \boxtimes \widetilde{\mathcal{N}}) \right) \boxtimes \widetilde{\mathcal{L}} \right) \right],$$

$$[\widetilde{\mathcal{M}}] \cdot \left([\widetilde{\mathcal{N}}] \cdot [\widetilde{\mathcal{L}}]\right) = \sum_k (-1)^k \left[ H^k \int_{p_{13}} r^* \left( \widetilde{\mathcal{M}} \boxtimes \left( \int_{p_{13}} r^*(\widetilde{\mathcal{N}} \boxtimes \widetilde{\mathcal{L}}) \right) \right) \right],$$

it suffices to prove that

$$\int_{p_{13}} r^* \left( \left( \int_{p_{13}} r^*(\widetilde{\mathcal{M}} \boxtimes \widetilde{\mathcal{N}}) \right) \boxtimes \widetilde{\mathcal{L}} \right) = \int_{p_{13}} r^* \left( \widetilde{\mathcal{M}} \boxtimes \left( \int_{p_{13}} r^*(\widetilde{\mathcal{N}} \boxtimes \widetilde{\mathcal{L}}) \right) \right).$$

The left-hand side (LHS) of the above formula is given by

$$(\mathrm{LHS}) = \int_{p_{13}} r^* \int_{p_{13}\times 1\times 1} (r \times 1 \times 1)^* (\widetilde{\mathcal{M}} \boxtimes \widetilde{\mathcal{N}} \boxtimes \widetilde{\mathcal{L}}).$$

Since the diagram

$$\begin{array}{ccc} X^4 & \xrightarrow{1\times r} & X^5 \\ {\scriptstyle p_{13}\times 1}\downarrow & & \downarrow{\scriptstyle p_{13}\times 1\times 1} \\ X^3 & \xrightarrow{r} & X^4 \end{array}$$

is a cartesian square (i.e., $X^4$ is the fiber product of $r$ and $p_{13} \times 1 \times 1$), we get

$$\begin{aligned}(\mathrm{LHS}) &= \int_{p_{13}} \int_{p_{13}\times 1} (1\times r)^*(r\times 1\times 1)^*(\widetilde{\mathcal{M}}\boxtimes\widetilde{\mathcal{N}}\boxtimes\widetilde{\mathcal{L}})\\ &= \int_{p_{13}\circ(p_{13}\times 1)} \big((r\times 1\times 1)\circ(1\times r)\big)^*(\widetilde{\mathcal{M}}\boxtimes\widetilde{\mathcal{N}}\boxtimes\widetilde{\mathcal{L}}).\end{aligned}$$

Similarly the right-hand side (RHS) can be rewritten as follows:

$$(\mathrm{RHS}) = \int_{p_{13}\circ(1\times p_{13})} \big((1\times 1\times r)\circ(r\times 1)\big)^*(\widetilde{\mathcal{M}}\boxtimes\widetilde{\mathcal{N}}\boxtimes\widetilde{\mathcal{L}}).$$

Therefore, the assertion (i) follows from $p_{13}\circ(p_{13}\times 1) = p_{13}\circ(1\times p_{13})$ and $(r\times 1\times 1)\circ(1\times r) = (1\times 1\times r)\circ(r\times 1)$.

(ii) It is enough to prove the following two formulas:

$$[\widetilde{\mathcal{M}}_y]\cdot[\widetilde{\mathcal{M}}_w] = [\widetilde{\mathcal{M}}_{yw}] \qquad (l(y)+l(w)=l(yw)), \tag{13.1.9}$$

$$[\widetilde{\mathcal{M}}_s]^2 = [\widetilde{\mathcal{M}}_e] \qquad (s\in S), \tag{13.1.10}$$

where $S=\{s_\alpha \mid \alpha\in\Pi\}$. We first show (13.1.9). By $[\widetilde{\mathcal{M}}_w] = [\widetilde{\mathcal{N}}_w] = [\int_{j_w}\mathcal{O}_{Z_w}]$ we get

$$\begin{aligned}[\widetilde{\mathcal{M}}_y]\cdot[\widetilde{\mathcal{M}}_w] &= \sum_k (-1)^k\left[H^k\int_{p_{13}} r^*\left(\int_{j_y}\mathcal{O}_{Z_y}\boxtimes\int_{j_w}\mathcal{O}_{Z_w}\right)\right]\\ &= \sum_k (-1)^k\left[H^k\int_{p_{13}} r^*\int_{j_y\times j_w}\mathcal{O}_{Z_y\times Z_w}\right].\end{aligned}$$

We see from $l(y)+l(w)=l(yw)$ that for any $(a,b)\in Z_{yw}$ there exists a unique $c\in X$ such that $(a,c)\in Z_y$ and $(c,b)\in Z_w$. Conversely, if $(a,c)\in Z_y$ and $(c,b)\in Z_w$, then $(a,b)\in Z_{yw}$. Hence if we define morphisms $\varphi: Z_{yw}\to Z_y\times Z_w$, $\psi: Z_{yw}\to X^3$ by $\varphi(a,b)=(a,c,c,b)$, $\psi(a,b)=(a,c,b)$, then the diagram

$$\begin{array}{ccc} Z_{yw} & \xrightarrow{\varphi} & Z_y\times Z_w\\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle j_y\times j_w}\\ X^3 & \xrightarrow{r} & X^4\end{array}$$

is the fiber product of $r$ and $j_y\times j_w$. Therefore, it follows from this and $p_{13}\circ\psi = j_{yw}$ that

$$\int_{p_{13}} r^*\int_{j_y\times j_w}\mathcal{O}_{Z_y\times Z_w} = \int_{p_{13}}\int_{\psi}\varphi^*\mathcal{O}_{Z_y\times Z_w} = \int_{j_{yw}}\mathcal{O}_{Z_{yw}} = \widetilde{\mathcal{N}}_{yw}.$$

Hence we obtain (13.1.9). We next show (13.1.10). Note that $\overline{Z}_s$ is smooth and $\overline{Z}_s = Z_s\sqcup Z_e$. Moreover, $Z_e$ is a smooth closed subvariety of $\overline{Z}_s$ of codimension one. Let $j'_s: Z_s\to\overline{Z}_s$, $j'_e: Z_e\to\overline{Z}_s$, $\overline{j_s}: \overline{Z}_s\to X\times X$ be the embeddings. Then we have an exact sequence

$$0 \longrightarrow \mathcal{O}_{\overline{Z}_s} \longrightarrow \int_{j'_s} \mathcal{O}_{Z_s} \longrightarrow \int_{j'_e} \mathcal{O}_{Z_e} \longrightarrow 0 \tag{13.1.11}$$

of $\mathcal{D}_{\overline{Z}_s}$-modules. Since $\overline{Z}_s$ is smooth, we have $\widetilde{\mathcal{L}}_s = \int_{\overline{j}_s} \mathcal{O}_{\overline{Z}_s}$. Hence by applying the functor $\int_{\overline{j}_s}$ to (13.1.11) we get a new exact sequence

$$0 \longrightarrow \widetilde{\mathcal{L}}_s \longrightarrow \widetilde{\mathcal{N}}_s \longrightarrow \widetilde{\mathcal{N}}_e \longrightarrow 0 \tag{13.1.12}$$

of $\mathcal{D}_{X\times X}$-modules. Therefore, we obtain $[\widetilde{\mathcal{M}}_s] = [\widetilde{\mathcal{N}}_s] = [\widetilde{\mathcal{L}}_s] + [\widetilde{\mathcal{N}}_e]$. Since $[\widetilde{\mathcal{N}}_e]$ is the identity element 1 of $K(\mathrm{Mod}_c(D_{X\times X}, \Delta G))$ by (13.1.9), it suffices to prove

$$[\widetilde{\mathcal{L}}_s]^2 = -2[\widetilde{\mathcal{L}}_s]. \tag{13.1.13}$$

By the definition of $[\widetilde{\mathcal{L}}_s]$ we have

$$\begin{aligned}[\widetilde{\mathcal{L}}_s]^2 &= \sum_k (-1)^k \left[ H^k \int_{p_{13}} r^* \left( \int_{\overline{j}_s} \mathcal{O}_{\overline{Z}_s} \boxtimes \int_{\overline{j}_s} \mathcal{O}_{\overline{Z}_s} \right) \right] \\ &= \sum_k (-1)^k \left[ H^k \int_{p_{13}} r^* \int_{\overline{j}_s \times \overline{j}_s} \mathcal{O}_{\overline{Z}_s \times \overline{Z}_s} \right].\end{aligned}$$

Now set $Y = \{(a,b,c) \in X^3 \mid (a,b), (b,c) \in \overline{Z}_s\}$ and let $j : Y \to X \times X \times X$ be the embedding. Then by virtue of the base change theorem, we have $r^* \int_{\overline{j}_s \times \overline{j}_s} \mathcal{O}_{\overline{Z}_s \times \overline{Z}_s} = \int_j \mathcal{O}_Y$ and hence

$$[\widetilde{\mathcal{L}}_s]^2 = \sum_k (-1)^k \left[ H^k \int_{p_{13}\circ j} \mathcal{O}_Y \right].$$

Note that the image of $p_{13} \circ j : Y \to X \times X$ is $\overline{Z}_s$ and $Y \to \overline{Z}_s$ is a $\mathbb{P}^1$-bundle on $\overline{Z}_s$. Consequently we get

$$H^k \int_{p_{13}\circ j} \mathcal{O}_Y = \begin{cases} \mathcal{L}_s & (k = \pm 1) \\ 0 & (k \neq \pm 1), \end{cases}$$

from which (13.1.13) follows immediately. □

Let us define morphisms $p_1 : X \times X \to X$, $q : X \times X \to X \times X \times X$ by

$$p_1(a,b) = a, \quad q(a,b) = (a,b,b). \tag{13.1.14}$$

Then we have the following similar result.

**Proposition 13.1.6.** *Let $K$ be a closed subgroup of $G$. Then a $K(\mathrm{Mod}_c(D_{X\times X}, \Delta G))$-module structure on $K(\mathrm{Mod}_c(D_X, K))$ is defined by*

$$[\widetilde{\mathcal{M}}] \cdot [\mathcal{N}] = \sum_k (-1)^k \left[ H^k \int_{p_1} q^*(\widetilde{\mathcal{M}} \boxtimes \mathcal{N}) \right]$$
$$\big(\widetilde{\mathcal{M}} \in \mathrm{Mod}_c(D_{X\times X}, \Delta G), \quad \mathcal{N} \in \mathrm{Mod}_c(D_X, K)\big).$$

The proof of this proposition is similar to that of Proposition 13.1.5 (i), and omitted. By Propositions 13.1.5 and 13.1.6 the Grothendieck group $K(\mathrm{Mod}_c(D_X, K))$ is a $W$-module. Hence combining this result with Theorem 11.5.3, we see that the Grothendieck group $K(\mathrm{Mod}_f(\mathfrak{g}, \chi_{-\rho}, K))$ of the abelian category of $K$-equivariant $\mathfrak{g}$-modules is a representation space of the Weyl group $W$. We can prove that it coincides with the *coherent continuation representation* due to G. Zuckerman and D. Vogan. Namely, Propositions 13.1.5 and 13.1.6 give a $D$-module-theoretical interpretation of coherent continuation representations.

If $K = B$, then we have

$$i_2^*([\widetilde{\mathcal{M}}] \cdot [\widetilde{\mathcal{N}}]) = [\widetilde{\mathcal{M}}] \cdot i_2^*([\widetilde{\mathcal{N}}]) \quad \big(\widetilde{\mathcal{M}}, \widetilde{\mathcal{N}} \in \mathrm{Mod}_c(D_{X\times X}, \Delta G)\big). \tag{13.1.15}$$

Consequently $K(\mathrm{Mod}_c(D_X, B))$ is isomorphic to the left regular representation of the Weyl group $W$ as a left $K(\mathrm{Mod}_c(D_{X\times X}, \Delta G))(\simeq \mathbb{Z}[W])$-module. One of the advantages of using $\mathrm{Mod}_c(D_{X\times X}, \Delta G)$ rather than $\mathrm{Mod}_c(D_X, B)$ is that we can define a ring structure on its Grothendieck group by the *convolution product*.

## 13.2 Hecke algebras and Hodge modules

In this section we show that Hecke algebras naturally appear in the context of the geometry of Schubert varieties. Among other things we give a proof of Theorem 12.2.5 which was stated without proof in Chapter 12.

Let us reconsider the proof of the main theorem of Chapter 12 (the Kazhdan–Lusztig conjecture, Theorem 12.2.7) in the view of the ring isomorphism

$$K(\mathrm{Mod}_c(D_{X\times X}, \Delta G)) \simeq \mathbb{Z}[W]$$

given in Section 13.1. The original problem was to determine the transfer matrix between the two bases $\{[M(-w\rho-\rho)]\}_{w\in W}$, $\{[L(-w\rho-\rho)]\}_{w\in W}$ of the Grothendieck group $K(\mathrm{Mod}_f(\mathfrak{g}, \chi_{-\rho}, B))$. By the correspondence between $U(\mathfrak{g})$-modules and $D_X$-modules this problem is equivalent to determining the transfer matrix between two bases $\{[\mathcal{M}_w]\}_{w\in W}$, $\{[\mathcal{L}_w]\}_{w\in W}$ of $K(\mathrm{Mod}_c(D_X, B))$, and its answer was given by Kazhdan–Lusztig's theorem (Theorem 12.2.5). Note that what we actually need in this process is the following result, which is weaker than Kazhdan–Lusztig's theorem (see the proof of Theorem 12.2.7 at the end of Chapter 12 and Proposition 13.1.2).

**Proposition 13.2.1.** *Under the ring isomorphism* $K(\mathrm{Mod}_c(D_{X\times X}, \Delta G)) \simeq \mathbb{Z}[W]$ $([\widetilde{\mathcal{M}}_w] \leftrightarrow (-1)^{l(w)}w)$ *in Proposition* 13.1.5*, we have*

$$[\widetilde{\mathcal{L}}_w] \leftrightarrow (-1)^{l(w)} \sum_{y\leqq w} P_{y,w}(1)y.$$

We reduced the proof of the Kazhdan–Lusztig conjecture to Proposition 13.2.1 above in Chapter 12 and derived it from Kazhdan–Lusztig's theorem (Theorem 12.2.5).

Recall that the Kazhdan–Lusztig polynomials $P_{y,w}(q) \in \mathbb{Z}[q]$ are the coefficients of the elements

$$C_w = \sum_{y \leqq w} P_{y,w}(q) T_y \in H(W) \tag{13.2.1}$$

in the Hecke algebra $H(W) = \bigoplus_{w \in W} \mathbb{Z}[q, q^{-1}]T_w$ which are characterized by the conditions (12.2.7),…,(12.2.10). Consider the ring $H(W)|_{q=1} = \mathbb{Z} \otimes_{\mathbb{Z}[q,q^{-1}]} H(W)$ given by the specialization $q \mapsto 1$. Then we have a ring isomorphism $H(W)|_{q=1} \simeq \mathbb{Z}[W]$ induced by $1 \otimes T_w \leftrightarrow w$. Therefore, Proposition 13.2.1 can be restated as follows.

**Proposition 13.2.2.** *There exists a ring isomorphism*

$$K(\mathrm{Mod}_c(D_{X\times X}, \Delta G)) \simeq H(W)|_{q=1}$$

*such that*

$$[\widetilde{\mathcal{M}}_w] \leftrightarrow (-1)^{l(w)} T_w|_{q=1}, \quad [\widetilde{\mathcal{L}}_w] \leftrightarrow (-1)^{l(w)} C_w|_{q=1}.$$

Note that the elements $C_w|_{q=1}$ appearing in this proposition are obtained by specializing the elements $C_w$ which are characterized inside the Hecke algebra. It indicates that the essence of our problem lies not in $H(W)|_{q=1}$ but in $H(W)$. Hence it would be natural to expect the existence of some superstructure of $K(\mathrm{Mod}_c(D_{X\times X}, \Delta G))$ which corresponds to $H(W)$. The main aim of this section is to construct such a superstructure using the theory of Hodge modules.

We need some results on Hodge modules. We use the notation in Section 8.3. Recall that for the one-point algebraic variety pt, the category $MHM(\mathrm{pt})$ coincides with $SHM^p$. We denote its Grothendieck group by

$$R = K(MHM(\mathrm{pt})) = K(SHM^p). \tag{13.2.2}$$

Since each object of $MHM(\mathrm{pt})$ has a weight filtration $W$ such that $\mathrm{gr}_n^W \in SH(n)^p$, we obtain a direct sum decomposition

$$R = \bigoplus_{n \in \mathbb{Z}} R_n, \tag{13.2.3}$$

where

$$R_n = K(SH(n)^p) \qquad (n \in \mathbb{Z}). \tag{13.2.4}$$

Note that $R_n$ is a free $\mathbb{Z}$-module with a basis consisting of the isomorphism classes of irreducible objects in $SH(n)^p$. The tensor product in $MHM(\mathrm{pt})$ induces a ring structure on $R$ such that $R$ is commutative and

$$R_i R_j \subset R_{i+j} \qquad (i, j \in \mathbb{Z}). \tag{13.2.5}$$

The unit element of $R$ is represented by the trivial Hodge structure

$$\mathbb{Q}^H = (\mathbb{C}, F, \mathbb{Q}, W) \in MHM(\mathrm{pt}), \tag{13.2.6}$$

$$\mathrm{gr}_p^F = 0 \quad (p \neq 0), \quad \mathrm{gr}_n^W = 0 \quad (n \neq 0).$$

Furthermore, the dualizing functor in $MHM(\mathrm{pt})$ induces an involution $r \mapsto \overline{r}$ of the ring $R$ with the property

$$\overline{R}_i = R_{-i} \qquad (i \in \mathbb{Z}). \tag{13.2.7}$$

Finally, consider an injective homomorphism $\mathbb{Z}[q, q^{-1}] \hookrightarrow R$ of rings defined by $q^n \mapsto [\mathbb{Q}^H(-n)]$. Then we have

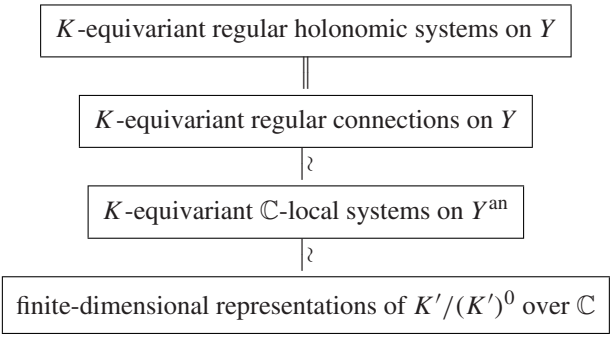
$$q \in R_2, \qquad \overline{q} = q^{-1}. \tag{13.2.8}$$

Now let $Y$ be a smooth algebraic variety endowed with an action of an algebraic group $K$. Then the notion of $K$-equivariant mixed Hodge modules on $Y$ can be defined similarly to the case of $K$-equivariant $D_Y$-modules because we also have inverse image functors in the category of mixed Hodge modules. We denote the abelian category of $K$-equivariant mixed Hodge modules on $Y$ by $MHM(Y, K)$. Its Grothendieck group is denoted by $K(MHM(Y, K))$. By the tensor product functor

$$MHM(\mathrm{pt}) \times MHM(Y, K) \to MHM(Y, K) \tag{13.2.9}$$

the Grothendieck group $K(MHM(Y, K))$ is naturally an $R$-module.

Let us consider the case where $Y$ consists of a single $K$-orbit. This is exactly the case when $Y$ is a homogeneous space $K/K'$ for a closed subgroup $K'$ of $K$. In this case we have the following equivalences of categories:




Therefore, $\mathrm{Mod}_c(D_Y, K) = \mathrm{Mod}_{rh}(D_Y, K)$ is equivalent to the category of finite-dimensional representations of the finite group $K'/(K')^0$ over $\mathbb{C}$. In the case of Hodge modules the situation is more complicated, because $\mathbb{Q}$-structures and filtrations are concerned. However, as in the case of $D$-modules the next proposition holds.

**Proposition 13.2.3.** *Assume that $Y$ is a homogeneous space $K/K'$ of an algebraic group $K$. Then we have*

$$MHM(Y, K) \simeq MHM(\mathrm{pt}, K'/(K')^0).$$

Namely, the problem of classifying equivariant mixed Hodge modules on a homogeneous space is equivalent to classifying mixed Hodge structures with actions of a fixed finite group; however, it is a hard task in general. For a finite group $\mathcal{G}$ we denote by $\mathrm{Irr}(\mathcal{G}, \mathbb{Q})$ the set of the isomorphism classes of irreducible representations of $\mathcal{G}$ over $\mathbb{Q}$.

**Proposition 13.2.4.** *Let $\mathcal{G}$ be a finite group and assume that all its irreducible representations over $\mathbb{Q}$ are absolutely irreducible. Then any $\mathcal{H} \in MHM(\mathrm{pt}, \mathcal{G})$ can be uniquely decomposed as*

$$\mathcal{H} = \bigoplus_{\sigma \in \mathrm{Irr}(\mathcal{G}, \mathbb{Q})} (\mathcal{H}_\sigma \otimes \sigma) \qquad \mathcal{H}_\sigma \in MHM(\mathrm{pt}).$$

*Proof.* Suppose that we are given an action of $\mathcal{G}$ on $\mathcal{H} = (H_{\mathbb{C}}, F, H, W) \in MHM(\mathrm{pt})$. By our assumption the $\mathcal{G}$-module $H$ over $\mathbb{Q}$ is decomposed as

$$H = \bigoplus_{\sigma \in \mathrm{Irr}(\mathcal{G}, \mathbb{Q})} (H_\sigma \otimes_{\mathbb{Q}} \sigma) \qquad H_\sigma = \mathrm{Hom}_{\mathcal{G}}(\sigma, H).$$

Hence, if we denote the complexifications of $H_\sigma$, $\sigma$ by $H_{\sigma,\mathbb{C}}$, $\sigma_{\mathbb{C}}$, respectively, then we have

$$H_{\mathbb{C}} = \bigoplus_{\sigma \in \mathrm{Irr}(\mathcal{G}, \mathbb{Q})} (H_{\sigma,\mathbb{C}} \otimes_{\mathbb{C}} \sigma_{\mathbb{C}}) = \bigoplus_{\sigma \in \mathrm{Irr}(\mathcal{G}, \mathbb{Q})} (H_{\sigma,\mathbb{C}} \otimes_{\mathbb{Q}} \sigma).$$

Since the filtration $F$ of $H_{\mathbb{C}}$ is $\mathcal{G}$-invariant, there exists a filtration $F$ of $H_{\sigma,\mathbb{C}}$ satisfying

$$F_p(H_{\mathbb{C}}) = \bigoplus_{\sigma} (F_p(H_{\sigma,\mathbb{C}}) \otimes_{\mathbb{Q}} \sigma).$$

It also follows from the $\mathcal{G}$-invariance of the weight filtration $W$ that we can define a filtration $W$ (we use the same letter $W$ for it) of $(H_\sigma, F)$ so that

$$W_n(H, F) = \bigoplus_{\sigma} (W_n(H_\sigma, F) \otimes \sigma)$$

holds. Hence our assertion holds for $\mathcal{H}_\sigma = (H_{\sigma,\mathbb{C}}, F, H_\sigma, W)$. □

Let $Z$ be a smooth algebraic variety. For a $\mathbb{Q}$-local system $S$ on $Z^{\mathrm{an}}$ we define an object

$$S^H[\dim Z] = (\mathcal{M}, F, S[\dim Z], W) \in MF_{rh}W(D_Z, \mathbb{Q}) \tag{13.2.10}$$

by the conditions

$$\begin{aligned} &\mathrm{DR}_Z(\mathcal{M}) = S \otimes_{\mathbb{Q}} \mathbb{C}[\dim Z], \\ &\mathrm{gr}_p^F = 0 \quad (p \neq 0), \quad \mathrm{gr}_n^W = 0 \quad (n \neq \dim Z). \end{aligned} \tag{13.2.11}$$

By Propositions 13.2.3 and 13.2.4 we have the following result.

**Corollary 13.2.5.** *Let $Y = K/K'$ be a homogeneous space of $K$ and assume that all irreducible representations of $K'/(K')^0$ over $\mathbb{Q}$ are absolutely irreducible. Denote by $S_\sigma$ the local system on $Y^{\mathrm{an}}$ which corresponds to $\sigma \in \mathrm{Irr}(K'/(K')^0, \mathbb{Q})$. Then for any $\sigma$ we have $S_\sigma^H[\dim Y] \in MHM(Y, K)$. Moreover, any $\mathcal{H} \in MHM(Y, K)$ is uniquely decomposed as*

$$\mathcal{H} = \bigoplus_{\sigma \in \mathrm{Irr}(K'/(K')^0, \mathbb{Q})} (\mathcal{H}_\sigma \otimes S_\sigma^H[\dim Y]) \quad (\mathcal{H}_\sigma \in MHM(\mathrm{pt})).$$

**Corollary 13.2.6.** *Under the assumptions of Corollary* 13.2.5, *$K(MHM(Y, K))$ is a free $R$-module with the basis $\{[S_\sigma^H[\dim Y]] \mid \sigma \in \mathrm{Irr}(K'/(K')^0, \mathbb{Q})\}$.*

Now let us return to consider the category $MHM(X \times X, \Delta G)$. For each $w \in W$, we have the $\Delta G$-equivariant Hodge module

$$\mathbb{Q}_{Z_w}^H[\dim Z_w] \in MHM(Z_w, \Delta G) \tag{13.2.12}$$

on the $\Delta G$-orbit $Z_w$, and by Corollary 13.2.6, the Grothendieck group $K(MHM(Z_w, \Delta G))$ is a free $R$-module of rank one, with basis $[\mathbb{Q}_{Z_w}^H[\dim Z_w]]$. Since the embedding $j_w : Z_w \hookrightarrow X \times X$ is an affine morphism, we obtain three embeddings

$$j_{w\bigstar},\ j_{w!},\ j_{w!\bigstar} : MHM(Z_w, \Delta G) \to MHM(X \times X, \Delta G) \tag{13.2.13}$$

of categories (see Section 8.3), which induce (at the level of Grothendieck groups) injective morphisms

$$j_{w\bigstar}, j_{w!}, j_{w!\bigstar} : K(MHM(Z_w, \Delta G)) \to K(MHM(X \times X, \Delta G)) \tag{13.2.14}$$

of $R$-modules. Moreover, we have

$$\begin{aligned} K(MHM(X \times X, \Delta G)) &= \bigoplus_{w \in W} j_{w\bigstar} K(MHM(Z_w, \Delta G)) \\ &= \bigoplus_{w \in W} j_{w!} K(MHM(Z_w, \Delta G)) \\ &= \bigoplus_{w \in W} j_{w!\bigstar} K(MHM(Z_w, \Delta G)). \end{aligned} \tag{13.2.15}$$

Now for $w \in W$ let us set

$$\begin{aligned} &\widetilde{\mathcal{N}}_w^H = j_{w\bigstar}(\mathbb{Q}_{Z_w}^H[\dim Z_w]), \quad \widetilde{\mathcal{M}}_w^H = j_{w!}(\mathbb{Q}_{Z_w}^H[\dim Z_w]), \\ &\widetilde{\mathcal{L}}_w^H = j_{w!\bigstar}(\mathbb{Q}_{Z_w}^H[\dim Z_w]) = \mathrm{IC}_{\overline{Z}_w}^H. \end{aligned} \tag{13.2.16}$$

Then in view of the above arguments we have three bases $\{[\widetilde{\mathcal{N}}_w^H]\}_{w \in W}$, $\{[\widetilde{\mathcal{M}}_w^H]\}_{w \in W}$, $\{[\widetilde{\mathcal{L}}_w^H]\}_{w \in W}$ of the free $R$-module $K(MHM(X \times X, \Delta G))$. The underlying $D$-modules of $\widetilde{\mathcal{N}}_w^H, \widetilde{\mathcal{M}}_w^H, \widetilde{\mathcal{L}}_w^H$ are $\widetilde{\mathcal{N}}_w, \widetilde{\mathcal{M}}_w, \widetilde{\mathcal{L}}_w$, respectively. At the level of $D$-modules,

we proved $[\widetilde{\mathcal{N}}_w] = [\widetilde{\mathcal{M}}_w]$. However, at the level of Hodge modules one has in general $[\widetilde{\mathcal{N}}_w^H] \neq [\widetilde{\mathcal{M}}_w^H]$.

Similar to the constructions in Section 13.1 we define a product on the $R$-module $K(MHM(X \times X, \Delta G))$ by

$$[\mathcal{V}_1] \cdot [\mathcal{V}_2] = (-1)^{\dim X} \sum_j (-1)^j [H^j(p_{13!} r^\bigstar (\mathcal{V}_1 \boxtimes \mathcal{V}_2))] \tag{13.2.17}$$
$$(\mathcal{V}_1, \mathcal{V}_2 \in MHM(X \times X, \Delta G)),$$

where $p_{13} : X \times X \times X \to X \times X$ and $r : X \times X \times X \to X \times X \times X \times X$ are the morphisms defined in Section 13.1. We also have the base change theorem in the category of Hodge modules, and hence it is proved similarly to Proposition 13.1.5 that $K(MHM(X \times X, \Delta G))$ is endowed with a structure of an $R$-algebra by the product defined above.

**Proposition 13.2.7.** *Let $\mathcal{V}_1, \mathcal{V}_2 \in MHM(X \times X, \Delta G)$. Then we have*

(i) $H^j r^\bigstar (\mathcal{V}_1 \boxtimes \mathcal{V}_2) = 0 \quad (j \neq -\dim X)$.
(ii) $r^!(\mathcal{V}_1 \boxtimes \mathcal{V}_2) = r^\bigstar (\mathcal{V}_1 \boxtimes \mathcal{V}_2)[-2\dim X](-\dim X)$. *Here $(\bullet)$ is the Tate twist.*

*Proof.* We define morphisms $\varphi_1 : X \times G \to X \times X$, $\varphi_2 : G \times X \to X \times X$, $\psi : X \times G \times X \to X \times X \times X$ by $\varphi_1(x, g) = (gx, gB)$, $\varphi_2(g, x) = (gB, gx)$, $\psi(x_1, g, x_2) = (gx_1, gB, gx_2)$, respectively. We first show that

$$\varphi_1^\bigstar \mathcal{V}_1 = \mathcal{V}_1' \boxtimes \mathbb{Q}_G^H[\dim X] \quad (\mathcal{V}_1' \in MHM(X)), \tag{13.2.18}$$
$$\varphi_2^\bigstar \mathcal{V}_2 = \mathbb{Q}_G^H \boxtimes \mathcal{V}_2'[\dim X] \quad (\mathcal{V}_2' \in MHM(X)). \tag{13.2.19}$$

Let $\sigma : G \times X \times X \to X \times X$ be the morphism defined by the $G$-action on $X \times X$ and let $p_2 : G \times X \times X \to X \times X$ be the projection. We also define morphisms $k : X \times G \to G \times X \times X$, $i : X \to X \times X$, $p : X \times G \to X$ by $k(x, g) = (g, x, eB)$, $i(x) = (x, eB)$, $p(x, g) = x$, respectively. Then we have a chain of isomorphisms

$$\varphi_1^\bigstar \mathcal{V}_1 = k^\bigstar \sigma^\bigstar \mathcal{V}_1 \simeq k^\bigstar p_2^\bigstar \mathcal{V}_1 = p^\bigstar i^\bigstar \mathcal{V}_1 = i^\bigstar \mathcal{V}_1 \boxtimes \mathbb{Q}_G^H.$$

Since $\varphi_1$ is smooth, we have $H^j \varphi_1^\bigstar \mathcal{V}_1 = 0$ for $j \neq \dim G - \dim X$. This implies that there exists an object $\mathcal{V}_1' \in MHM(X)$ such that

$$\varphi_1^\bigstar \mathcal{V}_1 = (\mathcal{V}_1' \boxtimes \mathbb{Q}_G^H[\dim G])[\dim X - \dim G] = \mathcal{V}_1' \boxtimes \mathbb{Q}_G^H[\dim X].$$

This shows (13.2.18). The assertion (13.2.19) is proved similarly. Now let us define a morphism $\widetilde{r} : X \times G \times X \to X \times G \times G \times X$ by $\widetilde{r}(x_1, g, x_2) = (x_1, g, g, x_2)$. Then by $r \circ \psi = (\varphi_1 \times \varphi_2) \circ \widetilde{r}$ we have

$$\begin{aligned} \psi^\bigstar r^\bigstar (\mathcal{V}_1 \boxtimes \mathcal{V}_2) &= \widetilde{r}^\bigstar (\mathcal{V}_1' \boxtimes \mathbb{Q}_G^H \boxtimes \mathbb{Q}_G^H \boxtimes \mathcal{V}_2')[2\dim X] \\ &= \mathcal{V}_1' \boxtimes \mathbb{Q}_G^H \boxtimes \mathcal{V}_2'[2\dim X]. \end{aligned}$$

This implies that $H^j\psi^\bigstar r^\bigstar(\mathcal{V}_1 \boxtimes \mathcal{V}_2) = 0$ for $j \neq \dim G - 2\dim X$. Since $\psi$ is smooth, we obtain from this that $H^j r^\bigstar(\mathcal{V}_1 \boxtimes \mathcal{V}_2) = 0$ for $j \neq -\dim X$. This completes the proof of (i). Moreover, the smoothness of $\psi$ implies that $\psi^! = \psi^\bigstar[2(\dim G - \dim X)](\dim G - \dim X)$. Therefore, we get

$$\begin{aligned}\psi^\bigstar r^!(\mathcal{V}_1 \boxtimes \mathcal{V}_2) &= \psi^! r^!(\mathcal{V}_1 \boxtimes \mathcal{V}_2)[2(\dim G - \dim X)](\dim G - \dim X)\\ &= \widetilde{r}^!(\mathcal{V}_1' \boxtimes \mathbb{Q}_G^H \boxtimes \mathbb{Q}_G^H \boxtimes \mathcal{V}_2')[2\dim G](\dim G - \dim X)\\ &= \mathcal{V}_1' \boxtimes \mathbb{Q}_G^H \boxtimes \mathcal{V}_2'(-\dim X).\end{aligned}$$

The assertion (ii) follows from this. □

Recall that we have an embedding $\mathbb{Z}[q, q^{-1}] \hookrightarrow R$ of the rings, where $R = K(SHM^p)$.

**Theorem 13.2.8.** *Define an isomorphism*

$$F : K(MHM(X \times X, \Delta G)) \to R \otimes_{\mathbb{Z}[q,q^{-1}]} H(W)$$

*of $R$-modules by $F([\widetilde{\mathcal{M}}_w^H]) = (-1)^{l(w)}T_w$ $(w \in W)$. Then $F$ is an isomorphism of $R$-algebras satisfying the conditions*

$$F([\widetilde{\mathcal{L}}_w^H]) = (-1)^{l(w)}C_w, \quad F([\widetilde{\mathcal{N}}_w^H]) = (-q)^{l(w)}T_{w^{-1}}^{-1} \quad (w \in W).$$

*Proof.* First let us show that $F$ is an isomorphism of $R$-algebras. By an argument similar to the one in Section 13.1 we can prove that

$$[\widetilde{\mathcal{M}}_{w_1}^H] \cdot [\widetilde{\mathcal{M}}_{w_2}^H] = [\widetilde{\mathcal{M}}_{w_1 w_2}^H] \qquad (l(w_1) + l(w_2) = l(w_1 w_2)), \tag{13.2.20}$$

$$[\widetilde{\mathcal{L}}_s^H]^2 = -(q+1)[\widetilde{\mathcal{L}}_s^H] \qquad (s \in S), \tag{13.2.21}$$

where $S = \{s_\alpha \mid \alpha \in \Pi\}$. In particular, $[\widetilde{\mathcal{M}}_e^H]$ is the unit element. Moreover, it follows from the exact sequence

$$0 \longrightarrow \widetilde{\mathcal{M}}_e^H \longrightarrow \widetilde{\mathcal{M}}_s^H \longrightarrow \widetilde{\mathcal{L}}_s^H \longrightarrow 0$$

in $MHM(X \times X, \Delta G)$ that

$$[\widetilde{\mathcal{M}}_s^H] = [\widetilde{\mathcal{L}}_s^H] + [\widetilde{\mathcal{M}}_e^H] \qquad (s \in S). \tag{13.2.22}$$

Hence, if we set $m_w = (-1)^{l(w)}[\widetilde{\mathcal{M}}_w^H]$ $(w \in W)$, then by (13.2.20), (13.2.21) and (13.2.22) we have the relations

$$m_{w_1} m_{w_2} = m_{w_1 w_2} \qquad (l(w_1) + l(w_2) = l(w_1 w_2)), \tag{13.2.23}$$

$$(m_s - 1)(m_s - q) = 0 \qquad (s \in S), \tag{13.2.24}$$

which are exactly the same as the relations defining the products in $H(W)$. This means that $F$ is an isomorphism of $R$-algebras.

For each $w \in W$ define a homomorphism $F_w : K(MHM(X \times X, \Delta G)) \to R$ of $R$-modules by

$$\sum_k (-1)^k [H^k j_w^\bigstar \mathcal{V}] = F_w([\mathcal{V}]) \left[ \mathbb{Q}^H_{Z_w}[\dim Z_w] \right]. \tag{13.2.25}$$

Since $F_w$ satisfies $F_w([\widetilde{\mathcal{M}}^H_y]) = \delta_{y,w}$ we have

$$F(m) = \sum_{w \in W} (-1)^{l(w)} F_w(m) T_w \quad (m \in K(MHM(X \times X, \Delta G))). \tag{13.2.26}$$

Next, consider the endomorphism of the Grothendieck group

$$d : K(MHM(X \times X, \Delta G)) \to K(MHM(X \times X, \Delta G)) \tag{13.2.27}$$

induced by the duality functor $\mathbb{D} : MHM(X \times X, \Delta G) \to MHM(X \times X, \Delta G)$. By the properties of the functor $\mathbb{D}$ we have

$$\begin{gathered} d^2 = 1, \quad d(r \cdot m) = \overline{r} \cdot d(m) \\ (r \in R,\ m \in K(MHM(X \times X, \Delta G))). \end{gathered} \tag{13.2.28}$$

Moreover, we have

$$\begin{aligned} d([\mathcal{V}_1] \cdot [\mathcal{V}_2]) &= (-1)^{\dim X} \sum_j (-1)^j [H^j (p_{13\bigstar} r^! (\mathbb{D}\mathcal{V}_1 \boxtimes \mathbb{D}\mathcal{V}_2))] \\ &= (-1)^{\dim X} q^{\dim X} \sum_j (-1)^j [H^j (p_{13!} r^\bigstar (\mathbb{D}\mathcal{V}_1 \boxtimes \mathbb{D}\mathcal{V}_2))] \end{aligned}$$

by Proposition 13.2.7 (ii) and the fact that $p_{13}$ is a projective morphism. Hence we obtain

$$\begin{gathered} d(m_1 \cdot m_2) = q^{\dim X} d(m_1) \cdot d(m_2) \\ (m_1, m_2 \in K(MHM(X \times X, \Delta G))). \end{gathered} \tag{13.2.29}$$

If we set $\overline{m} = q^{\dim X} d(m)$ for each $m \in K(MHM(X \times X, \Delta G))$, then the map $m \mapsto \overline{m}$ is an involution of the ring $K(MHM(X \times X, \Delta G))$ and we have

$$\overline{r \cdot m} = \overline{r} \cdot \overline{m} \quad (r \in R,\ m \in K(MHM(X \times X, \Delta G))). \tag{13.2.30}$$

Now define an involution $h \mapsto \overline{h}$ of the ring $H(W)$ by $\overline{T}_w = T^{-1}_{w^{-1}}$ $(w \in W)$, $\overline{q} = q^{-1}$ and extend it to $R \otimes_{\mathbb{Z}[q,q^{-1}]} H(W)$ by $\overline{r \otimes h} = \overline{r} \otimes \overline{h}$ $(r \in R,\ h \in H(W))$. Let us prove the relation

$$F(\overline{m}) = \overline{F(m)} \qquad (m \in K(MHM(X \times X, \Delta G))). \tag{13.2.31}$$

Note that for any $w \in W$ we have

$$\mathbb{D}\widetilde{\mathcal{L}}_w^H = \widetilde{\mathcal{L}}_w^H(\dim X + l(w)). \tag{13.2.32}$$

Let us consider the special case of $w = s \in S$. Then it follows from (13.2.24) that

$$F(\overline{m}_s) = q^{-1}T_s + (q^{-1} - 1) = T_s^{-1} = \overline{F(m_s)}. \tag{13.2.33}$$

Since $K(\mathrm{Mod}(X \times X, \Delta G))$ is generated by the elements $\{m_s\}_{s\in S}$ as an $R$-module, we see from this that (13.2.31) holds.

Now we can prove the remaining assertions. The proof of $F([\widetilde{\mathcal{N}}_w^H]) = (-q)^{l(w)}T_{w^{-1}}^{-1}$ is easy. It follows immediately from $\widetilde{\mathcal{N}}_w^H = (\mathbb{D}\widetilde{\mathcal{M}}_w^H)(-\dim X - l(w))$ and (13.2.31). So let us calculate $F([\widetilde{\mathcal{L}}_w^H])$. First set

$$C'_w = (-1)^{l(w)}F([\widetilde{\mathcal{L}}_w^H]) = \sum_{y\leqq w} P'_{y,w}T_y \quad (P'_{y,w} \in R). \tag{13.2.34}$$

Then we have

$$\overline{C'_w} = q^{l(w)}C'_w, \tag{13.2.35}$$

$$P'_{w,w} = 1, \tag{13.2.36}$$

$$P'_{y,w} \in \bigoplus_{i\leqq l(w)-l(y)-1} R_i \quad \text{for} \quad y < w. \tag{13.2.37}$$

Here (13.2.35) follows from (13.2.32). By the definition of $\widetilde{\mathcal{L}}_w^H$ (13.2.36) is also clear. Let us prove (13.2.37). Note that $H^k j_y^\bigstar \widetilde{\mathcal{L}}_w^H$ has mixed weights $\leqq \dim Z_w + k$ because $\widetilde{\mathcal{L}}_w^H$ has a pure weight $\dim Z_w$. Moreover, by a property of intersection cohomology complexes, we have $H^k j_y^\bigstar \widetilde{\mathcal{L}}_w^H = 0$ for any $k \geqq 0$. Therefore, if $H^k j_y^\bigstar \widetilde{\mathcal{L}}_w^H \neq 0$, then it has mixed weights $< \dim Z_w$. On the other hand, by (13.2.26) we have

$$(-1)^{l(w)-l(y)}\sum_k (-1)^k [H^k j_y^\bigstar \widetilde{\mathcal{L}}_w^H] = P'_{y,w}\left[\mathbb{Q}_{Z_y}^H[\dim Z_y]\right]. \tag{13.2.38}$$

Since $\mathbb{Q}_{Z_y}^H[\dim Z_y]$ has the pure weight $\dim Z_y$, we obtain $P'_{y,w} \in \bigoplus_{i<l(w)-l(y)} R_i$. This shows (13.2.37). Using an argument completely similar to the one in [KL1], we can prove that the element $C'_w \in R \otimes H(W)$, satisfying the conditions (13.2.35), (13.2.36), (13.2.37), should be the same as $C_w$ (we omit the details). Hence we get $C'_w = C_w$ and $F([\widetilde{\mathcal{L}}_w^H]) = (-1)^{l(w)}C_w$. □

We note that Proposition 13.2.1 follows from Theorem 13.2.8 by the specialization $q \mapsto 1$.

Finally, let us give a proof of Theorem 12.2.5 making use of the theory of Hodge modules. The argument below is a faithful translation of the one in Kazhdan–Lusztig [KL2] which uses the theory of weights for étale sheaves in positive characteristic instead of Hodge modules.

**Proposition 13.2.9.** *If $y \leqq w$, then $H^k j_y^\bigstar \widetilde{\mathcal{L}}_w^H$ has the pure weight* $\dim Z_w + k$.

In order to prove this we need the following.

**Lemma 13.2.10.** *For $a_1, \dots, a_N \in \mathbb{N}^+$ define an action of $\mathbb{C}^\times$ on $\mathbb{C}^N$ by*

$$z \cdot (x_1, \dots, x_N) = (z^{a_1} x_1, \dots, z^{a_N} x_N).$$

*Let $Z$ be a $\mathbb{C}^\times$-invariant irreducible closed subvariety of $\mathbb{C}^N$ and set $j : \{0\} \hookrightarrow \mathbb{C}^N$. Then $j^\bigstar(\mathrm{IC}_Z^H)$ has the pure weight* $\dim Z$.

*Proof.* By a standard property of intersection cohomology complexes we have

$$H^k j^\bigstar(\mathrm{IC}_Z^H) = 0 \qquad (k \geqq 0). \tag{13.2.39}$$

Since the origin 0 is the unique fixed point of the action of $\mathbb{C}^\times$ on $Z$, we also have that

$$H^k j^\bigstar(\mathrm{IC}_Z^H) = H^k(\mathbb{C}^N, \mathrm{IC}_Z^H). \tag{13.2.40}$$

Therefore, it suffices to show that $H^k(\mathbb{C}^N, \mathrm{IC}_Z^H)$ has the pure weight $\dim Z + k$ for any $k < 0$. Set $Z' = Z \setminus \{0\}$, and denote by $i : \mathbb{C}^N \setminus \{0\} \hookrightarrow \mathbb{C}^N$ the embedding. Then we have a distinguished triangle

$$j_\bigstar j^!(\mathrm{IC}_Z^H) \longrightarrow \mathrm{IC}_Z^H \longrightarrow i_\bigstar(\mathrm{IC}_{Z'}^H) \xrightarrow{+1} .$$

By

$$j^!(\mathrm{IC}_Z^H) = \mathbb{D} j^\bigstar \mathbb{D}(\mathrm{IC}_Z^H) = \mathbb{D} j^\bigstar(\mathrm{IC}_Z^H(\dim Z))$$

we have

$$H^k(j^!(\mathrm{IC}_Z^H)) = (\mathbb{D} H^{-k}(j^\bigstar(\mathrm{IC}_Z^H)))(-\dim Z).$$

Hence we obtain $H^k(j^!(\mathrm{IC}_Z^H)) = 0$ $(k \leqq 0)$ by (13.2.39). Therefore, it follows from the distinguished triangle above that

$$H^k(\mathbb{C}^N, \mathrm{IC}_Z^H) \xrightarrow{\sim} H^k(\mathbb{C}^N \setminus \{0\}, \mathrm{IC}_{Z'}^H) \quad (k < 0).$$

Now it remains to prove that $H^k(\mathbb{C}^N \setminus \{0\}, \mathrm{IC}_{Z'}^H)$ has the pure weight $\dim Z + k$ for $k < 0$.

By the assumption on the action of $\mathbb{C}^\times$ there exists a geometric quotients $P = (\mathbb{C}^N \setminus \{0\})/\mathbb{C}^\times$, $Z'' = Z'/\mathbb{C}^\times$ of $\mathbb{C}^N \setminus \{0\}$, $Z'$, respectively. Then $Z''$ is a closed subvariety of the projective variety $P$. Let $\varphi : \mathbb{C}^N \setminus \{0\} \to P$ be the canonical quotient morphism with respect to the $\mathbb{C}^\times$-action. By $\mathrm{IC}_{Z'}^H = \varphi^\bigstar \mathrm{IC}_{Z''}^H[1]$ we have a distinguished triangle

$$\mathrm{IC}_{Z''}^H[-1](-1) \longrightarrow \mathrm{IC}_{Z''}^H[1] \longrightarrow \varphi_\bigstar \mathrm{IC}_{Z'}^H \xrightarrow{+1} .$$

From this we obtain a cohomology long exact sequence

$$\begin{aligned} \cdots \longrightarrow H^{k-1}(P, \mathrm{IC}_{Z''}^H)(-1) &\xrightarrow{\psi_k} H^{k+1}(P, \mathrm{IC}_{Z''}^H) \\ &\longrightarrow H^k(\mathbb{C}^N \setminus \{0\}, \mathrm{IC}_{Z'}^H) \longrightarrow \cdots . \end{aligned}$$

By the hard Lefschetz theorem for Hodge modules, the homomorphism $\psi_k$ is injective for $k \leqq 0$. Therefore, for $k < 0$, $H^k(\mathbb{C}^N \setminus \{0\}, \mathrm{IC}_{Z'}^H)$ is a quotient of $H^{k+1}(P, \mathrm{IC}_{Z''}^H)$. The variety $Z''$ being projective, $H^{k+1}(P, \mathrm{IC}_{Z''}^H)$ and hence $H^k(\mathbb{C}^N \setminus \{0\}, \mathrm{IC}_{Z'}^H)$ must have the pure weight $\dim Z'' + k + 1 = \dim Z + k$. □

*Proof of Proposition* 13.2.9. Taking a suitable open subset $U$ of $X \times X$ we have a commutative diagram

$$\begin{array}{ccccc} Z_y \cap U & \longrightarrow & \overline{Z}_w \cap U & \longrightarrow & U \\ \wr\downarrow & & \wr\downarrow & & \wr\downarrow \\ \{0\} \times \mathbb{C}^M & \longrightarrow & Z \times \mathbb{C}^M & \longrightarrow & \mathbb{C}^N \times \mathbb{C}^M, \end{array}$$

where the horizontal arrows are embeddings. If we restrict the $G$-action on $X \times X$ to a one-dimensional torus in $G$, we can show that $\{0\} \subset Z \subset \mathbb{C}^N$ satisfies the assumption of Lemma 13.2.10. Hence the assertion follows from Lemma 13.2.10. □

**Theorem 13.2.11.** *Suppose that $y \leqq w$. We define integers $c_{y,w,j} \in \mathbb{Z}$ by $P_{y,w}(q) = \sum_j c_{y,w,j} q^j \in \mathbb{Z}[q]$. Then*

(i) *For $k + l(w) - l(y) \notin 2\mathbb{N}$ we have $H^k j_y^{\bigstar}(\widetilde{\mathcal{L}}_w^H) = 0$.*
(ii) *The integers $c_{y,w,j} \in \mathbb{Z}$ are non-negative. Moreover, for $k + l(w) - l(y) = 2j$ $(j \geqq 0)$ we have an isomorphism*

$$H^k j_y^{\bigstar}(\widetilde{\mathcal{L}}_w^H) = \left(\mathbb{Q}_{Z_y}^H[\dim Z_y]\right)^{\oplus c_{y,w,j}}.$$

*Proof.* By Proposition 13.2.9 we have an expression

$$H^k j_y^{\bigstar}(\widetilde{\mathcal{L}}_w^H) = N_k \otimes \mathbb{Q}_{Z_y}^H[\dim Z_y] \quad (N_k \in SH(k + l(w) - l(y))^p).$$

Therefore, by (13.2.38)

$$\sum_k (-1)^{l(w)+l(y)-k}[N_k] = \sum_j c_{y,w,j} q^j.$$

Since we have $[N_k] \in R_{k+l(w)-l(y)}$ and $R = \bigoplus_n R_n$, $q \in R_2$, we obtain

$$\begin{aligned} k + l(w) - l(y) = 2j + 1 \quad &\Longrightarrow \quad [N_k] = 0, \\ k + l(w) - l(y) = 2j \quad &\Longrightarrow \quad [N_k] = c_{y,w,j} q^j. \end{aligned}$$

Hence our assertion follows from the fact that an object in $SH(k)^p$ is a direct sum of finitely many irreducible objects. □

Now Theorem 12.2.5 is an immediate consequence of Theorem 13.2.11.

# A

# Algebraic Varieties

## A.1 Basic definitions

Let $k[X_1, X_2, \ldots, X_n]$ be a polynomial algebra over an algebraically closed field $k$ with $n$ indeterminates $X_1, \ldots, X_n$. We sometimes abbreviate it as $k[X] = k[X_1, X_2, \ldots, X_n]$. Let us associate to each polynomial $f(X) \in k[X]$ its zero set

$$V(f) := \{x = (x_1, x_2, \ldots, x_n) \in k^n \mid f(x) = f(x_1, x_2, \ldots, x_n) = 0\}$$

in the $n$-fold product set $k^n$ of $k$. For any subset $S \subset k[X]$ we also set $V(S) = \bigcap_{f \in S} V(f)$. Then we have the following properties:

(i) $V(1) = \emptyset$, $V(0) = k^n$.
(ii) $\bigcap_{i \in I} V(S_i) = V(\bigcup_{i \in I} S_i)$.
(iii) $V(S_1) \cup V(S_2) = V(S_1 S_2)$, where $S_1 S_2 := \{fg \mid f \in S_1,\ g \in S_2\}$.

The inclusion $\subset$ of (iii) is clear. We will prove only the inclusion $\supset$. For $x \in V(S_1 S_2) \setminus V(S_2)$ there is an element $g \in S_2$ such that $g(x) \neq 0$. On the other hand, it follows from $x \in V(S_1 S_2)$ that $f(x)g(x) = 0\ ({}^\forall f \in S_1)$. Hence $f(x) = 0\ ({}^\forall f \in S_1)$ and $x \in V(S_1)$. So the part $\supset$ was also proved.

By (i), (ii), (iii) the set $k^n$ is endowed with the structure of a topological space by taking $\{V(S) \mid S \subset k[X]\}$ to be its closed subsets. We call this topology of $k^n$ the *Zariski topology*. The closed subsets $V(S)$ of $k^n$ with respect to it are called *algebraic sets* in $k^n$. Note that $V(S) = V(\langle S \rangle)$, where $\langle S \rangle$ denotes the ideal of $k[X]$ generated by $S$. Hence we may assume from the beginning that $S$ is an ideal of $k[X]$. Conversely, for a subset $W \subset k^n$ the set

$$I(W) := \left\{ f \in k[X] \mid f(x) = 0\ ({}^\forall x \in W) \right\}$$

is an ideal of $k[X]$. When $W$ is a (Zariski) closed subset of $k^n$, we have clearly $V(I(W)) = W$. Namely, in the diagram

$$\boxed{\text{ideals in } k[X]} \underset{I}{\overset{V}{\rightleftarrows}} \boxed{\text{closed subsets in } k^n}$$

we have $V \circ I = \mathrm{Id}$. However, for an ideal $J \subset k[X]$ the equality $I(V(J)) = J$ does not hold in general. We have only $I(V(J)) \supset J$. The difference will be clarified later by Hilbert's Nullstellensatz.

Let $V$ be a Zariski closed subset of $k^n$ (i.e., an algebraic set in $k^n$). We regard it as a topological space by the relative topology induced from the Zariski topology of $k^n$. We denote by $k[V]$ the $k$-algebra of $k$-valued functions on $V$ obtained by restricting polynomial functions to $V$. It is called the *coordinate ring* of $V$. The restriction map $\rho_V : k[X] \to k[V]$ given by $\rho_V(f) := f|_V$ is a surjective homomorphism of $k$-algebras with $\mathrm{Ker}\, \rho_V = I(V)$, and hence we have $k[V] \simeq k[X]/I(V)$. For each point $x = (x_1, x_2, \ldots, x_n) \in k^n$ of $V$ define a homomorphism $e_x : k[V] \to k$ of $k$-algebras by $e_x(f) = f(x)$. Then we get a map

$$e : V \longrightarrow \mathrm{Hom}_{k\text{-alg}}(k[V],\, k) \quad (x \longmapsto e_x),$$

where $\mathrm{Hom}_{k\text{-alg}}(k[V],\, k)$ denotes the set of the $k$-algebra homomorphisms from $k[V]$ to $k$. Conversely, for a $k$-algebra homomorphism $\phi : k[V] \to k$ define $x = (x_1, x_2, \ldots, x_n) \in k^n$ by $x_i = \phi\rho_V(X_i)$ $(1 \leq i \leq n)$. Then we have $x \in V$ and $e_x = \phi$. Hence we have an identification $V = \mathrm{Hom}_{k\text{-alg}}(k[V],\, k)$ as a set. Moreover, the closed subsets of $V$ are of the form $V(\rho_V^{-1}(J)) = \{\, x \in V \mid e_x(J) = 0 \,\} \subset V$ for ideals $J$ of $k[V]$. Therefore, the topological space $V$ is recovered from the $k$-algebra $k[V]$. It indicates the possibility of defining the notion of algebraic sets starting from certain $k$-algebras without using the embedding into $k^n$. Note that the coordinate ring $A = k[V]$ is finitely generated over $k$, and *reduced* (i.e., does not contain non-zero nilpotent elements) because $k[V]$ is a subring of the ring of functions on $V$ with values in the field $k$.

In this chapter we give an account of the classical theory of "algebraic varieties" based on reduced finitely generated (commutative) algebras over algebraically closed fields (in the modern language of schemes one allows general commutative rings as "coordinate algebras").

The following two theorems are fundamental.

**Theorem A.1.1 (A weak form of Hilbert' Nullstellensatz).** *Any maximal ideal of the polynomial ring $k[X_1, X_2, \ldots, X_n]$ is generated by the elements $X_i - x_i$ $(1 \leq i \leq n)$ for a point $x = (x_1, x_2, \ldots, x_n) \in k^n$.*

**Theorem A.1.2 (Hilbert's Nullstellensatz).** *We have $I(V(J)) = \sqrt{J}$, where $\sqrt{J}$ is the* **radical** *$\{\, f \in k[X] \mid f^N \in J$ for some $N \gg 0\}$ of $J$.*

For the proofs, see, for example, [Mu]. □

For a finitely generated $k$-algebra $A$ denote by $\mathrm{Specm}\, A$ the set of the maximal ideals of $A$. For an algebraic set $V \subset k^n$ we have a bijection

$$\mathrm{Hom}_{k\text{-alg}}(k[V],\, k) \simeq \mathrm{Specm}\, k[V] \quad (e \longmapsto \mathrm{Ker}\, e)$$

by Theorem A.1.1. Under this correspondence, the closed subsets in $\mathrm{Specm}\, k[V] \simeq V$ are the sets

$$V(I) = \{\, \mathfrak{m} \in \mathrm{Specm}\, k[V] \mid \mathfrak{m} \supset I \,\},$$

where $I$ ranges through the ideals of $k[V]$. Theorem A.1.2 implies that there is a one-to-one correspondence between the closed subsets in Specm $k[V]$ and the *radical ideals* $I$ ($I = \sqrt{I}$) of $k[V]$.

## A.2 Affine varieties

Motivated by the arguments in the previous section, we start from a finitely generated *reduced* commutative $k$-algebra $A$ to define an algebraic variety. Namely, we set $V = \text{Specm } A$ and define its topology so that the closed subsets are given by

$$\big\{V(I) = \{\mathfrak{m} \in \text{Specm } A \mid I \subset \mathfrak{m}\} \mid I\text{: ideals of } A\big\}.$$

By Hilbert's Nullstellensatz (its weak form), we get the identification

$$V \simeq \text{Hom}_{k\text{-alg}}(A,\ k).$$

We sometimes write a point $x \in V$ as $\mathfrak{m}_x \in \text{Specm } A$ or $e_x \in \text{Hom}_{k\text{-alg}}(A,\ k)$. Under this notation we have

$$f(x) = e_x(f) = (f \mod \mathfrak{m}_x) \in k$$

for $f \in A$. Here, we used the identification $k \simeq A/\mathfrak{m}_x$ obtained by the composite of the morphisms $k \hookrightarrow A \to A/\mathfrak{m}_x$. Hence the ring $A$ is regarded as a $k$-algebra consisting of certain $k$-valued functions on $V$.

Recall that any open subset of $V$ is of the form $D(I) = V \setminus V(I)$, where $I$ is an ideal of $A$. Since $A$ is a noetherian ring (finitely generated over $k$), the ideal $I$ is generated by a finite subset $\{f_1, f_2, \ldots, f_r\}$ of $I$. Then we have

$$D(I) = V \setminus \Big(\bigcap_{i=1}^{r} V(f_i)\Big) = \bigcup_{i=1}^{r} D(f_i),$$

where $D(f) = \{x \in V \mid f(x) \neq 0\} = V \setminus V(f)$ for $f \in A$. We call an open subset of the form $D(f)$ for $f \in A$ a *principal open subset* of $V$. Principal open subsets form a basis of the open subsets of $V$. Note that we have the equivalence

$$D(f) \subset D(g) \Longleftrightarrow \sqrt{(f)} \subset \sqrt{(g)} \Longleftrightarrow f \in \sqrt{(g)}$$

by Hilbert's Nullstellensatz.

Assume that we are given an $A$-module $M$. We introduce a sheaf $\widetilde{M}$ on the topological space $V = \text{Specm } A$ as follows. For a multiplicatively closed subset $S$ of $A$ we denote by $S^{-1}M$ the localization of $M$ with respect to $S$. It consists of the equivalence classes with respect to the equivalence relation $\sim$ on the set of pairs $(s,\ m) = m/s$ ($s \in S$, $m \in M$) given by $m/s \sim m'/s' \Longleftrightarrow t(s'm - sm') = 0$ ($\exists t \in S$). By the ordinary operation rule of fractional numbers $S^{-1}A$ is endowed with a ring structure and $S^{-1}M$ turns out to be an $S^{-1}A$-module. For $f \in A$ we set

$M_f = S_f^{-1}M$, where $S_f := \{1, f, f^2, \dots\}$. Note that for two principal open subsets $D(f) \subset D(g)$ a natural homomorphism $r_f^g : M_g \to M_f$ is defined as follows. We have $f^n = hg$ ($h \in A$, $n \in \mathbb{N}$), and then the element $m/g^l \in M_g$ is mapped to $m/g^l = h^l m/h^l g^l = h^l m/f^{nl} \in M_f$. In the case of $D(f) = D(g)$ we easily see $M_f \simeq M_g$ by considering the inverse.

**Theorem A.2.1.**

(i) *For an $A$-module $M$ there exists a unique sheaf $\widetilde{M}$ on $V = \operatorname{Specm} A$ such that for any principal open subset $D(f)$ we have $\widetilde{M}(D(f)) = M_f$, and the restriction homomorphism $\widetilde{M}(D(f)) \to \widetilde{M}(D(g))$ for $D(f) \subset D(g)$ is given by $r_f^g$.*

(ii) *The sheaf $\mathcal{O}_V := \widetilde{A}$ is naturally a sheaf of $k$-algebras on $V$.*

(iii) *For an $A$-module $M$ the sheaf $\widetilde{M}$ is naturally a sheaf of $\mathcal{O}_V$-module. The stalk of $\widetilde{M}$ at $x \in V$ is given by*

$$M_{\mathfrak{m}_x} = (A \setminus \mathfrak{m}_x)^{-1}M = \varinjlim_{f(x)\neq 0} M_f.$$

*This is a module over the local ring $\mathcal{O}_{V,x} := A_{\mathfrak{m}_x} = (A \setminus \mathfrak{m}_x)^{-1}A$.*

The key point of the proof is the fact that the functor $D(f) \mapsto M_f$ on the category of principal open subsets $\{D(f) \mid f \in A\}$ satisfies the "axioms of sheaves (for a basis of open subsets)," which is assured by the next lemma.

**Lemma A.2.2.** *Assume that the condition $\langle f_1, f_2, \dots, f_r\rangle \ni 1$ is satisfied in the ring $A$. Then for an $A$-module $M$ we have an exact sequence*

$$0 \longrightarrow M \xrightarrow{\alpha} \prod_{i=1}^{r} M_{f_i} \xrightarrow{\beta} \prod_{i,j} M_{f_i f_j},$$

*where the last arrow maps $(s_i)_{i=1}^r$ to $s_i - s_j \in M_{f_i f_j}$ $(1 \le i, j \le r)$.*

*Proof.* We first show that for any $N \in \mathbb{N}$ there exists $g_1, \dots, g_r$ satisfying $\sum_i g_i f_i^N = 1$. Note that our assumption $\langle f_1, f_2, \dots, f_r\rangle \ni 1$ is equivalent to saying that for any $\mathfrak{m} \in \operatorname{Specm} A$ there exists at least one element $f_i$ such that $f_i \notin \mathfrak{m}$ ($\operatorname{Specm} A = \bigcup_{i=1}^r D(f_i)$). If $f_i \notin \mathfrak{m}$, then we have $f_i^N \notin \mathfrak{m}$ for any $N$ since $\mathfrak{m}$ is a prime ideal. Therefore, we get $\langle f_1^N, f_2^N, \dots, f_r^N\rangle \ni 1$, and the assertion is proved.

Let us show the injectivity of $\alpha$. Assume $m \in \operatorname{Ker}\alpha$. Then there exists $N \gg 0$ such that $f_i^N m = 0$ $(1 \le i \le r)$. Combining it with the equality $\sum_i g_i f_i^N = 1$ we get $m = \sum_i g_i f_i^N m = 0$.

Next assume that $(m_l) \in \prod_l M_{f_l} \in \operatorname{Ker}\beta$. We will show that there is an element $m \in M$ such that $\alpha(m) = (m_l)$. It follows from our assumption that $m_i = m_j$ in $M_{f_i f_j}$ $(1 \le i, j \le r)$. This is equivalent to saying that $(f_i f_j)^N (m_i - m_j) = 0$ $(1 \le i, j \le r)$ for a large number $N$. That is, $f_j^N f_i^N m_i = f_i^N f_j^N m_j$. Now set $m = \sum_{i=1}^r g_i (f_i^N m_i)$ $(\sum g_i f_i^N = 1)$. Then for any $1 \le l \le r$ we have $f_l^N m = f_l^N \sum_i g_i (f_i^N m_i) = \sum_i g_i f_i^N (f_l^N m_i) = \sum_i g_i f_i^N f_l^N m_l = f_l^N m_l$ and $f_l^N (m - m_l) = 0$. Hence we have $\alpha(m) = (m_l)$. □

For a general open subset $U = \bigcup_{i=1}^{r} D(f_i)$ of $U$ we have

$$\Gamma(U, \widetilde{M}) = \{(s_i) \in M_{f_i} \mid s_i = s_j \text{ in } M_{f_i f_j} \ (1 \leq i, j \leq r)\}.$$

In particular, for $\mathcal{O}_V = \widetilde{A}$ we have

$$\Gamma(U, \mathcal{O}_V) = \{f : U \to k \mid \text{for each point of } U, \text{ there is an open neighborhood } D(g) \text{ such that } f|_{D(g)} \in A_g.\}$$

Let $(X, \mathcal{O}_X)$ be a pair of a topological space $X$ and a sheaf $\mathcal{O}_X$ of $k$-algebras on $X$ consisting of certain $k$-valued functions. We say that the pair $(X, \mathcal{O}_X)$ (or simply $X$) is an *affine variety* if $(X, \mathcal{O}_X)$ is isomorphic to some $(V, \mathcal{O}_V)$ ($V = \operatorname{Specm} A$, $\mathcal{O}_V = \widetilde{A}$) in the sense that there exists a homeomorphism $\phi : X \xrightarrow{\sim} V$ such that the correspondence $f \mapsto f \circ \phi$ gives an isomorphism $\Gamma(\phi(U), \mathcal{O}_V) \xrightarrow{\sim} \Gamma(U, \mathcal{O}_X)$ for any open subset $U$ of $V$. In this case we have a natural isomorphism $\phi^\sharp : \phi^{-1}\mathcal{O}_V \xrightarrow{\sim} \mathcal{O}_X$ of a sheaf of $k$-algebras. In particular, we have an isomorphism $\phi_x^\sharp : \mathcal{O}_{V,\phi(x)} \xrightarrow{\sim} \mathcal{O}_{X,x}$ of local rings for any $x \in X$.

## A.3 Algebraic varieties

Let $(X, \mathcal{O}_X)$ be a pair of a topological space $X$ and a sheaf $\mathcal{O}_X$ of $k$-algebras consisting of certain $k$-valued functions. We say that the pair $(X, \mathcal{O}_X)$ (or simply $X$) is called a *prevariety* over $k$ if it is locally an affine variety (i.e., if for any point $x \in X$ there is an open neighborhood $U \ni x$ such that $(U, \mathcal{O}_X|_U)$ is isomorphic to an affine variety). In such cases, we call the sheaf $\mathcal{O}_X$ the *structure sheaf* of $X$ and sections of $\mathcal{O}_X$ are called *regular functions*. A morphism $\phi : X \to Y$ between prevarieties $X$, $Y$ is a continuous map so that for any open subset $U$ of $Y$ and any $f \in \Gamma(U, \mathcal{O}_Y)$ we have $f \circ \phi \in \Gamma(f^{-1}U, \mathcal{O}_X)$.

A prevariety $X$ is called an *algebraic variety* if it is *quasi-compact* and *separated*. Let us explain more precisely these two conditions.

The quasi-compactness is a purely topological condition. We say that a topological space $X$ is quasi-compact if any open covering of $X$ admits a finite subcovering. We say "quasi" because $X$ is not assumed to be Hausdorff. In fact, an algebraic variety is not Hausdorff unless it consists of finitely many points. In particular, an algebraic variety $X$ is covered by finitely many affine varieties.

The condition of separatedness plays an alternative role to that of the Hausdorff condition. To define it we need the notion of products of prevarieties.

We define the product $V_1 \times V_2$ of two affine varieties $V_1$, $V_2$ to be the affine variety associated to the tensor product $A_1 \otimes_k A_2$ ($A_i = \Gamma(V_i, \mathcal{O}_{V_i})$) of $k$-algebras. This operation is possible thanks to the following proposition.

**Proposition A.3.1.** *For any pair $A_1$, $A_2$ of finitely generated reduced $k$-algebras the tensor product $A_1 \otimes_k A_2$ is also reduced.*

*Proof.* Consider the embeddings $V_i \subset \mathbb{A}^{n_i}$ $(i = 1, 2)$ and $V_1 \times V_2 \subset \mathbb{A}^{n_1+n_2}$. Next, define the ring $k[V_1 \times V_2]$ to be the restriction of the polynomial ring $k[\mathbb{A}^{n_1+n_2}]$ to $V_1 \times V_2$. Then the restriction map $\varphi : k[V_1] \otimes_k k[V_2] \to k[V_1 \times V_2]$ is bijective (the right-hand side is obviously reduced). The surjectivity is clear. We can also prove the injectivity, observing that for any linearly independent elements $\{f_i\}$ (resp. $\{g_j\}$) in $k[V_1]$ (resp. $k[V_2]$) over $k$ the elements $\{\varphi(f_i g_j)\}$ are again linearly independent in $k[V_1 \times V_2]$. □

By definition the product $V_1 \times V_2$ of affine varieties $V_1$, $V_2$ has a finer topology than the usual product topology.

Now let us give the definition of the product of two prevarieties $X$, $Y$. Let $X = \bigcup_i V_i$, $Y = \bigcup_j U_j$ be affine open coverings of $X$ and $Y$, respectively. Then the product set $X \times Y$ is covered by $\{ V_i \times U_j \}_{(i,j)}$ $(X \times Y = \bigcup_{(i,j)} V_i \times U_j)$. Note that we regard the product sets $V_i \times U_j$ as affine varieties by the above arguments. Namely, the structure sheaf $\mathcal{O}_{V_i \times U_j}$ is associated to the tensor product $\Gamma(V_i, \mathcal{O}_{V_i}) \otimes_k \Gamma(U_j, \mathcal{O}_{U_j})$. Then we can glue $(V_i \times U_j, \mathcal{O}_{V_i \times U_j})$ to get a topology of $X \times Y$ and a sheaf $\mathcal{O}_{X \times Y}$ of $k$-algebras consisting of certain $k$-valued functions on $X \times Y$, for which $(X \times Y, \mathcal{O}_{X \times Y})$ is a prevariety. This prevariety is called the product of two prevarieties $X$ and $Y$. It is the "fiber product" in the category of prevarieties.

Using these definitions, we say that a prevariety $X$ is *separated* if the diagonal set $\Delta = \{(x, x) \in X \times X\}$ is closed in the self-product $X \times X$.

Let us add some remarks.

(i) If $\phi : X \to Y$ is a morphism of algebraic varieties, then its graph $\Gamma_\phi = \{ (x, \phi(x)) \in X \times Y \}$ is a closed subset of $X \times Y$.
(ii) Affine varieties are separated (hence they are algebraic varieties).

## A.4 Quasi-coherent sheaves

Let $(X, \mathcal{O}_X)$ be an algebraic variety. We say that a sheaf $F$ of $\mathcal{O}_X$-module (hereafter, we simply call $F$ an $\mathcal{O}_X$-module) is *quasi-coherent* over $\mathcal{O}_X$ if for each point $x \in X$ there exists an affine open neighborhood $V \ni x$ and a module $M_V$ over $A_V = \mathcal{O}_X(V)$ such that $F|_V \simeq \widetilde{M}_V$ as $\mathcal{O}_V$-modules ($\widetilde{M}_V$ is an $\mathcal{O}_V$-module on $V = \operatorname{Specm} A_V$ constructed from the $A_V$-module $M_V$ by Theorem A.2.1). If, moreover, every $M_V$ is finitely generated over $A_V$, we say that $F$ is *coherent* over $\mathcal{O}_X$. The next theorem is fundamental.

**Theorem A.4.1.**

(i) (Chevalley) *The following conditions on an algebraic variety $X$ are equivalent:*
  (a) *$X$ is an affine variety.*
  (b) *For any quasi-coherent $\mathcal{O}_X$-module $F$ we have $H^i(X, F) = 0$ $(i \geq 1)$.*
  (c) *For any quasi-coherent $\mathcal{O}_X$-module $F$ we have $H^1(X, F) = 0$.*

(ii) *Let $X$ be an affine variety and $A = \mathcal{O}_X(X)$ its coordinate ring. Then the functor*

$$\operatorname{Mod}(A) \ni M \longmapsto \widetilde{M} \in \operatorname{Mod}_{qc}(\mathcal{O}_X)$$

*from the category* $\mathrm{Mod}(A)$ *of* $A$*-modules to the category* $\mathrm{Mod}_{qc}(\mathcal{O}_X)$ *of quasi-coherent* $\mathcal{O}_X$*-modules induces an equivalence of categories. Namely, any quasi-coherent* $\mathcal{O}_X$*-module is isomorphic to the sheaf* $\widetilde{M}$ *constructed from an* $A$*-module* $M$*, and there exists an isomorphism*

$$\mathrm{Hom}_A(M,\ N) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_X}(\widetilde{M},\ \widetilde{N}).$$

*In particular, for a quasi-coherent* $\mathcal{O}_X$*-module* $F$ *we have*

$$F \simeq \widetilde{F(X)}.$$

By Theorem A.4.1 (ii) local properties of a quasi-coherent $\mathcal{O}_X$-module $F$ can be deduced from those of the $\mathcal{O}_X(V)$-module $F(V)$ for an affine neighborhood $V$. For example, $F$ is locally free (resp. coherent) if and only if every point $x \in X$ has an affine open neighborhood $V$ such that $F(V)$ is free (resp. finitely generated) over $\mathcal{O}_X(V)$.

Let us give some examples.

**Example A.4.2.** *Tangent sheaf* $\Theta_X$ and *cotangent sheaf* $\Omega^1_X$ (In this book, $\Omega_X$ stands for the sheaf $\Omega^n_X := \bigwedge^n \Omega^1_X$ ($n = \dim X$) of differential forms of top degree). We denote by $\mathcal{E}nd_k\mathcal{O}_X$ the sheaf of $k$-linear endomorphisms of $\mathcal{O}_X$. We say that a section $\theta \in (\mathcal{E}nd_k\mathcal{O}_X)(X)$ is a vector field on $X$ if for each open subset $U \subset X$ the restriction $\theta(U) := \theta|_U \in (\mathcal{E}nd_k\mathcal{O}_X)(U)$ satisfies the condition

$$\theta(U)(fg) = \theta(U)(f)g + f\theta(U)(g) \quad (f,\ g \in \mathcal{O}_X(U)).$$

For an open subset $U$ of $X$, denote the set of the vector fields $\theta$ ($\in (\mathcal{E}nd_k\mathcal{O}_U)(U)$) on $U$ by $\Theta(U)$. Then $\Theta(U)$ is an $\mathcal{O}_X(U)$-module, and the presheaf $U \mapsto \Theta(U)$ turns out to be a sheaf (of $\mathcal{O}_X$-modules). We denote this sheaf by $\Theta_X$ and call it the *tangent sheaf* of $X$. When $U$ is affine, we have $\Theta_U \simeq \widetilde{\mathrm{Der}_k(A)}$ for $A = \mathcal{O}_X(U)$, where the right-hand side is the $\mathcal{O}_U$-module associated to the $A$-module

$$\mathrm{Der}_k(A) := \{\theta \in \mathrm{End}_k A \mid \theta(fg) = \theta(f)g + f\theta(g)\ (f,\ g \in A)\}$$

of the derivations of $A$ over $k$. It follows from this fact that $\Theta_X$ is a coherent $\mathcal{O}_X$-module. Indeed, if $A = k[X]/I$ (here $k[X] = k[X_1, X_2, \ldots, X_n]$ is a polynomial ring), then we have

$$\mathrm{Der}_k(k[X]) = \bigoplus_{i=1}^{n} k[X]\partial_i \quad \left(\partial_i := \frac{\partial}{\partial X_i}\right)$$

(free $k[X]$-module of rank $n$) and

$$\mathrm{Der}_k(A) \simeq \{\theta \in \mathrm{Der}_k(k[X]) \mid \theta(I) \subset I\}.$$

Hence $\mathrm{Der}_k(A)$ is finitely generated over $A$.

On the other hand we define the *cotangent sheaf* of $X$ by $\Omega^1_X := \delta^{-1}(\mathcal{J}/\mathcal{J}^2)$, where $\delta : X \to X \times X$ is the diagonal embedding, $\mathcal{J}$ is the ideal sheaf of $\delta(X)$ in $X \times X$ defined by

$$\mathcal{J}(V) = \{f \in \mathcal{O}_{X\times X}(V) \mid f(V \cap \delta(X)) = \{0\}\}$$

for any open subset $V$ of $X \times X$, and $\delta^{-1}$ stands for the sheaf-theoretical inverse image functor. Sections of the sheaf $\Omega^1_X$ are called differential forms. By the canonical morphism $\mathcal{O}_X \to \delta^{-1}\mathcal{O}_{X\times X}$ of sheaf of $k$-algebras $\Omega^1_X$ is naturally an $\mathcal{O}_X$-module. We have a morphism $d : \mathcal{O}_X \to \Omega^1_X$ of $\mathcal{O}_X$-modules defined by $df = f \otimes 1 - 1 \otimes f \mod \delta^{-1}\mathcal{J}^2$. It satisfies $d(fg) = d(f)g + f(dg)$ for any $f, g \in \mathcal{O}_X$. For $\alpha \in \mathcal{H}om_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)$ we have $\alpha \circ d \in \Theta_X$, which gives an isomorphism $\mathcal{H}om_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X) \simeq \Theta_X$ of $\mathcal{O}_X$-modules.

## A.5 Smoothness, dimensions and local coordinate systems

Let $x$ be a point of an algebraic variety $X$. We say that $X$ is *smooth* (or non-singular) at $x \in X$ if the stalk $\mathcal{O}_{X,x}$ is a regular local ring. This condition is satisfied if and only if the cotangent sheaf $\Omega^1_X$ is a free $\mathcal{O}_X$-module on an open neighborhood of $x$. The smooth points of $X$ form an open subset of $X$. Let us denote this open subset by $X_{\mathrm{reg}}$. An algebraic variety is called *smooth* (or non-singular) if all of its points are smooth. It is equivalent to saying that $\Omega^1_X$ is a locally free $\mathcal{O}_X$-module. In this case $\Theta_X$ is also locally free of the same rank by $\mathcal{H}om_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X) \simeq \Theta_X$. For a smooth point $x \in X$ the *dimension* of $X$ at $x$ is defined by

$$\dim_x X := \mathrm{rank}_{\mathcal{O}_{X,x}} \Theta_{X,x} = \mathrm{rank}_{\mathcal{O}_{X,x}} \Omega^1_{X,x},$$

where $\Theta_{X,x}$ and $\Omega^1_{X,x}$ are the stalks of $\Theta_X$ and $\Omega^1_X$ at $x$, respectively. It also coincides with the Krull dimension of the regular local ring $\mathcal{O}_{X,x}$. We define the *dimension* of $X$ to be the locally constant function on $X_{\mathrm{reg}}$ defined by

$$(\dim X)(x) := \dim_x X.$$

If $X$ is irreducible, the value $\dim_x X$ does not depend on the point $x \in X_{\mathrm{reg}}$.

**Theorem A.5.1.** *Let $X$ be a smooth algebraic variety of dimension $n$. Then for each point $p \in X$, there exist an affine open neighborhood $V$ of $p$, regular functions $x_i \in k[V] = \mathcal{O}_X(V)$, and vector fields $\partial_i \in \Theta_X(V)$ $(1 \le i \le n)$ satisfying the conditions*

$$\begin{cases} [\partial_i, \partial_j] = 0, \quad \partial_i(x_j) = \delta_{ij} \ (1 \le i, j \le n) \\ \Theta_V = \bigoplus_{i=1}^n \mathcal{O}_V \partial_i. \end{cases}$$

*Moreover, we can choose the functions $x_1, x_2, \ldots, x_n$ so that they generate the maximal ideal $\mathfrak{m}_p$ of the local ring $\mathcal{O}_{X,p}$ at $p$.*

*Proof.* By the theory of regular local rings there exist $n(= \dim_x X)$ functions $x_1, \ldots, x_n \in \mathfrak{m}_p$ generating the ideal $\mathfrak{m}_p$. Then $dx_1, \ldots, dx_n$ is a basis of the free $\mathcal{O}_{X,p}$-module $\Omega^1_{X,p}$. Hence we can take an affine open neighborhood $V$ of $p$ such that $\Omega^1_X(V)$ is a free module with basis $dx_1, \ldots, dx_n$ over $\mathcal{O}_X(V)$. Taking the dual basis $\partial_1, \ldots, \partial_n \in \Theta_X(V) \simeq \mathrm{Hom}_{\mathcal{O}_X(V)}(\Omega^1_X(V), \mathcal{O}_X(V))$ we get $\partial_i(x_j) = \delta_{ij}$. Write $[\partial_i, \partial_j]$ as $[\partial_i, \partial_j] = \sum_{l=1}^n g_{ij}^l \partial_l$ $(g_{ij}^l \in \mathcal{O}_X(V))$. Then we have $g_{ij}^l = [\partial_i, \partial_j] x_l = \partial_i \partial_j x_l - \partial_j \partial_i x_l = 0$. Hence $[\partial_i, \partial_j] = 0$. □

**Definition A.5.2.** The set $\{ x_i, \partial_i \mid 1 \leq i \leq n \}$ defined over an affine open neighborhood of $p$ satisfying the conditions of Theorem A.5.1 is called a *local coordinate system* at $p$.

It is clear that this notion is a counterpart of the local coordinate system of a complex manifold. Note that the local coordinate system $\{x_i\}$ defined on an affine open subset $V$ of a smooth algebraic variety does not necessarily separate the points in $V$. We only have an *étale morphism* $V \to k^n$ given by $q \mapsto (x_1(q), \ldots, x_n(q))$.

We have the following relative version of Theorem A.5.1.

**Theorem A.5.3.** *Let $Y$ be a smooth subvariety of a smooth algebraic variety $X$. Assume that* $\dim_p Y = m$, $\dim_p X = n$ *at $p \in Y$. Then we can take an affine open neighborhood $V$ of $p$ in $X$ and a local coordinate system $\{x_i, \partial_i \mid 1 \leq i \leq n\}$ such that $Y \cap V = \{q \in V \mid x_i(q) = 0 \ (m < i \leq n)\}$ (hence $k[Y \cap V] = k[V]/\sum_{i>m} k[V] x_i$) and $\{x_i, \partial_i \mid 1 \leq i \leq m\}$ is a local coordinate system of $Y \cap V$. Here we regard $\partial_i$ $(1 \leq i \leq m)$ as derivations on $k[Y \cap V]$ by using the relation $\partial_i x_j = 0$ $(j > m)$.*

*Proof.* The result follows from the fact that smooth $\Rightarrow$ locally complete intersection. □

# B

# Derived Categories and Derived Functors

In this appendix, we give a brief account of the theory of derived categories without proofs. The basic references are Hartshorne [Ha1], Verdier [V2], Borel et al. [Bor3, Chapter 1], Gelfand–Manin [GeM], Kashiwara–Schapira [KS2], [KS4]. We especially recommend the reader to consult Kashiwara–Schapira [KS4] for details on this subject.

## B.1 Motivation

The notion of derived categories is indispensable if one wants to fully understand the theory of $D$-modules. Many operations of $D$-modules make sense only in derived categories, and the Riemann–Hilbert correspondence, which is the main subject of Part I, cannot be formulated without this notion. Derived categories were introduced by A. Grothendieck [Ha1], [V2]. We hear that M. Sato arrived at the same notion independently in his way of creating algebraic analysis. In this section we explain the motivation of the theory of derived categories and give an outline of the theory.

Let us first recall the classical definition of right derived functors. Let $\mathcal{C}$, $\mathcal{C}'$ be abelian categories and $F : \mathcal{C} \to \mathcal{C}'$ a left exact functor. Assume that the category $\mathcal{C}$ has *enough injectives*, i.e., for any object $X \in \mathrm{Ob}(\mathcal{C})$ there exists a monomorphism $X \to I$ into an injective object $I$. Then for any $X \in \mathrm{Ob}(\mathcal{C})$ there exists an exact sequence

$$0 \longrightarrow X \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots$$

such that $I^k$ is an injective object for any $k \in \mathbb{Z}$. Such an exact sequence is called an *injective resolution* of $X$. Next consider the complex

$$I^\cdot = \left[0 \longrightarrow I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow I^3 \cdots\right]$$

in $\mathcal{C}$ and apply to it the functor $F$. Then we obtain a complex

$$F(I^\cdot) = \left[0 \longrightarrow F(I^0) \longrightarrow F(I^1) \longrightarrow F(I^2) \longrightarrow F(I^3) \cdots\right]$$

in $\mathcal{C}'$. As is well known in homological algebra the $n$th cohomology group of $F(I^\cdot)$:

$$H^n F(I^\cdot) = \mathrm{Ker}[F(I^n) \longrightarrow F(I^{n+1})]/\,\mathrm{Im}[F(I^{n-1}) \longrightarrow F(I^n)]$$

does not depend on the choice of injective resolutions, and is uniquely determined up to isomorphisms. Set $R^n F(X) = H^n F(I^\cdot) \in \mathrm{Ob}(\mathcal{C}')$. Then $R^n F$ defines a functor $R^n F : \mathcal{C} \to \mathcal{C}'$. We call $R^n F$ the $n$th *derived functor* of $F$. For $n < 0$ we have $R^n F = 0$ and $R^0 F = F$. Similar construction can be applied also to complexes in $\mathcal{C}$ which are bounded below. Indeed, consider a complex

$$X^\cdot = \left[\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow X^k \longrightarrow X^{k+1} \longrightarrow X^{k+2} \longrightarrow \cdots\right]$$

in $\mathcal{C}$ such that $X^i = 0$ for any $i < k$. Then there exists a complex

$$I^\cdot = \left[\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow I^k \longrightarrow I^{k+1} \longrightarrow I^{k+2} \longrightarrow \cdots\right]$$

of injective objects in $\mathcal{C}$ and a *quasi-isomorphism* $f : X^\cdot \to I^\cdot$, i.e., a morphism of complexes $f : X^\cdot \to I^\cdot$ which induces an isomorphism $H^i(X^\cdot) \simeq H^i(I^\cdot)$ for any $i \in \mathbb{Z}$. We call $I^\cdot$ an injective resolution of $X^\cdot$. Since the injective resolution $I^\cdot$ and the complex $F(I^\cdot)$ are uniquely determined up to homotopy equivalences, the $n$th cohomology group $H^n F(I^\cdot) \in \mathrm{Ob}(\mathcal{C}')$ of $F(I^\cdot)$ is uniquely determined up to isomorphisms. Set $R^n F(X^\cdot) = H^n F(I^\cdot) \in \mathrm{Ob}(\mathcal{C}')$. If we introduce the homotopy category $K^+(\mathcal{C})$ of complexes in $\mathcal{C}$ which are bounded below (for the definition see Section B.2 below), this gives a functor $R^n F : K^+(\mathcal{C}) \to \mathcal{C}'$. Such derived functors for "complexes" are frequently used in algebraic geometry.

However, this classical construction of derived functors has some defects. Since we treat only cohomology groups $\{H^n F(I^\cdot)\}_{n\in\mathbb{Z}}$ of $F(I^\cdot)$, we lose various important information of the complex $F(I^\cdot)$ itself. Moreover, the above construction of derived functors is not convenient for the composition of functors. For example, let $G : \mathcal{C}' \longrightarrow \mathcal{C}''$ be another left exact functor. Then, for $X \in \mathrm{Ob}(\mathcal{C})$ the equality $R^{i+j}(G \circ F)(X) = R^i G(R^j F(X))$ cannot be expected in general. The theory of spectral sequences was invented as a remedy for such problems, but the best way is to treat everything at the level of complexes without taking cohomology groups. Namely, we want to introduce certain categories of complexes and define a lifting $RF$ (which will be also called a derived functor of $F$) of $R^n F$'s between such categories of complexes. This is the theory of derived categories. Indeed, the language of derived categories allows one to formulate complicated relations among various functors in a very beautiful and efficient way.

Now let us briefly explain the construction of derived categories. Let $C(\mathcal{C})$ be the category of complexes in $\mathcal{C}$. Since the injective resolutions $f : X^\cdot \to I^\cdot$ of $X^\cdot$ are just quasi-isomorphisms in $C(\mathcal{C})$, we should change the family of morphisms of $C(\mathcal{C})$ so that quasi-isomorphisms are isomorphisms in the new category. For this purpose we use a general theory of localizations of categories (see Section B.4). However, this localization cannot be applied directly to the category $C(\mathcal{C})$. So we first define the homotopy category $K(\mathcal{C})$ by making homotopy equivalences in $C(\mathcal{C})$ invertible, and then apply the localization. The derived category $D(\mathcal{C})$ thus obtained is an additive category and not an abelian category any more. Therefore, we cannot consider short exact sequences $0 \to X^\cdot \to Y^\cdot \to Z^\cdot \to 0$ of complexes in $D(\mathcal{C})$ as in $C(\mathcal{C})$.

Nevertheless, we can define the notion of distinguished triangles in $D(\mathcal{C})$ which is a substitute for that of short exact sequences of complexes. In other words, the derived category $D(\mathcal{C})$ is a triangulated category in the sense of Definition B.3.6. As in the case of short exact sequences in $C(\mathcal{C})$, from a distinguished triangle in $D(\mathcal{C})$ we can deduce a cohomology long exact sequence in $\mathcal{C}$. Let $F : \mathcal{C} \to \mathcal{C}'$ be a left exact functor and assume that $\mathcal{C}$ has enough injectives. Denote by $D^+(\mathcal{C})$ (resp. $D^+(\mathcal{C}')$) the full subcategory of $D(\mathcal{C})$ (resp. $D(\mathcal{C})$) consisting of complexes which are bounded below. Then we can construct a (right) derived functor $RF : D^+(\mathcal{C}) \to D^+(\mathcal{C}')$ of $F$, which sends distinguished triangles to distinguished triangles. If we identify an object $X$ of $\mathcal{C}$ with a complex

$$\big[\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow X \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots\big]$$

concentrated in degree 0 and hence with an object of $D^+(\mathcal{C})$, we have an isomorphism $H^n(RF(X)) \simeq R^nF(X)$ in $\mathcal{C}'$. From this we see that the new derived functor $RF : D^+(\mathcal{C}) \to D^+(\mathcal{C}')$ extends classical ones. Moreover, this new construction of derived functors turns out to be very useful for the compositions of various functors. Let $G : \mathcal{C}' \to \mathcal{C}''$ be another left exact functor and assume that $\mathcal{C}'$ has enough injectives. Then also the derived functors $RG$ and $R(G \circ F)$ exist and (under a sufficiently weak hypothesis) we have a beautiful composition rule $RG \circ RF = R(G \circ F)$. Since in the theory of $D$-modules we frequently use the compositions of various derived functors, such a nice property is very important.

## B.2 Categories of complexes

Let $\mathcal{C}$ be an abelian category, e.g., the category of $R$-modules over a ring $R$, the category $\mathrm{Sh}(T_0)$ of sheaves on a topological space $T_0$. Denote by $C(\mathcal{C})$ the category of complexes in $\mathcal{C}$. More precisely, an object $X^\bullet$ of $C(\mathcal{C})$ consists of a family of objects $\{X^n\}_{n\in\mathbb{Z}}$ in $\mathcal{C}$ and that of morphisms $\{d^n_{X^\bullet} : X^n \longrightarrow X^{n+1}\}_{n\in\mathbb{Z}}$ in $\mathcal{C}$ satisfying $d^{n+1}_{X^\bullet} \circ d^n_{X^\bullet} = 0$ for any $n \in \mathbb{Z}$. A morphism $f : X^\bullet \longrightarrow Y^\bullet$ in $C(\mathcal{C})$ is a family of morphisms $\{f^n : X^n \longrightarrow Y^n\}_{n\in\mathbb{Z}}$ in $\mathcal{C}$ satisfying the condition $d^n_{Y^\bullet} \circ f^n = f^{n+1} \circ d^n_{X^\bullet}$ for any $n \in \mathbb{Z}$. Namely, a morphism in $C(\mathcal{C})$ is just a chain map between two complexes in $\mathcal{C}$. For an object $X^\bullet \in \mathrm{Ob}(C(\mathcal{C}))$ of $C(\mathcal{C})$ we say that $X^\bullet$ is bounded below (resp. bounded above, resp. bounded) if it satisfies the condition $X^i = 0$ for $i \ll 0$ (resp. $i \gg 0$, resp. $|i| \gg 0$). We denote by $C^+(\mathcal{C})$ (resp. $C^-(\mathcal{C})$, resp. $C^b(\mathcal{C})$) the full subcategory of $C(\mathcal{C})$ consisting of objects which are bounded below (resp. bounded above, resp. bounded). These are naturally abelian categories. Moreover, we identify $\mathcal{C}$ with the full subcategory of $C(\mathcal{C})$ consisting of complexes concentrated in degree 0.

**Definition B.2.1.** We say that a morphism $f : X^\bullet \to Y^\bullet$ in $C(\mathcal{C})$ is a *quasi-isomorphism* if it induces an isomorphism $H^n(X^\bullet) \simeq H^n(Y^\bullet)$ between cohomology groups for any $n \in \mathbb{Z}$.

**Definition B.2.2.**

(i) For a complex $X^\cdot \in \mathrm{Ob}(C(\mathcal{C}))$ with differentials $d^n_{X^\cdot} : X^n \to X^{n+1}$ $(n \in \mathbb{Z})$ and an integer $k \in \mathbb{Z}$, we define the *shifted complex* $X^\cdot[k]$ by

$$\begin{cases} X^n[k] = X^{n+k}, \\ d^n_{X^\cdot[k]} = (-1)^k d^{n+k}_{X^\cdot} : X^n[k] = X^{n+k} \longrightarrow X^{n+1}[k] = X^{n+k+1}. \end{cases}$$

(ii) For a morphism $f : X^\cdot \to Y^\cdot$ in $C(\mathcal{C})$, we define the *mapping cone* $M_f{}^\cdot \in \mathrm{Ob}(C(\mathcal{C}))$ by

$$\begin{cases} M^n_f = X^{n+1} \oplus Y^n, \\ d^n_{M_f{}^\cdot} : M^n_f = X^{n+1} \oplus Y^n \longrightarrow M^{n+1}_f = X^{n+2} \oplus Y^{n+1} \\ \qquad\qquad \cup \qquad\qquad\qquad\qquad\qquad\quad \cup \\ \qquad\quad (x^{n+1}, y^n) \longmapsto (-d^{n+1}_{X^\cdot}(x^{n+1}),\ f^{n+1}(x^{n+1}) + d^n_{Y^\cdot}(y^n)). \end{cases}$$

There exists a natural short exact sequence

$$0 \longrightarrow Y^\cdot \xrightarrow{\alpha(f)} M_f{}^\cdot \xrightarrow{\beta(f)} X^\cdot[1] \longrightarrow 0$$

in $C(\mathcal{C})$, from which we obtain the cohomology long exact sequence

$$\cdots \longrightarrow H^{n-1}(M_f{}^\cdot) \longrightarrow H^n(X^\cdot) \longrightarrow H^n(Y^\cdot) \longrightarrow H^n(M_f{}^\cdot) \longrightarrow \cdots$$

in $\mathcal{C}$. Since the connecting homomorphisms $H^n(X^\cdot) \longrightarrow H^n(Y^\cdot)$ in this long exact sequence coincide with $H^n(f) : H^n(X^\cdot) \longrightarrow H^n(Y^\cdot)$ induced by $f : X^\cdot \to Y^\cdot$, we obtain the following useful result.

**Lemma B.2.3.** *A morphism $f : X^\cdot \to Y^\cdot$ in $C(\mathcal{C})$ is a quasi-isomorphism if and only if $H^n(M_f{}^\cdot) = 0$ for any $n \in \mathbb{Z}$.*

**Definition B.2.4.** For a complex $X^\cdot \in \mathrm{Ob}(C(\mathcal{C}))$ in $\mathcal{C}$ and an integer $k \in \mathbb{Z}$ we define the *truncated complexes* by

$$\tau^{\leqslant k} X^\cdot = \tau^{<k+1} X^\cdot := [\cdots \to X^{k-1} \to Z^k = \mathrm{Ker}\, d^k_{X^\cdot} \to 0 \to 0 \to \cdots],$$
$$\tau^{\geqslant k+1} X^\cdot = \tau^{>k} X^\cdot := [\cdots \to 0 \to 0 \to B^{k+1} = \mathrm{Im}\, d^k_{X^\cdot} \to X^{k+1} \to \cdots].$$

For $X^\cdot \in \mathrm{Ob}(C(\mathcal{C}))$ there exists a short exact sequence

$$0 \longrightarrow \tau^{\leqslant k} X^\cdot \longrightarrow X^\cdot \longrightarrow \tau^{>k} X^\cdot \longrightarrow 0$$

in $C(\mathcal{C})$ for each $k \in \mathbb{Z}$. Note that the complexes $\tau^{\leqslant k} X^\cdot$ and $\tau^{>k} X^\cdot$ satisfy the following conditions, which explain the reason why we call them "truncated" complexes:

$$H^j(\tau^{\leqslant k} X^\cdot) \simeq \begin{cases} H^j(X^\cdot) & j \leq k \\ 0 & j > k, \end{cases}$$
$$H^j(\tau^{>k} X^\cdot) \simeq \begin{cases} H^j(X^\cdot) & j > k \\ 0 & j \leq k. \end{cases}$$

## B.3 Homotopy categories

In this section, before constructing derived categories, we define homotopy categories. Derived categories are obtained by applying a localization of categories to homotopy categories. In order to apply the localization, we need a family of morphisms called a multiplicative system (see Definition B.4.2 below). But the quasi-isomorphisms in $C(\mathcal{C})$ do not form a multiplicative system. Therefore, for the preparation of the localization, we define the *homotopy categories* $K^{\#}(\mathcal{C})$ ($\# = \emptyset, +, -, b$) of an abelian category $\mathcal{C}$ as follows. First recall that a morphism $f : X^{\cdot} \to Y^{\cdot}$ in $C^{\#}(\mathcal{C})$ ($\# = \emptyset, +, -, b$) is *homotopic to* 0 (we write $f \sim 0$ for short) if there exists a family $\{s_n : X^n \to Y^{n-1}\}_{n\in\mathbb{Z}}$ of morphisms in $\mathcal{C}$ such that $f^n = s^{n+1} \circ d_{X^{\cdot}}^n + d_{Y^{\cdot}}^{n-1} \circ s^n$ for any $n \in \mathbb{Z}$:

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & X^{n-1} & \xrightarrow{d_{X^{\cdot}}^{n-1}} & X^n & \xrightarrow{d_{X^{\cdot}}^{n}} & X^{n+1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle f^{n-1}} & \swarrow{\scriptstyle s^n} & \downarrow{\scriptstyle f^n} & \swarrow{\scriptstyle s^{n+1}} & \downarrow{\scriptstyle f^{n+1}} & & \\
\cdots & \longrightarrow & Y^{n-1} & \xrightarrow{d_{Y^{\cdot}}^{n-1}} & Y^n & \xrightarrow{d_{Y^{\cdot}}^{n}} & Y^{n+1} & \longrightarrow & \cdots .
\end{array}$$

We say also that two morphisms $f, g \in \mathrm{Hom}_{C^{\#}(\mathcal{C})}(X^{\cdot}, Y^{\cdot})$ in $C^{\#}(\mathcal{C})$ are *homotopic* (we write $f \sim g$ for short) if the difference $f - g \in \mathrm{Hom}_{C^{\#}(\mathcal{C})}(X^{\cdot}, Y^{\cdot})$ is homotopic to 0.

**Definition B.3.1.** For $\# = \emptyset, +, -, b$ we define the homotopy category $K^{\#}(\mathcal{C})$ of $\mathcal{C}$ by

$$\begin{cases} \mathrm{Ob}(K^{\#}(\mathcal{C})) = \mathrm{Ob}(C^{\#}(\mathcal{C})), \\ \mathrm{Hom}_{K^{\#}(\mathcal{C})}(X^{\cdot}, Y^{\cdot}) = \mathrm{Hom}_{C^{\#}(\mathcal{C})}(X^{\cdot}, Y^{\cdot}) \big/ \mathrm{Ht}(X^{\cdot}, Y^{\cdot}), \end{cases}$$

where $\mathrm{Ht}(X^{\cdot}, Y^{\cdot})$ is a subgroup of $\mathrm{Hom}_{C^{\#}(\mathcal{C})}(X^{\cdot}, Y^{\cdot})$ defined by $\mathrm{Ht}(X^{\cdot}, Y^{\cdot}) = \big\{ f \in \mathrm{Hom}_{C^{\#}(\mathcal{C})}(X^{\cdot}, Y^{\cdot}) \mid f \sim 0 \big\}$.

The homotopy categories $K^{\#}(\mathcal{C})$ are not abelian, but they are still additive categories. We may regard the categories $K^{\#}(\mathcal{C})$ ($\# = +, -, b$) as full subcategories of $K(\mathcal{C})$. Moreover, $\mathcal{C}$ is naturally identified with the full subcategories of these homotopy categories consisting of complexes concentrated in degree 0. Since morphisms which are homotopic to 0 induce zero maps in cohomology groups, the additive functors $H^n : K^{\#}(\mathcal{C}) \to \mathcal{C}$ ($X^{\cdot} \longmapsto H^n(X^{\cdot})$) are well defined. We say that a morphism $f : X^{\cdot} \to Y^{\cdot}$ in $K^{\#}(\mathcal{C})$ is a quasi-isomorphism if it induces an isomorphism $H^n(X^{\cdot}) \simeq H^n(Y^{\cdot})$ for any $n \in \mathbb{Z}$. Recall that a morphism $f : X^{\cdot} \to Y^{\cdot}$ in $C^{\#}(\mathcal{C})$ is called a *homotopy equivalence* if there exists a morphism $g : Y^{\cdot} \to X^{\cdot}$ in $C^{\#}(\mathcal{C})$ such that $g \circ f \sim \mathrm{id}_{X^{\cdot}}$ and $f \circ g \sim \mathrm{id}_{Y^{\cdot}}$. Homotopy equivalences in $C^{\#}(\mathcal{C})$ are isomorphisms in $K^{\#}(\mathcal{C})$ and hence quasi-isomorphisms. As in the case of the categories $C^{\#}(\mathcal{C})$, we can also define truncation functors $\tau^{\geqslant k} : K(\mathcal{C}) \to K^{+}(\mathcal{C})$ and $\tau^{\leqslant k} : K(\mathcal{C}) \to K^{-}(\mathcal{C})$.

Since the homotopy category $K^{\#}(\mathcal{C})$ is not abelian, we cannot consider short exact sequences in it any more. So we introduce the notion of distinguished triangles in

$K^{\#}(\mathcal{C})$ which will be a substitute for that of short exact sequences in the derived category $D^{\#}(\mathcal{C})$.

**Definition B.3.2.**

(i) A sequence $X^{\cdot} \longrightarrow Y^{\cdot} \longrightarrow Z^{\cdot} \longrightarrow X^{\cdot}[1]$ of morphisms in $K^{\#}(\mathcal{C})$ is called a *triangle*.

(ii) A *morphism of triangles* between two triangles $X_1^{\cdot} \longrightarrow Y_1^{\cdot} \longrightarrow Z_1^{\cdot} \longrightarrow X_1^{\cdot}[1]$ and $X_2^{\cdot} \longrightarrow Y_2^{\cdot} \longrightarrow Z_2^{\cdot} \longrightarrow X_2^{\cdot}[1]$ in $K^{\#}(\mathcal{C})$ is a commutative diagram

$$\begin{array}{ccccccc} X_1^{\cdot} & \longrightarrow & Y_1^{\cdot} & \longrightarrow & Z_1^{\cdot} & \longrightarrow & X_1^{\cdot}[1] \\ {\scriptstyle h}\downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle h[1]} \\ X_2^{\cdot} & \longrightarrow & Y_2^{\cdot} & \longrightarrow & Z_2^{\cdot} & \longrightarrow & X_2^{\cdot}[1] \end{array}$$

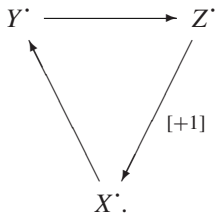
in $K^{\#}(\mathcal{C})$.

(iii) We say that a triangle $X^{\cdot} \longrightarrow Y^{\cdot} \longrightarrow Z^{\cdot} \longrightarrow X^{\cdot}[1]$ in $K^{\#}(\mathcal{C})$ is a *distinguished triangle* if it is isomorphic to a mapping cone triangle $X_0^{\cdot} \xrightarrow{f} Y_0^{\cdot} \xrightarrow{\alpha(f)} M_f^{\cdot} \xrightarrow{\beta(f)} X_0^{\cdot}[1]$ associated to a morphism $f : X_0^{\cdot} \longrightarrow Y_0^{\cdot}$ in $C^{\#}(\mathcal{C})$, i.e., there exists a commutative diagram

$$\begin{array}{ccccccc} X_0^{\cdot} & \xrightarrow{f} & Y_0^{\cdot} & \xrightarrow{\alpha(f)} & M_f^{\cdot} & \xrightarrow{\beta(f)} & X_0^{\cdot}[1] \\ \wr\downarrow & & \wr\downarrow & & \wr\downarrow & & \wr\downarrow \\ X^{\cdot} & \longrightarrow & Y^{\cdot} & \longrightarrow & Z^{\cdot} & \longrightarrow & X^{\cdot}[1] \end{array}$$

in which all vertical arrows are isomorphisms in $K^{\#}(\mathcal{C})$.

A distinguished triangle $X^{\cdot} \longrightarrow Y^{\cdot} \longrightarrow Z^{\cdot} \longrightarrow X^{\cdot}[1]$ is sometimes denoted by $X^{\cdot} \longrightarrow Y^{\cdot} \longrightarrow Z^{\cdot} \xrightarrow{+1}$ or by




**Proposition B.3.3.** *The family of distinguished triangles in $\mathcal{C}_0 = K^{\#}(\mathcal{C})$ satisfies the following properties* (TR0) $\sim$ (TR5)*:*

(TR0) *A triangle which is isomorphic to a distinguished triangle is also distinguished.*

(TR1) *For any $X^{\cdot} \in \mathrm{Ob}(\mathcal{C}_0)$, $X^{\cdot} \xrightarrow{\mathrm{id}_{X^{\cdot}}} X^{\cdot} \longrightarrow 0 \longrightarrow X^{\cdot}[1]$ is a distinguished triangle.*

(TR2) *Any morphism* $f : X^\cdot \longrightarrow Y^\cdot$ *in* $\mathcal{C}_0$ *can be embedded into a distinguished triangle* $X^\cdot \xrightarrow{f} Y^\cdot \longrightarrow Z^\cdot \longrightarrow X^\cdot[1]$.

(TR3) *A triangle* $X^\cdot \xrightarrow{f} Y^\cdot \xrightarrow{g} Z^\cdot \xrightarrow{h} X^\cdot[1]$ *in* $\mathcal{C}_0$ *is distinguished if and only if* $Y^\cdot \xrightarrow{g} Z^\cdot \xrightarrow{h} X^\cdot[1] \xrightarrow{-f[1]} Y^\cdot[1]$ *is distinguished.*

(TR4) *Given two distinguished triangles* $X_1^\cdot \xrightarrow{f_1} Y_1^\cdot \longrightarrow Z_1^\cdot \longrightarrow X_1^\cdot[1]$ *and* $X_2^\cdot \xrightarrow{f_2} Y_2^\cdot \longrightarrow Z_2^\cdot \longrightarrow X_2^\cdot[1]$ *and a commutative diagram*

$$\begin{array}{ccc} X_1^\cdot & \xrightarrow{f_1} & Y_1^\cdot \\ {\scriptstyle h}\downarrow & & \downarrow \\ X_2^\cdot & \xrightarrow{f_2} & Y_2^\cdot \end{array}$$

*in* $\mathcal{C}_0$*, then we can embed them into a morphism of triangles, i.e., into a commutative diagram in* $\mathcal{C}_0$*:*

$$\begin{array}{ccccccc} X_1^\cdot & \longrightarrow & Y_1^\cdot & \longrightarrow & Z_1^\cdot & \longrightarrow & X_1^\cdot[1] \\ {\scriptstyle h}\downarrow & & \downarrow & & {\scriptstyle \psi}\vdots\downarrow & & {\scriptstyle h[1]}\downarrow \\ X_2^\cdot & \longrightarrow & Y_2^\cdot & \longrightarrow & Z_2^\cdot & \longrightarrow & X_2^\cdot[1]. \end{array}$$

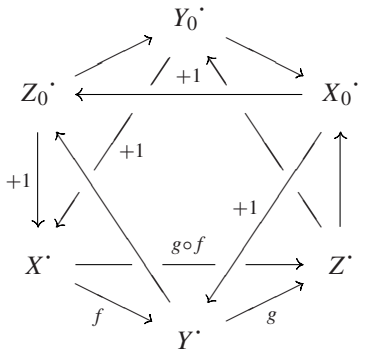
(TR5) *Let*

$$\begin{cases} X^\cdot \xrightarrow{f} Y^\cdot \longrightarrow Z_0^\cdot \longrightarrow X^\cdot[1] \\ Y^\cdot \xrightarrow{g} Z^\cdot \longrightarrow X_0^\cdot \longrightarrow Y^\cdot[1] \\ X^\cdot \xrightarrow{g\circ f} Z^\cdot \longrightarrow Y_0^\cdot \longrightarrow X^\cdot[1], \end{cases}$$

*be three distinguished triangles. Then there exists a distinguished triangle* $Z_0^\cdot \longrightarrow Y_0^\cdot \longrightarrow X_0^\cdot \longrightarrow Z_0^\cdot[1]$ *which can be embedded into the commutative diagram*

$$\begin{array}{ccccccc} X^\cdot & \xrightarrow{f} & Y^\cdot & \longrightarrow & Z_0^\cdot & \longrightarrow & X^\cdot[1] \\ {\scriptstyle \mathrm{id}}\| & & {\scriptstyle g}\downarrow & & \downarrow & & {\scriptstyle \mathrm{id}}\| \\ X^\cdot & \xrightarrow{g\circ f} & Z^\cdot & \longrightarrow & Y_0^\cdot & \longrightarrow & X^\cdot[1] \\ {\scriptstyle f}\downarrow & & {\scriptstyle \mathrm{id}}\| & & \downarrow & & {\scriptstyle f[1]}\downarrow \\ Y^\cdot & \xrightarrow{g} & Z^\cdot & \longrightarrow & X_0^\cdot & \longrightarrow & Y^\cdot[1] \\ \downarrow & & \downarrow & & {\scriptstyle \mathrm{id}}\| & & \downarrow \\ Z_0^\cdot & \longrightarrow & Y_0^\cdot & \longrightarrow & X_0^\cdot & \longrightarrow & Z_0^\cdot[1]. \end{array}$$

For the proof, see [KS2, Proposition 1.4.4]. The property (TR5) is called the *octahedral axiom*, because it can be visualized by the following figure:




**Corollary B.3.4.** *Set $\mathcal{C}_0 = K^{\#}(\mathcal{C})$ and let $X^\bullet \xrightarrow{f} Y^\bullet \xrightarrow{g} Z^\bullet \xrightarrow{h} X^\bullet[1]$ be a distinguished triangle in $\mathcal{C}_0$.*

(i) *For any $n \in \mathbb{Z}$ the sequence $H^n(X^\bullet) \xrightarrow{H^n(f)} H^n(Y^\bullet) \xrightarrow{H^n(g)} H^n(Z^\bullet)$ in $\mathcal{C}$ is exact.*
(ii) *The composite $g \circ f$ is zero.*
(iii) *For any $W^\bullet \in \mathrm{Ob}(\mathcal{C}_0)$, the sequences*

$$\begin{cases} \mathrm{Hom}_{\mathcal{C}_0}(W^\bullet, X^\bullet) \longrightarrow \mathrm{Hom}_{\mathcal{C}_0}(W^\bullet, Y^\bullet) \longrightarrow \mathrm{Hom}_{\mathcal{C}_0}(W^\bullet, Z^\bullet) \\ \mathrm{Hom}_{\mathcal{C}_0}(Z^\bullet, W^\bullet) \longrightarrow \mathrm{Hom}_{\mathcal{C}_0}(Y^\bullet, W^\bullet) \longrightarrow \mathrm{Hom}_{\mathcal{C}_0}(X^\bullet, W^\bullet) \end{cases}$$

*associated to the distinguished triangle $X^\bullet \xrightarrow{f} Y^\bullet \xrightarrow{g} Z^\bullet \xrightarrow{h} X^\bullet[1]$ are exact in the abelian category $\mathcal{A}b$ of abelian groups.*

**Corollary B.3.5.** *Set $\mathcal{C}_0 = K^{\#}(\mathcal{C})$.*

(i) *Let*

$$\begin{array}{ccccccc} X_1^\bullet & \longrightarrow & Y_1^\bullet & \longrightarrow & Z_1^\bullet & \longrightarrow & X_1^\bullet[1] \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle f[1]} \\ X_2^\bullet & \longrightarrow & Y_2^\bullet & \longrightarrow & Z_2^\bullet & \longrightarrow & X_2^\bullet[1]. \end{array}$$

*be a morphism of distinguished triangles in $\mathcal{C}_0$. Assume that $f$ and $g$ are isomorphisms. Then $h$ is also an isomorphism.*
(ii) *Let $X^\bullet \xrightarrow{u} Y^\bullet \longrightarrow Z^\bullet \longrightarrow X^\bullet[1]$ and $X^\bullet \xrightarrow{u} Y^\bullet \longrightarrow Z'^\bullet \longrightarrow X^\bullet[1]$ be two distinguished triangles in $\mathcal{C}_0$. Then $Z^\bullet \simeq Z'^\bullet$.*
(iii) *Let $X^\bullet \xrightarrow{u} Y^\bullet \longrightarrow Z^\bullet \longrightarrow X^\bullet[1]$ be a distinguished triangle in $\mathcal{C}_0$. Then $u$ is an isomorphism if and only if $Z^\bullet \simeq 0$.*

By abstracting the properties of the homotopy categories $K^{\#}(\mathcal{C})$ let us introduce the notion of triangulated categories as follows. In the case when $\mathcal{C}_0 = K^{\#}(\mathcal{C})$ for an abelian category $\mathcal{C}$, the automorphism $T : \mathcal{C}_0 \longrightarrow \mathcal{C}_0$ in the definition below is the degree shift functor $(\bullet)[1] : K^{\#}(\mathcal{C}) \to K^{\#}(\mathcal{C})$.

**Definition B.3.6.** Let $\mathcal{C}_0$ be an additive category and $T : \mathcal{C}_0 \to \mathcal{C}_0$ an automorphism of $\mathcal{C}_0$.

(i) A sequence of morphisms $X \longrightarrow Y \longrightarrow Z \longrightarrow T(X)$ in $\mathcal{C}_0$ is called a triangle in $\mathcal{C}_0$.
(ii) Consider a family $\mathcal{T}$ of triangles in $\mathcal{C}_0$, called distinguished triangles. We say that the pair $(\mathcal{C}_0, \mathcal{T})$ is a *triangulated category* if the family $\mathcal{T}$ of distinguished triangles satisfies the axioms obtained from (TR0) $\sim$ (TR5) in Proposition B.3.3 by replacing $(\bullet)[1]$'s with $T(\bullet)$'s everywhere.

It is clear that Corollary B.3.4 (ii), (iii) and Corollary B.3.5 are true for any triangulated category $(\mathcal{C}_0, \mathcal{T})$. Derived categories that we introduce in the next section are also triangulated categories. Note also that the morphism $\psi$ in (TR4) is not unique in general, which is the source of some difficulties in using triangulated categories.

**Definition B.3.7.** Let $(\mathcal{C}_0, \mathcal{T})$, $(\mathcal{C}_0', \mathcal{T}')$ be two triangulated categories and $T : \mathcal{C}_0 \to \mathcal{C}_0$, $T' : \mathcal{C}_0' \to \mathcal{C}_0'$ the corresponding automorphisms. Then we say that an additive functor $F : \mathcal{C}_0 \to \mathcal{C}_0'$ is a *functor of triangulated categories* (or a *$\partial$-functor*) if $F \circ T = T' \circ F$ and $F$ sends distinguished triangles in $\mathcal{C}_0$ to those in $\mathcal{C}_0'$.

**Definition B.3.8.** An additive functor $F : \mathcal{C}_0 \longrightarrow \mathcal{A}$ from a triangulated category $(\mathcal{C}_0, \mathcal{T})$ into an abelian category $\mathcal{A}$ is called a *cohomological functor* if for any distinguished triangle $X \longrightarrow Y \longrightarrow Z \longrightarrow T(X)$, the associated sequence $F(X) \longrightarrow F(Y) \longrightarrow F(Z)$ in $\mathcal{A}$ is exact.

The assertions (i) and (iii) of Corollary B.3.4 imply that the functors $H^n : K^{\#}(\mathcal{C}) \longrightarrow \mathcal{C}$ and $\mathrm{Hom}_{K^{\#}(\mathcal{C})}(W^{\cdot}, \bullet) : K^{\#}(\mathcal{C}) \longrightarrow \mathcal{A}b$ are cohomological functors, respectively. Let $F : \mathcal{C}_0 \longrightarrow \mathcal{A}$ be a cohomological functor. Then by using the axiom (TR3) repeatedly, from a distinguished triangle $X \longrightarrow Y \longrightarrow Z \longrightarrow T(X)$ in $\mathcal{C}_0$ we obtain a long exact sequence

$$\cdots \longrightarrow F(T^{-1}Z) \longrightarrow F(X) \longrightarrow F(Y) \longrightarrow F(Z) \longrightarrow F(TX) \longrightarrow F(TY) \longrightarrow \cdots$$

in the abelian category $\mathcal{A}$.

## B.4 Derived categories

In this section, we shall construct derived categories $D^{\#}(\mathcal{C})$ from homotopy categories $K^{\#}(\mathcal{C})$ by adding morphisms so that quasi-isomorphisms are invertible in $D^{\#}(\mathcal{C})$. For this purpose, we need the general theory of localizations of categories. Now let $\mathcal{C}_0$ be a category and $S$ a family of morphisms in $\mathcal{C}_0$. In what follows, for two functors $F_1, F_2 : \mathcal{A}_1 \to \mathcal{A}_2$ $(\in Fun(\mathcal{A}_1, \mathcal{A}_2))$ we denote by $\mathrm{Hom}_{Fun(\mathcal{A}_1, \mathcal{A}_2)}(F_1, F_2)$ the set of *natural transformations* (i.e., morphisms of functors) from $F_1$ to $F_2$.

**Definition B.4.1.** A *localization* of the category $\mathcal{C}_0$ by $S$ is a pair $((\mathcal{C}_0)_S, Q)$ of a category $(\mathcal{C}_0)_S$ and a functor $Q : \mathcal{C}_0 \to (\mathcal{C}_0)_S$ which satisfies the following universal properties:

(i) $Q(s)$ is an isomorphism for any $s \in S$.
(ii) For any functor $F : \mathcal{C}_0 \to \mathcal{C}_1$ such that $F(s)$ is an isomorphism for any $s \in S$, there exists a functor $F_S : (\mathcal{C}_0)_S \to \mathcal{C}_1$ and an isomorphism $F \simeq F_S \circ Q$ of functors:

$$\begin{array}{ccc} \mathcal{C}_0 & \xrightarrow{Q} & (\mathcal{C}_0)_S \\ {\scriptstyle F}\downarrow & \swarrow^{\exists F_S} & \\ \mathcal{C}_1. & & \end{array}$$

(iii) Let $G_1, G_2 : (\mathcal{C}_0)_S \to \mathcal{C}_1$ be two functors. Then the natural morphism

$$\mathrm{Hom}_{Fun((\mathcal{C}_0)_S, \mathcal{C}_1)}(G_1, G_2) \longrightarrow \mathrm{Hom}_{Fun(\mathcal{C}_0, \mathcal{C}_1)}(G_1 \circ Q, G_2 \circ Q)$$

is a bijection.

By the property (iii), $F_S$ in (ii) is unique up to isomorphisms. Moreover, since the localization $((\mathcal{C}_0)_S, Q)$ is characterized by universal properties (if it exists) it is unique up to equivalences of categories. We call this operation "a localization of categories'' because it is similar to the more familiar localization of (non-commutative) rings. As we need the so-called "Ore conditions'' for the construction of localizations of rings, we have to impose some conditions on $S$ to ensure the existence of the localization $((\mathcal{C}_0)_S, Q)$.

**Definition B.4.2.** Let $\mathcal{C}_0$ be a category and $S$ a family of morphisms. We call the family $S$ a *multiplicative system* if it satisfies the following axioms:

(M1) $\mathrm{id}_X \in S$ for any $X \in \mathrm{Ob}(\mathcal{C}_0)$.
(M2) If $f, g \in S$ and their composite $g \circ f$ exists, then $g \circ f \in S$.
(M3) Any diagram

$$\begin{array}{ccc} & & Y' \\ & & {\scriptstyle s}\downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

in $\mathcal{C}_0$ with $s \in S$ fits into a commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{g} & Y' \\ {\scriptstyle t}\downarrow & & {\scriptstyle s}\downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

in $\mathcal{C}_0$ with $t \in S$. We impose also the condition obtained by reversing all arrows.
(M4) For $f, g \in \mathrm{Hom}_{\mathcal{C}_0}(X, Y)$ the following two conditions are equivalent:

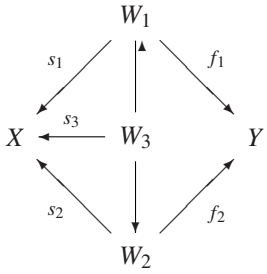
(i) $^{\exists}s : Y \longrightarrow Y', s \in S$ such that $s \circ f = s \circ g$.
(ii) $^{\exists}t : X' \longrightarrow X, t \in S$ such that $f \circ t = g \circ t$.

Let $\mathcal{C}_0$ be a category and $S$ a multiplicative system in it. Then we can define a category $(\mathcal{C}_0)_S$ by

(objects): $\mathrm{Ob}((\mathcal{C}_0)_S) = \mathrm{Ob}(\mathcal{C}_0)$.
(morphisms): For $X, Y \in \mathrm{Ob}((\mathcal{C}_0)_S) = \mathrm{Ob}(\mathcal{C}_0)$, we set

$$\mathrm{Hom}_{(\mathcal{C}_0)_S}(X, Y) = \left\{(X \xleftarrow{s} W \xrightarrow{f} Y) \mid s \in S\right\} \Big/ \sim$$

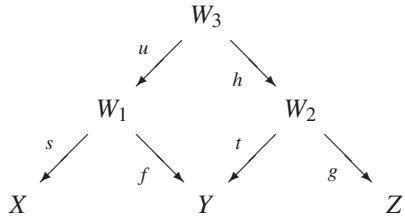
where two diagrams $(X \xleftarrow{s_1} W_1 \xrightarrow{f_1} Y)$ $(s_1 \in S)$ and $(X \xleftarrow{s_2} W_2 \xrightarrow{f_2} Y)$ $(s_2 \in S)$ are equivalent $(\sim)$ if and only if they fit into a commutative diagram




with $s_3 \in S$. We omit the details here. Let us just explain how we compose morphisms in the category $(\mathcal{C}_0)_S$. Assume that we are given two morphisms

$$\begin{cases} \left[(X \xleftarrow{s} W_1 \xrightarrow{f} Y)\right] \in \mathrm{Hom}_{(\mathcal{C}_0)_S}(X, Y) \\ \left[(Y \xleftarrow{t} W_2 \xrightarrow{g} Z)\right] \in \mathrm{Hom}_{(\mathcal{C}_0)_S}(Y, Z) \end{cases}$$

$(s, t \in S)$ in $(\mathcal{C}_0)_S$. Then by the axiom (M3) of multiplicative systems we can construct a commutative diagram




with $u \in S$ and the composite of these two morphisms in $(\mathcal{C}_0)_S$ is given by

$$\left[(X \xleftarrow{s \circ u} W_3 \xrightarrow{g \circ h} Z)\right] \in \mathrm{Hom}_{(\mathcal{C}_0)_S}(X, Z).$$

Moreover, there exists a natural functor $Q : \mathcal{C}_0 \longrightarrow (\mathcal{C}_0)_S$ defined by

$$\begin{cases} Q(X) = X \quad \text{for } X \in \mathrm{Ob}(\mathcal{C}_0), \\ \mathrm{Hom}_{\mathcal{C}_0}(X, Y) \longrightarrow \mathrm{Hom}_{(\mathcal{C}_0)_S}(Q(X), Q(Y)) \\ \quad \cup \qquad\qquad\qquad\qquad \cup \\ \quad f \qquad \longmapsto \quad \left[(X \xleftarrow{\mathrm{id}_X} X \xrightarrow{f} Y)\right] \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for any } X, Y \in \mathrm{Ob}(\mathcal{C}_0). \end{cases}$$

We can easily check that the pair $((\mathcal{C}_0)_S, Q)$ satisfies the conditions of the localization of $\mathcal{C}_0$ by $S$. Since morphisms in $S$ are invertible in the localized category $(\mathcal{C}_0)_S$, a morphism $[(X \xleftarrow{s} W \xrightarrow{f} Y)]$ in $(\mathcal{C}_0)_S$ can be written also as $Q(f) \circ Q(s)^{-1}$. If, moreover, $\mathcal{C}_0$ is an additive category, then we can show that $(\mathcal{C}_0)_S$ is also an additive category and $Q : \mathcal{C}_0 \to (\mathcal{C}_0)_S$ is an additive functor. Since we defined the localization $(\mathcal{C}_0)_S$ of $\mathcal{C}_0$ by universal properties, also the following category $(\mathcal{C}_0)^S$ satisfies the conditions of the localization:

(objects): $\mathrm{Ob}((\mathcal{C}_0)^S) = \mathrm{Ob}(\mathcal{C}_0)$.
(morphisms): For $X, Y \in \mathrm{Ob}((\mathcal{C}_0)^S) = \mathrm{Ob}(\mathcal{C}_0)$, we set

$$\mathrm{Hom}_{(\mathcal{C}_0)^S}(X, Y) = \left\{(X \xrightarrow{f} W \xleftarrow{s} Y) \mid s \in S\right\} \Big/ \sim$$

where we define the equivalence $\sim$ of diagrams similarly. Namely, a morphism in the localization $(\mathcal{C}_0)_S$ can be written also as $Q(s)^{-1} \circ Q(f)$ for $s \in S$. The following elementary lemma will be effectively used in the next section.

**Lemma B.4.3.** *Let $\mathcal{C}_0$ be a category and $S$ a multiplicative system in it. Let $\mathcal{J}_0$ be a full subcategory of $\mathcal{C}_0$ and denote by $T$ the family of morphisms in $\mathcal{J}_0$ which belong to $S$. Assume, moreover, that for any $X \in \mathrm{Ob}(\mathcal{C}_0)$ there exists a morphism $s : X \to J$ in $S$ such that $J \in \mathrm{Ob}(\mathcal{J}_0)$. Then $T$ is a multiplicative system in $\mathcal{J}_0$, and the natural functor $(\mathcal{J}_0)_T \to (\mathcal{C}_0)_S$ gives an equivalence of categories.*

Now let us return to the original situation and consider a homotopy category $\mathcal{C}_0 = K^{\#}(\mathcal{C})$ $(\# = \emptyset, +, -, b)$ of an abelian category $\mathcal{C}$. Denote by $S$ the family of quasi-isomorphisms in it. Then we can prove that $S$ is a multiplicative system.

**Definition B.4.4.** We set $D^{\#}(\mathcal{C}) = (K^{\#}(\mathcal{C}))_S$ and call it a *derived category* of $\mathcal{C}$. The canonical functor $Q : K^{\#}(\mathcal{C}) \to D^{\#}(\mathcal{C})$ is called the *localization functor*.

By construction, quasi-isomorphisms are isomorphisms in the derived category $D^{\#}(\mathcal{C})$. Moreover, if we define distinguished triangles in $D^{\#}(\mathcal{C})$ to be the triangles isomorphic to the images of distinguished triangles in $K^{\#}(\mathcal{C})$ by $Q$, then $D^{\#}(\mathcal{C})$ is a triangulated category and $Q : K^{\#}(\mathcal{C}) \to D^{\#}(\mathcal{C})$ is a functor of triangulated categories. We can also prove that the canonical morphisms $\mathcal{C} \to D(\mathcal{C})$ and $D^{\#}(\mathcal{C}) \to D(\mathcal{C})$ $(\# = +, -, b)$ are fully faithful. Namely, the categories $\mathcal{C}$ and $D^{\#}(\mathcal{C})$ $(\# = +, -, b)$ can be identified with full subcategories of $D(\mathcal{C})$. By the results in the previous section, to a distinguished triangle $X^{\cdot} \longrightarrow Y^{\cdot} \longrightarrow Z^{\cdot} \longrightarrow X^{\cdot}[1]$ in the derived category $D^{\#}(\mathcal{C})$ we can associate a cohomology long exact sequence

$$\cdots \longrightarrow H^{-1}(Z^\cdot) \longrightarrow H^0(X^\cdot) \longrightarrow H^0(Y^\cdot) \longrightarrow H^0(Z^\cdot) \longrightarrow H^1(X^\cdot) \longrightarrow \cdots$$

in $\mathcal{C}$. The following lemma is very useful to construct examples of distinguished triangles in $D^\#(\mathcal{C})$.

**Lemma B.4.5.** *Any short exact sequence* $0 \longrightarrow X^\cdot \xrightarrow{f} Y^\cdot \xrightarrow{g} Z^\cdot \longrightarrow 0$ *in* $C^\#(\mathcal{C})$ *can be embedded into a distinguished triangle* $X^\cdot \xrightarrow{f} Y^\cdot \xrightarrow{g} Z^\cdot \longrightarrow X^\cdot[1]$ *in* $D^\#(\mathcal{C})$.

*Proof.* Consider the short exact sequence

$$0 \longrightarrow M_{\mathrm{id}_{X^\cdot}}{}^\cdot \xrightarrow{\begin{pmatrix} \mathrm{id}_{X^\cdot} & 0 \\ 0 & f \end{pmatrix}} M_f{}^\cdot \xrightarrow{(0,\, g)} Z^\cdot \longrightarrow 0$$

in $C^\#(\mathcal{C})$. Since the mapping cone $M_{\mathrm{id}_{X^\cdot}}{}^\cdot$ is quasi-isomorphic to 0 by Lemma B.2.3, we obtain an isomorphism $\varphi : M_f{}^\cdot \simeq Z^\cdot$ in $D^\#(\mathcal{C})$. Hence there exists a commutative diagram

$$\begin{array}{ccccccc} X^\cdot & \xrightarrow{f} & Y^\cdot & \xrightarrow{\alpha(f)} & M_f{}^\cdot & \xrightarrow{\beta(f)} & X^\cdot[1] \\ {\scriptstyle \mathrm{id}}\| & & {\scriptstyle \mathrm{id}}\| & & {\scriptstyle \varphi}\downarrow & & {\scriptstyle \mathrm{id}}\| \\ X^\cdot & \xrightarrow{f} & Y^\cdot & \xrightarrow{g} & Z^\cdot & \xrightarrow{\beta(f)\circ\varphi^{-1}} & X^\cdot[1] \end{array}$$

in $D^\#(\mathcal{C})$, which shows that $X^\cdot \xrightarrow{f} Y^\cdot \xrightarrow{g} Z^\cdot \xrightarrow{\beta(f)\circ\varphi^{-1}} X^\cdot[1]$ is a distinguished triangle. □

**Definition B.4.6.** An abelian subcategory $\mathcal{C}'$ of $\mathcal{C}$ is called a *thick subcategory* if for any exact sequence $X_1 \to X_2 \to X_3 \to X_4 \to X_5$ in $\mathcal{C}$ with $X_i \in \mathrm{Ob}(\mathcal{C}')$ $(i = 1, 2, 4, 5)$, $X_3$ belongs to $\mathcal{C}'$.

**Proposition B.4.7.** *Let* $\mathcal{C}'$ *be a thick abelian subcategory of an abelian category* $\mathcal{C}$ *and* $D^\#_{\mathcal{C}'}(\mathcal{C})$ *the full subcategory of* $D^\#(\mathcal{C})$ *consisting of objects* $X^\cdot$ *such that* $H^n(X^\cdot) \in \mathrm{Ob}(\mathcal{C}')$ *for any* $n \in \mathbb{Z}$. *Then* $D^\#_{\mathcal{C}'}(\mathcal{C})$ *is a triangulated category.*

## B.5 Derived functors

Let $\mathcal{C}$ and $\mathcal{C}'$ be abelian categories and $F : \mathcal{C} \to \mathcal{C}'$ an additive functor. Let us consider the problem of constructing a $\partial$-functor $\tilde{F} : D^\#(\mathcal{C}) \to D^\#(\mathcal{C}')$ between their derived categories which is naturally associated to $F : \mathcal{C} \to \mathcal{C}'$. This problem can be easily solved if $\# = +$ or $-$ and $F$ is an exact functor. Indeed, let $Q : K^\#(\mathcal{C}) \to D^\#(\mathcal{C})$, $Q' : K^\#(\mathcal{C}') \to D^\#(\mathcal{C}')$ be the localization functors and consider the functor $K^\#F : K^\#(\mathcal{C}) \to K^\#(\mathcal{C}')$ defined by $X^\cdot \mapsto F(X^\cdot)$. Then by Lemma B.2.3 the functor $K^\#F$ sends quasi-isomorphisms in $K^\#(\mathcal{C})$ to those in $K^\#(\mathcal{C}')$. Hence it follows from the

universal properties of the localization $Q : K^{\#}(\mathcal{C}) \to D^{\#}(\mathcal{C})$ that there exists a unique functor $\tilde{F} : D^{\#}(\mathcal{C}) \to D^{\#}(\mathcal{C}')$ which makes the following diagram commutative:

$$\begin{array}{ccc} K^{\#}(\mathcal{C}) & \xrightarrow{K^{\#}F} & K^{\#}(\mathcal{C}') \\ {\scriptstyle Q}\downarrow & & \downarrow{\scriptstyle Q'} \\ D^{\#}(\mathcal{C}) & \xrightarrow[\tilde{F}]{} & D^{\#}(\mathcal{C}'). \end{array}$$

In this situation, we call the functor $\tilde{F} : D^{\#}(\mathcal{C}) \to D^{\#}(\mathcal{C}')$ a "localization" of $Q' \circ K^{\#}F : K^{\#}(\mathcal{C}) \to D^{\#}(\mathcal{C}')$. However, many important additive functors that we encounter in sheaf theory or homological algebra are not exact. They are only left exact or right exact. So in such cases the functor $Q' \circ K^{\#}F : K^{\#}(\mathcal{C}) \to D^{\#}(\mathcal{C}')$ does not factorize through $Q : K^{\#}(\mathcal{C}) \to D^{\#}(\mathcal{C})$ in general. In other words, there is no localization of the functor $Q' \circ K^{\#}F$. As a remedy for this problem we will introduce the following notion of right (or left) localizations. In what follows, let $\mathcal{C}_0$ be a general category, $S$ a multiplicative system in $\mathcal{C}_0$, $F : \mathcal{C}_0 \to \mathcal{C}_1$ a functor. As before we denote by $Q : \mathcal{C}_0 \to (\mathcal{C}_0)_S$ the canonical functor.

**Definition B.5.1.** A *right localization* of $F$ is a pair $(F_S, \tau)$ of a functor $F_S : (\mathcal{C}_0)_S \to \mathcal{C}_1$ and a morphism of functors $\tau : F \to F_S \circ Q$ such that for any functor $G : (\mathcal{C}_0)_S \to \mathcal{C}_1$ the morphism

$$\mathrm{Hom}_{Fun((\mathcal{C}_0)_S, \mathcal{C}_1)}(F_S, G) \longrightarrow \mathrm{Hom}_{Fun(\mathcal{C}_0, \mathcal{C}_1)}(F, G \circ Q)$$

is bijective. Here the morphism above is obtained by the composition of

$$\mathrm{Hom}_{Fun((\mathcal{C}_0)_S, \mathcal{C}_1)}(F_S, G) \longrightarrow \mathrm{Hom}_{Fun(\mathcal{C}_0, \mathcal{C}_1)}(F_S \circ Q, G \circ Q)$$

$$\xrightarrow{\tau} \mathrm{Hom}_{Fun(\mathcal{C}_0, \mathcal{C}_1)}(F, G \circ Q).$$

We say that $F$ is *right localizable* if it has a right localization.

The notion of left localizations can be defined similarly. Note that by definition the functor $F_S$ is a representative of the functor

$$G \longrightarrow \mathrm{Hom}_{Fun(\mathcal{C}_0, \mathcal{C}_1)}(F, G \circ Q).$$

Therefore, if a right localization $(F_S, \tau)$ of $F$ exists, it is unique up to isomorphisms. Let us give a useful criterion for the existence of the right localization of $F : \mathcal{C}_0 \to \mathcal{C}_1$.

**Proposition B.5.2.** *Let $\mathcal{J}_0$ be a full subcategory of $\mathcal{C}_0$ and denote by $T$ the family of morphisms in $\mathcal{J}_0$ which belong to $S$. Assume the following conditions:*

(i) *For any $X \in \mathrm{Ob}(\mathcal{C}_0)$ there exists a morphism $s : X \to J$ in $S$ such that $J \in \mathrm{Ob}(\mathcal{J}_0)$.*
(ii) *$F(t)$ is an isomorphism for any $t \in T$.*

*Then $F$ is localizable.*

A very precise proof of this proposition can be found in Kashiwara–Schapira [KS4, Proposition 7.3.2]. Here we just explain how the functor $F_S : (\mathcal{C}_0)_S \to \mathcal{C}_1$ is defined.

First, by Lemma B.4.3 there exists an equivalence of categories $\Phi : (\mathcal{J}_0)_T \xrightarrow{\sim} (\mathcal{C}_0)_S$. Let $\iota : \mathcal{J}_0 \to \mathcal{C}_0$ be the inclusion. Then by the condition (ii) above the functor $F \circ \iota : \mathcal{J}_0 \to \mathcal{C}_1$ factorizes through the localization functor $\mathcal{J}_0 \to (\mathcal{J}_0)_T$ and we obtain a functor $F_T : (\mathcal{J}_0)_T \to \mathcal{C}_1$. The functor $F_S : (\mathcal{C}_0)_S \to \mathcal{C}_1$ is defined by $F_S = F_T \circ \Phi^{-1}$:

$$\begin{array}{ccc} (\mathcal{J}_0)_T & \xrightarrow{F_T} & \mathcal{C}_1 \\ {\scriptstyle \Phi}\wr\downarrow & \nearrow_{F_S} & \\ (\mathcal{C}_0)_S. & & \end{array}$$

Now let us return to the original situation and assume that $F : \mathcal{C} \longrightarrow \mathcal{C}'$ is a left exact functor. In this situation, by Proposition B.5.2 we can give a criterion for the existence of a right localization of the functor $Q' \circ K^+F : K^+(\mathcal{C}) \to D^+(\mathcal{C}')$ as follows.

**Definition B.5.3.** A *right derived functor* of $F$ is a pair $(RF, \tau)$ of a $\partial$-functor $RF : D^+(\mathcal{C}) \to D^+(\mathcal{C}')$ and a morphism of functors $\tau : Q' \circ K^+(F) \to RF \circ Q$

$$\begin{array}{ccc} K^+(\mathcal{C}) & \xrightarrow{K^+(F)} & K^+(\mathcal{C}') \\ {\scriptstyle Q}\downarrow & \swarrow_{\tau} & \downarrow{\scriptstyle Q'} \\ D^+(\mathcal{C}) & \xrightarrow{RF} & D^+(\mathcal{C}') \end{array}$$

such that for any functor $\tilde{G} : D^+(\mathcal{C}) \to D^+(\mathcal{C}')$ the morphism

$$\mathrm{Hom}_{Fun(D^+(\mathcal{C}), D^+(\mathcal{C}'))}(RF, \tilde{G}) \longrightarrow \mathrm{Hom}_{Fun(K^+(\mathcal{C}), D^+(\mathcal{C}'))}(Q' \circ K^+(F), \tilde{G} \circ Q)$$

induced by $\tau$ is an isomorphism. We say that $F$ is *right derivable* if it admits a right derived functor.

Similarly, for right exact functors $F$ we can define the notion of *left derived functors* $LF : D^-(\mathcal{C}) \to D^-(\mathcal{C}')$. By definition, if a right derived functor $(RF, \tau)$ of a left exact functor $F$ exists, it is unique up to isomorphisms. Moreover, for an exact functor $F$ the natural functor $D^+(\mathcal{C}) \to D^+(\mathcal{C}')$ defined simply by $X^\cdot \mapsto F(X^\cdot)$ gives a right derived functor. In other words, any exact functor is right (and left) derivable.

**Definition B.5.4.** Let $F : \mathcal{C} \to \mathcal{C}'$ be an additive functor between abelian categories. We say that a full additive subcategory $\mathcal{J}$ of $\mathcal{C}$ is *$F$-injective* if the following conditions are satisfied:

(i) For any $X \in \mathrm{Ob}(\mathcal{C})$, there exists an object $I \in \mathrm{Ob}(\mathcal{J})$ and an exact sequence $0 \to X \to I$.

(ii) If $0 \to X' \to X \to X'' \to 0$ is an exact sequence in $\mathcal{C}$ and $X', X \in \mathrm{Ob}(\mathcal{J})$, then $X'' \in \mathrm{Ob}(\mathcal{J})$.
(iii) For any exact sequence $0 \to X' \to X \to X'' \to 0$ in $\mathcal{C}$ such that $X', X, X'' \in \mathrm{Ob}(\mathcal{J})$, the sequence $0 \to F(X') \to F(X) \to F(X'') \to 0$ in $\mathcal{C}'$ is also exact.

Similarly we define *$F$-projective* subcategories of $\mathcal{C}$ by reversing all arrows in the conditions above.

**Example B.5.5.**
(i) Assume that the abelian category $\mathcal{C}$ has enough injectives. Denote by $\mathcal{I}$ the full additive subcategory of $\mathcal{C}$ consisting of injective objects in $\mathcal{C}$. Then $\mathcal{I}$ is $F$-injective for any additive functor $F : \mathcal{C} \to \mathcal{C}'$ (use the fact that any exact sequence $0 \to X' \to X \to X'' \to 0$ in $\mathcal{C}$ with $X' \in \mathrm{Ob}(\mathcal{I})$ splits).
(ii) Let $T_0$ be a topological space and set $\mathcal{C} = \mathrm{Sh}(T_0)$. Let $F = \Gamma(T_0, \bullet) : \mathrm{Sh}(T_0) \to \mathcal{A}b$ be the global section functor. Then $F = \Gamma(T_0, \bullet)$ is left exact and

$$\mathcal{J} = \{\text{flabby sheaves on } T_0\} \subset \mathrm{Sh}(T_0)$$

is an $F$-injective subcategory of $\mathcal{C}$.
(iii) Let $T_0$ be a topological space and $\mathcal{R}$ a sheaf of rings on $T_0$. Denote by $\mathrm{Mod}(\mathcal{R})$ the abelian category of sheaves of left $\mathcal{R}$-modules on $T_0$ and let $\mathcal{P}$ be the full subcategory of $\mathrm{Mod}(\mathcal{R})$ consisting of flat $\mathcal{R}$-modules, i.e., objects $M \in \mathrm{Mod}(\mathcal{R})$ such that the stalk $M_x$ at $x$ is a flat $\mathcal{R}_x$-module for any $x \in T_0$. For a right $\mathcal{R}$-module $N$, consider the functor $F_N = N \otimes_{\mathcal{R}} (\bullet) : \mathrm{Mod}(\mathcal{R}) \to \mathrm{Sh}(T_0)$. Then the category $\mathcal{P}$ is $F_N$-projective. For the details see Section C.1.

Assume that for the given left exact functor $F : \mathcal{C} \to \mathcal{C}'$ there exists an $F$-injective subcategory $\mathcal{J}$ of $\mathcal{C}$. Then it is well known that for any $X^\cdot \in \mathrm{Ob}(K^+(\mathcal{C}))$ we can construct a quasi-isomorphism $X^\cdot \to J^\cdot$ into an object $J^\cdot$ of $K^+(\mathcal{J})$. Such an object $J^\cdot$ is called an *$F$-injective resolution* of $X^\cdot$. Moreover, by Lemma B.2.3 we see that the functor $Q' \circ K^+F : K^+(\mathcal{J}) \to D^+(\mathcal{C}')$ sends quasi-isomorphisms in $K^+(\mathcal{J})$ to isomorphisms in $D^+(\mathcal{C}')$. Therefore, applying Proposition B.5.2 to the special case when $\mathcal{C}_0 = K^+(\mathcal{C})$, $\mathcal{C}_1 = D^+(\mathcal{C}')$, $\mathcal{J}_0 = K^+(\mathcal{J})$ and $S$ is the family of quasi-isomorphisms in $K^+(\mathcal{C})$, we obtain the fundamental important result.

**Theorem B.5.6.** *Let $F : \mathcal{C} \to \mathcal{C}'$ be a left (resp. right) exact functor and assume that there exists an $F$-injective (resp. $F$-projective) subcategory of $\mathcal{C}$. Then the right (resp. left) derived functor $RF : D^+(\mathcal{C}) \to D^+(\mathcal{C}')$ (resp. $LF : D^-(\mathcal{C}) \to D^-(\mathcal{C}')$) of $F$ exists.*

Let $F : \mathcal{C} \to \mathcal{C}'$ be a left exact functor and $\mathcal{J}$ an $F$-injective subcategory of $\mathcal{C}$. Then the right derived functor $RF : D^+(\mathcal{C}) \to D^+(\mathcal{C}')$ is constructed as follows. Denote by $S$ (resp. $T$) the family of quasi-isomorphisms in $K^+(\mathcal{C})$ (resp. $K^+(\mathcal{J})$). Then there exist an equivalence of categories $\Phi : K^+(\mathcal{J})_T \xrightarrow{\sim} K^+(\mathcal{C})_S = D^+(\mathcal{C})$ and a natural functor $\Psi : K^+(\mathcal{J})_T \to D^+(\mathcal{C}')$ induced by $K^+F : K^+(\mathcal{C}) \to K^+(\mathcal{C}')$ such that $RF = \Psi \circ \Phi^{-1}$. Consequently the right derived functor $RF : D^+(\mathcal{C}) \to D^+(\mathcal{C}')$ sends $X^\cdot \in \mathrm{Ob}(D^+(\mathcal{C}))$ to $F(J^\cdot) \in \mathrm{Ob}(D^+(\mathcal{C}'))$, where $J^\cdot \in \mathrm{Ob}(K^+(\mathcal{J}))$

is an $F$-injective resolution of $X^\cdot$. If $\mathcal{C}$ has enough injectives and denote by $\mathcal{I}$ the full subcategory of $\mathcal{C}$ consisting of injective objects in $\mathcal{C}$, then we have, moreover, an equivalence of categories $K^+(\mathcal{I}) \simeq D^+(\mathcal{C})$. This follows from the basic fact that quasi-isomorphisms in $K^+(\mathcal{I})$ are homotopy equivalences. Using this explicit description of derived functors, we obtain the following useful composition rule.

**Proposition B.5.7.** *Let $\mathcal{C}, \mathcal{C}', \mathcal{C}''$ be abelian categories and $F : \mathcal{C} \to \mathcal{C}'$, $G : \mathcal{C}' \to \mathcal{C}''$ left exact functors. Assume that $\mathcal{C}$ (resp. $\mathcal{C}'$) has an $F$-injective (resp. $G$-injective) subcategory $\mathcal{J}$ (resp. $\mathcal{J}'$) and $F(X) \in \mathrm{Ob}(\mathcal{J}')$ for any $X \in \mathrm{Ob}(\mathcal{J})$. Then $\mathcal{J}$ is $(G \circ F)$-injective and*

$$R(G \circ F) = RG \circ RF : D^+(\mathcal{C}) \longrightarrow D^+(\mathcal{C}'').$$

**Definition B.5.8.** Assume that a left exact functor $F : \mathcal{C} \to \mathcal{C}'$ is right derivable. Then for each $n \in \mathbb{Z}$ we set

$$R^n F = H^n \circ RF : D^+(\mathcal{C}) \to \mathcal{C}'.$$

Since $RF : D^+(\mathcal{C}) \to D^+(\mathcal{C}')$ sends distinguished triangles to distinguished triangles, the functors $R^n F : D^+(\mathcal{C}) \to \mathcal{C}'$ defined above are cohomological functors. Now let us identify $\mathcal{C}$ with the full subcategory of $D^+(\mathcal{C})$ consisting of complexes concentrated in degree 0. Then we find that for $X \in \mathrm{Ob}(\mathcal{C})$, $R^n F(X) \in \mathrm{Ob}(\mathcal{C}')$ coincides with the $n$th derived functor of $F$ in the classical literature.

## B.6 Bifunctors in derived categories

In this section we introduce some important bifunctors in derived categories which will be frequently used throughout this book. First, let us explain the bifunctor $\mathrm{RHom}(\bullet, \bullet)$. Let $\mathcal{C}$ be an abelian category. For two complexes $X^\cdot, Y^\cdot \in \mathrm{Ob}(C(\mathcal{C}))$ in $\mathcal{C}$ define a new complex $\mathrm{Hom}^\cdot(X^\cdot, Y^\cdot) \in \mathrm{Ob}(C(\mathcal{A}b))$ by

$$\begin{cases} \mathrm{Hom}^n(X^\cdot, Y^\cdot) = \prod_{j-i=n} \mathrm{Hom}_{\mathcal{C}}(X^i, Y^j) \\ d^n = d^n_{\mathrm{Hom}^\cdot(X^\cdot, Y^\cdot)} : \prod_{j-i=n} \mathrm{Hom}_{\mathcal{C}}(X^i, Y^j) \longrightarrow \prod_{j-i=n+1} \mathrm{Hom}_{\mathcal{C}}(X^i, Y^j) \\ \qquad\qquad \cup \qquad\qquad\qquad\qquad\qquad\qquad\qquad \cup \\ \qquad\qquad \{f_{ij}\} \longmapsto \quad \{g_{ij} = (-1)^{n+1} f_{i+1,j} \circ d^i_{X^\cdot} + d^{j-1}_{Y^\cdot} \circ f_{i,j-1}\}. \end{cases}$$

This is the simple complex associated to the double complex $\mathrm{Hom}(X^\cdot, Y^\cdot)$, which satisfies the conditions

$$\begin{cases} \mathrm{Ker}\, d^n \simeq \mathrm{Hom}_{C(\mathcal{C})}(X^\cdot, Y^\cdot[n]) \\ \mathrm{Im}\, d^{n-1} \simeq \mathrm{Ht}(X^\cdot, Y^\cdot[n]) \\ H^n[\mathrm{Hom}^\cdot(X^\cdot, Y^\cdot)] \simeq \mathrm{Hom}_{K(\mathcal{C})}(X^\cdot, Y^\cdot[n]). \end{cases}$$

for any $n \in \mathbb{Z}$. We thus defined a bifunctor

$$\mathrm{Hom}^{\cdot}(\bullet,\bullet) : C(\mathcal{C})^{\mathrm{op}} \times C(\mathcal{C}) \longrightarrow C(\mathcal{A}b),$$

where $(\bullet)^{\mathrm{op}}$ denotes the *opposite category*. It is easy to check that it induces also a bifunctor

$$\mathrm{Hom}^{\cdot}(\bullet,\bullet) : K(\mathcal{C})^{\mathrm{op}} \times K(\mathcal{C}) \longrightarrow K(\mathcal{A}b)$$

in homotopy categories. Similarly we also obtain

$$\mathrm{Hom}^{\cdot}(\bullet,\bullet) : K^{-}(\mathcal{C})^{\mathrm{op}} \times K^{+}(\mathcal{C}) \longrightarrow K^{+}(\mathcal{A}b),$$

taking boundedness into account. From now on, assume that the category $\mathcal{C}$ has enough injectives and denote by $\mathcal{I}$ the full subcategory of $\mathcal{C}$ consisting of injective objects. The following lemma is elementary.

**Lemma B.6.1.** *Let $X^{\cdot} \in \mathrm{Ob}(K(\mathcal{C}))$ and $I^{\cdot} \in \mathrm{Ob}(K^{+}(\mathcal{I}))$. Assume that $X^{\cdot}$ or $I^{\cdot}$ is quasi-isomorphic to 0. Then the complex $\mathrm{Hom}^{\cdot}(X^{\cdot}, Y^{\cdot}) \in \mathrm{Ob}(K(\mathcal{A}b))$ is also quasi-isomorphic to 0.*

Let $X^{\cdot} \in \mathrm{Ob}(K(\mathcal{C}))$ and consider the functor

$$\mathrm{Hom}^{\cdot}(X^{\cdot},\bullet) : K^{+}(\mathcal{C}) \longrightarrow K(\mathcal{A}b).$$

Then by Lemmas B.2.3 and B.6.1 and Proposition B.5.2, we see that this functor induces a functor

$$R_{II}\,\mathrm{Hom}^{\cdot}(X^{\cdot},\bullet) : D^{+}(\mathcal{C}) \longrightarrow D(\mathcal{A}b)$$

between derived categories. Here we write "$R_{II}$" to indicate that we localize the bifunctor $\mathrm{Hom}^{\cdot}(\bullet,\bullet)$ with respect to the second factor. Since this construction is functorial with respect to $X^{\cdot}$, we obtain a bifunctor

$$R_{II}\,\mathrm{Hom}^{\cdot}(\bullet,\bullet) : K(\mathcal{C})^{\mathrm{op}} \times D^{+}(\mathcal{C}) \longrightarrow D(\mathcal{A}b).$$

By the universal properties of the localization $Q : K(\mathcal{C}) \to D(\mathcal{C})$, this bifunctor factorizes through $Q$ and we obtain a bifunctor

$$R_I R_{II}\,\mathrm{Hom}^{\cdot}(\bullet,\bullet) : D(\mathcal{C})^{\mathrm{op}} \times D^{+}(\mathcal{C}) \longrightarrow D(\mathcal{A}b).$$

We set $\mathrm{RHom}_{\mathcal{C}}(\bullet,\bullet) = R_I R_{II}\,\mathrm{Hom}^{\cdot}(\bullet,\bullet)$ and call it the functor RHom. Similarly taking boundedness into account, we also obtain a bifunctor

$$\mathrm{RHom}_{\mathcal{C}}(\bullet,\bullet) : D^{-}(\mathcal{C})^{\mathrm{op}} \times D^{+}(\mathcal{C}) \longrightarrow D^{+}(\mathcal{A}b).$$

These are bifunctors of triangulated categories. The following proposition is very useful to construct canonical morphisms in derived categories.

**Proposition B.6.2.** *For $Z^{\cdot} \in \mathrm{Ob}(K^{+}(\mathcal{C}))$ and $I^{\cdot} \in \mathrm{Ob}(K^{+}(\mathcal{I}))$ the natural morphism*

$$Q : \mathrm{Hom}_{K^{+}(\mathcal{C})}(Z^{\cdot}, I^{\cdot}) \longrightarrow \mathrm{Hom}_{D^{+}(\mathcal{C})}(Z^{\cdot}, I^{\cdot})$$

*is an isomorphism. In particular for any $X^{\cdot}, Y^{\cdot} \in \mathrm{Ob}(D^{+}(\mathcal{C}))$ and $n \in \mathbb{Z}$ there exists a natural isomorphism*

$$H^n\mathrm{RHom}_{\mathcal{C}}(X^{\cdot}, Y^{\cdot}) = H^n\,\mathrm{Hom}^{\cdot}(X^{\cdot}, I^{\cdot}) \xrightarrow{\sim} \mathrm{Hom}_{D^{+}(\mathcal{C})}(X^{\cdot}, Y^{\cdot}[n]),$$

*where $I^{\cdot}$ is an injective resolution of $Y^{\cdot}$.*

In the classical literature we denote $H^n\mathrm{RHom}_{\mathcal{C}}(X^\cdot, Y^\cdot)$ by $Ext^n_{\mathcal{C}}(X^\cdot, Y^\cdot)$ and call it the $n$th *hyperextension group* of $X^\cdot$ and $Y^\cdot$.

Next we shall explain the bifunctor $\bullet \otimes^L \bullet$. Let $T_0$ be a topological space and $\mathcal{R}$ a sheaf of rings on $T_0$. Denote by $\mathrm{Mod}(\mathcal{R})$ (resp. $\mathrm{Mod}(\mathcal{R}^{\mathrm{op}})$) the abelian category of sheaves of left (resp. right) $\mathcal{R}$-modules on $T_0$. Here $\mathcal{R}^{\mathrm{op}}$ denotes the opposite ring of $\mathcal{R}$. Set $\mathcal{C}_1 = \mathrm{Mod}(\mathcal{R}^{\mathrm{op}})$ and $\mathcal{C}_2 = \mathrm{Mod}(\mathcal{R})$. Then there exists a bifunctor of tensor products

$$\bullet \otimes_{\mathcal{R}} \bullet : \mathcal{C}_1 \times \mathcal{C}_2 \longrightarrow \mathrm{Sh}(T_0).$$

For two complexes $X^\cdot \in \mathrm{Ob}(C(\mathcal{C}_1))$ and $Y^\cdot \in \mathrm{Ob}(C(\mathcal{C}_2))$ we define a new complex $(X^\cdot \otimes_{\mathcal{R}} Y^\cdot)^\cdot \in \mathrm{Ob}(C(\mathrm{Sh}(T_0)))$ by

$$\begin{cases} (X^\cdot \otimes_{\mathcal{R}} Y^\cdot)^n = \displaystyle\prod_{i+j=n} X^i \otimes_{\mathcal{R}} Y^j \\ d^n = d^n_{(X^\cdot \otimes_{\mathcal{R}} Y^\cdot)^\cdot} : \displaystyle\prod_{i+j=n} X^i \otimes_{\mathcal{R}} Y^j \longrightarrow \prod_{i+j=n+1} X^i \otimes_{\mathcal{R}} Y^j \\ \qquad\qquad \cup \qquad\qquad\qquad\qquad \cup \\ \qquad \{x_i \otimes y_j\} \longmapsto \quad \big\{d^i_{X^\cdot}(x_i) \times y_j + (-1)^i x_i \otimes d^j_{Y^\cdot}(y_j)\big\}. \end{cases}$$

This is the simple complex associated to the double complex $X^\cdot \otimes_{\mathcal{R}} Y^\cdot$. We thus defined a bifunctor

$$(\bullet \otimes_{\mathcal{R}} \bullet)^\cdot : C(\mathcal{C}_1) \times C(\mathcal{C}_2) \longrightarrow C(\mathrm{Sh}(T_0))$$

which also induces a bifunctor

$$(\bullet \otimes_{\mathcal{R}} \bullet)^\cdot : K(\mathcal{C}_1) \times K(\mathcal{C}_2) \longrightarrow K(\mathrm{Sh}(T_0))$$

in homotopy categories. Similarly we also obtain

$$(\bullet \otimes_{\mathcal{R}} \bullet)^\cdot : K^-(\mathcal{C}_1) \times K^-(\mathcal{C}_2) \longrightarrow K^-(\mathrm{Sh}(T_0)),$$

taking boundedness into account. Note that in general the abelian categories $\mathcal{C}_1 = \mathrm{Mod}(\mathcal{R}^{\mathrm{op}})$ and $\mathcal{C}_2 = \mathrm{Mod}(\mathcal{R})$ do not have enough projectives (unless the topological space $T_0$ consists of a point). So we cannot use projective objects in these categories to derive the above bifunctor. However, it is well known that for any $Y \in \mathrm{Ob}(\mathcal{C}_2)$ there exist a flat $\mathcal{R}$-module $P \in \mathrm{Ob}(\mathcal{C}_2)$ and an epimorphism $P \to Y$. Therefore, we can use the full subcategory $\mathcal{P}$ of $\mathcal{C}_2$ consisting of flat $\mathcal{R}$-modules.

**Lemma B.6.3.** *Let $X^\cdot \in \mathrm{Ob}(K(\mathcal{C}_1))$ and $P^\cdot \in \mathrm{Ob}(K^-(\mathcal{P}))$. Assume that $X^\cdot$ or $P^\cdot$ is quasi-isomorphic to 0. Then the complex $(X^\cdot \otimes_{\mathcal{R}} P^\cdot)^\cdot \in \mathrm{Ob}(K(\mathrm{Sh}(T_0)))$ is also quasi-isomorphic to 0.*

By this lemma and previous arguments we obtain a bifunctor

$$\bullet \otimes^L_{\mathcal{R}} \bullet : D(\mathcal{C}_1) \times D^-(\mathcal{C}_2) \longrightarrow D(\mathrm{Sh}(T_0))$$

in derived categories. Taking boundedness into account, we also obtain a bifunctor

$$\bullet \otimes^L_{\mathcal{R}} \bullet : D^-(\mathcal{C}_1) \times D^-(\mathcal{C}_2) \longrightarrow D^-(\mathrm{Sh}(T_0)).$$

These are bifunctors of triangulated categories. In the classical literature we denote $H^{-n}(X^\cdot \otimes^L_{\mathcal{R}} Y^\cdot)$ by $Tor^{\mathcal{R}}_n(X^\cdot, Y^\cdot)$ and call it the $n$th *hypertorsion group* of $X^\cdot$ and $Y^\cdot$.

# C

# Sheaves and Functors in Derived Categories

In this appendix, assuming only few prerequisites for sheaf theory, we introduce basic operations of sheaves in derived categories and their main properties without proofs. For the details we refer to Hartshorne [Ha1], Iversen [Iv], Kashiwara–Schapira [KS2], [KS4]. We also give a proof of Kashiwara's non-characteristic deformation lemma.

## C.1 Sheaves and functors

In this section we quickly recall basic operations in sheaf theory. For a topological space $X$ we denote by $\mathrm{Sh}(X)$ the abelian category of sheaves on $X$. The abelian group of sections of $F \in \mathrm{Sh}(X)$ on an open subset $U \subset X$ is denoted by $F(U)$ or $\Gamma(U, F)$, and the subgroup of $\Gamma(U, F)$ consisting of sections with compact supports is denoted by $\Gamma_c(U, F)$. We thus obtain left exact functors $\Gamma(U, \bullet), \Gamma_c(U, \bullet) : \mathrm{Sh}(X) \to \mathcal{A}b$ for each open subset $U \subset X$, where $\mathcal{A}b$ denotes the abelian category of abelian groups. If $\mathcal{R}$ is a sheaf of rings on $X$, we denote by $\mathrm{Mod}(\mathcal{R})$ (resp. $\mathrm{Mod}(\mathcal{R}^{\mathrm{op}})$) the abelian category of sheaves of left (resp. right) $\mathcal{R}$-modules on $X$. Here $\mathcal{R}^{\mathrm{op}}$ denotes the opposite ring of $\mathcal{R}$. For example, in the case where $\mathcal{R}$ is the constant sheaf $\mathbb{Z}_X$ with germs $\mathbb{Z}$ the category $\mathrm{Mod}(\mathcal{R})$ is $\mathrm{Sh}(X)$. For $F, G \in \mathrm{Sh}(X)$ (resp. $M, N \in \mathrm{Mod}(\mathcal{R})$) we denote by $\mathrm{Hom}(F, G)$ (resp. $\mathrm{Hom}_{\mathcal{R}}(M, N)$) the abelian group of sheaf homomorphisms (resp. sheaf homomorphisms commuting with the actions of $\mathcal{R}$) on $X$ from $F$ to $G$ (resp. from $M$ to $N$). We thus obtain left exact bifunctors

$$\begin{cases} \mathrm{Hom}(\bullet, \bullet) : \mathrm{Sh}(X)^{\mathrm{op}} \times \mathrm{Sh}(X) \longrightarrow \mathcal{A}b, \\ \mathrm{Hom}_{\mathcal{R}}(\bullet, \bullet) : \mathrm{Mod}(\mathcal{R})^{\mathrm{op}} \times \mathrm{Mod}(\mathcal{R}) \longrightarrow \mathcal{A}b. \end{cases}$$

For a subset $Z \subset X$, we denote by $i_Z : Z \to X$ the inclusion map.

**Definition C.1.1.** Let $f : X \to Y$ be a morphism of topological spaces, $F \in \mathrm{Sh}(X)$ and $G \in \mathrm{Sh}(Y)$.

(i) The *direct image* $f_*F \in \mathrm{Sh}(Y)$ of $F$ by $f$ is defined by $f_*F(V) = F(f^{-1}(V))$ for each open subset $V \subset Y$. This gives a left exact functor $f_* : \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$. If $Y$ is the space pt consisting of one point, the functor $f_*$ is the global section functor $\Gamma(X, \bullet) : \mathrm{Sh}(X) \to \mathcal{A}b$.

(ii) The *proper direct image* $f_!F \in \mathrm{Sh}(Y)$ of $F$ by $f$ is defined by $f_!F(V) = \{s \in F(f^{-1}(V)) | f|_{\mathrm{supp}\, s} : \mathrm{supp}\, s \to V \quad \text{is proper}\}$ for each open subset $V \subset Y$. This gives a left exact functor $f_! : \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$. If $Y$ is pt, the functor $f_!$ is the global section functor with compact supports $\Gamma_c(X, \bullet) : \mathrm{Sh}(X) \to \mathcal{A}b$.

(iii) The *inverse image* $f^{-1}G \in \mathrm{Sh}(X)$ of $G$ by $f$ is the sheaf associated to the presheaf $(f^{-1}G)'$ defined by $(f^{-1}G)'(U) = \varinjlim G(V)$ for each open subset $U \subset X$, where $V$ ranges through the family of open subsets of $Y$ containing $f(U)$. Since we have an isomorphism $(f^{-1}G)_x \simeq G_{f(x)}$ for any $x \in X$, we obtain an exact functor $f^{-1} : \mathrm{Sh}(Y) \to \mathrm{Sh}(X)$.

When we treat proper direct images $f_!$ in this book, all topological spaces are assumed to be locally compact and Hausdorff. For two morphisms $f : X \to Y$, $g : Y \to Z$ of topological spaces, we have obvious relations $g_* \circ f_* = (g \circ f)_*$, $g_! \circ f_! = (g \circ f)_!$ and $f^{-1} \circ g^{-1} = (g \circ f)^{-1}$. For $F \in \mathrm{Sh}(X)$ and a subset $Z \subset X$ the inverse image $i_Z^{-1}F \in \mathrm{Sh}(Z)$ of $F$ by the inclusion map $i_Z : Z \to X$ is sometimes denoted by $F|_Z$. If $Z$ is a locally closed subset of $X$ (i.e., a subset of $X$ which is written as an intersection of an open subset and a closed subset), then it is well known that the functor $(i_Z)_! : \mathrm{Sh}(Z) \to \mathrm{Sh}(X)$ is exact.

**Proposition C.1.2.** *Let*

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array}$$

*be a* **cartesian square** *of topological spaces, i.e., $X'$ is homeomorphic to the fiber product $X \times_Y Y'$. Then there exists an isomorphism of functors $g^{-1} \circ f_! \simeq f'_! \circ g'^{-1} : \mathrm{Sh}(X) \to \mathrm{Sh}(Y')$.*

**Definition C.1.3.** Let $X$ be a topological space, $Z \subset X$ a locally closed subset and $F \in \mathrm{Sh}(X)$.

(i) Set $F_Z = (i_Z)_!(i_Z)^{-1}F \in \mathrm{Sh}(X)$. Since we have $(F_Z)_x \simeq F_x$ (resp. $(F_Z)_x \simeq 0$) for any $x \in Z$ (resp. $x \in X \setminus Z$), we obtain an exact functor $(\bullet)_Z : \mathrm{Sh}(X) \to \mathrm{Sh}(X)$.

(ii) Take an open subset $W$ of $X$ containing $Z$ as a closed subset of $W$. Since the abelian group $\mathrm{Ker}[F(W) \to F(W \setminus Z)]$ does not depend on the choice of $W$, we denote it by $\Gamma_Z(X, F)$. This gives a left exact functor $\Gamma_Z(X, \bullet) : \mathrm{Sh}(X) \to \mathcal{A}b$.

(iii) The subsheaf $\Gamma_Z(F)$ of $F$ is defined by $\Gamma_Z(F)(U) = \Gamma_{Z \cap U}(U, F|_U)$ for each open subset $U \subset X$. This gives a left exact functor $\Gamma_Z(\bullet) : \mathrm{Sh}(X) \to \mathrm{Sh}(X)$. By construction we have an isomorphism of functors $\Gamma(X, \bullet) \circ \Gamma_Z(\bullet) = \Gamma_Z(X, \bullet)$.

Note that if $U$ is an open subset of $X$ and $j = i_U : U \to X$, then there exists an isomorphism of functors $j_* \circ j^{-1} \simeq \Gamma_U(\bullet)$.

**Lemma C.1.4.** *Let $X$ be a topological space, $Z$ a locally closed subset of $X$ and $Z'$ a closed subset of $Z$. Also let $Z_1, Z_2$ (resp. $U_1, U_2$) be closed (resp. open) subsets of $X$ and $F \in \mathrm{Sh}(X)$.*

(i) *There exists a natural exact sequence*

$$0 \longrightarrow F_{Z\setminus Z'} \longrightarrow F_Z \longrightarrow F_{Z'} \longrightarrow 0 \tag{C.1.1}$$

*in* $\mathrm{Sh}(X)$.

(ii) *There exist natural exact sequences*

$$0 \longrightarrow \Gamma_{Z'}(F) \longrightarrow \Gamma_Z(F) \longrightarrow \Gamma_{Z\setminus Z'}(F), \tag{C.1.2}$$

$$0 \longrightarrow \Gamma_{Z_1\cap Z_2}(F) \longrightarrow \Gamma_{Z_1}(F) \oplus \Gamma_{Z_2}(F) \longrightarrow \Gamma_{Z_1\cup Z_2}(F), \tag{C.1.3}$$

$$0 \longrightarrow \Gamma_{U_1\cup U_2}(F) \longrightarrow \Gamma_{U_1}(F) \oplus \Gamma_{U_2}(F) \longrightarrow \Gamma_{U_1\cap U_2}(F). \tag{C.1.4}$$

*in* $\mathrm{Sh}(X)$.

Recall that a sheaf $F \in \mathrm{Sh}(X)$ on $X$ is called *flabby* if the restriction morphism $F(X) \to F(U)$ is surjective for any open subset $U \subset X$.

**Lemma C.1.5.**

(i) *Let $Z$ be a locally closed subset of $X$ and $F \in \mathrm{Sh}(X)$ a flabby sheaf. Then the sheaf $\Gamma_Z(F)$ is flabby. Moreover, for any morphism $f : X \to Y$ of topological spaces the direct image $f_*F$ is flabby.*

(ii) *Let $0 \to F' \to F \to F'' \to 0$ be an exact sequence in $\mathrm{Sh}(X)$ and $Z$ a locally closed subset of $X$. Assume that $F'$ is flabby. Then the sequences $0 \to \Gamma_Z(X, F') \to \Gamma_Z(X, F) \to \Gamma_Z(X, F'') \to 0$ and $0 \to \Gamma_Z(F') \to \Gamma_Z(F) \to \Gamma_Z(F'') \to 0$ are exact.*

(iii) *Let $0 \to F' \to F \to F'' \to 0$ be an exact sequence in $\mathrm{Sh}(X)$ and assume that $F'$ and $F$ are flabby. Then $F''$ is also flabby.*

(iv) *In the situation of Lemma* C.1.4*, assume, moreover, that $F$ is flabby. Then there are natural exact sequences*

$$0 \longrightarrow \Gamma_{Z'}(F) \longrightarrow \Gamma_Z(F) \longrightarrow \Gamma_{Z\setminus Z'}(F) \longrightarrow 0, \tag{C.1.5}$$

$$0 \longrightarrow \Gamma_{Z_1\cap Z_2}(F) \longrightarrow \Gamma_{Z_1}(F) \oplus \Gamma_{Z_2}(F) \longrightarrow \Gamma_{Z_1\cup Z_2}(F) \longrightarrow 0, \tag{C.1.6}$$

$$0 \longrightarrow \Gamma_{U_1\cup U_2}(F) \longrightarrow \Gamma_{U_1}(F) \oplus \Gamma_{U_2}(F) \longrightarrow \Gamma_{U_1\cap U_2}(F) \longrightarrow 0. \tag{C.1.7}$$

*in* $\mathrm{Sh}(X)$.

**Definition C.1.6.** Let $X$ be a topological space and $\mathcal{R}$ a sheaf of rings on $X$.

(i) For $M, N \in \mathrm{Mod}(\mathcal{R})$ the sheaf $\mathcal{H}om_{\mathcal{R}}(M, N) \in \mathrm{Sh}(X)$ of $\mathcal{R}$-linear homomorphisms from $M$ to $N$ is defined by $\mathcal{H}om_{\mathcal{R}}(M, N)(U) = \mathrm{Hom}_{\mathcal{R}|_U}(M|_U, N|_U)$ for each open subset $U \subset X$. This gives a left exact bifunctor $\mathcal{H}om_{\mathcal{R}}(\bullet, \bullet) : \mathrm{Mod}(\mathcal{R})^{\mathrm{op}} \times \mathrm{Mod}(\mathcal{R}) \to \mathrm{Sh}(X)$. By definition we have $\Gamma(X; \mathcal{H}om_{\mathcal{R}}(M, N)) = \mathrm{Hom}_{\mathcal{R}}(M, N)$.

(ii) For $M \in \mathrm{Mod}(\mathcal{R}^{\mathrm{op}})$, $N \in \mathrm{Mod}(\mathcal{R})$ the tensor product $M \otimes_{\mathcal{R}} N \in \mathrm{Sh}(X)$ of $M$ and $N$ is the sheaf associated to the presheaf $(M \otimes_{\mathcal{R}} N)'$ defined by $(M \otimes_{\mathcal{R}} N)'(U) = M(U) \otimes_{\mathcal{R}(U)} N(U)$ for each open subset $U \subset X$. Since by definition we have an isomorphism $(M \otimes_{\mathcal{R}} N)_x \simeq M_x \otimes_{\mathcal{R}_x} N_x$ for any $x \in X$, we obtain a right exact bifunctor $\bullet \otimes_{\mathcal{R}} \bullet : \mathrm{Mod}(\mathcal{R}^{\mathrm{op}}) \times \mathrm{Mod}(\mathcal{R}) \to \mathrm{Sh}(X)$.

Note that for any $M \in \mathrm{Mod}(\mathcal{R})$ the sheaf $\mathcal{H}om_{\mathcal{R}}(\mathcal{R}, M)$ is a left $\mathcal{R}$-module by the right multiplication of $\mathcal{R}$ on $\mathcal{R}$ itself, and there exists an isomorphism $\mathcal{H}om_{\mathcal{R}}(\mathcal{R}, M) \simeq M$ of left $\mathcal{R}$-modules. Now recall that $M \in \mathrm{Mod}(\mathcal{R})$ is an injective (resp. a projective) object of $\mathrm{Mod}(\mathcal{R})$ if the functor $\mathrm{Hom}_{\mathcal{R}}(\bullet, M)$ (resp. $\mathrm{Hom}_{\mathcal{R}}(M, \bullet)$) is exact.

**Proposition C.1.7.** *Let $\mathcal{R}$ be a sheaf of rings on $X$. Then the abelian category* $\mathrm{Mod}(\mathcal{R})$ *has* **enough injectives**.

An injective object in $\mathrm{Mod}(\mathcal{R})$ is sometimes called an *injective sheaf* or an injective $\mathcal{R}$-module.

**Definition C.1.8.** We say that $M \in \mathrm{Mod}(\mathcal{R})$ is *flat* (or a flat $\mathcal{R}$-module) if the functor $\bullet \otimes_{\mathcal{R}} M : \mathrm{Mod}(\mathcal{R}^{\mathrm{op}}) \to \mathrm{Sh}(X)$ is exact.

By the definition of tensor products, $M \in \mathrm{Mod}(\mathcal{R})$ is flat if and only if the stalk $M_x$ is a flat $\mathcal{R}_x$-module for any $x \in X$. Although in general the category $\mathrm{Mod}(\mathcal{R})$ does not have enough projectives (unless $X$ is the space pt consisting of one point), we have the following useful result.

**Proposition C.1.9.** *Let $\mathcal{R}$ be a sheaf of rings on $X$. Then for any $M \in \mathrm{Mod}(\mathcal{R})$ there exist a flat $\mathcal{R}$-module $P$ and an epimorphism $P \to M$.*

**Lemma C.1.10.** *Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence in* $\mathrm{Mod}(\mathcal{R})$.

(i) *Assume that $M'$ and $M$ are injective. Then $M''$ is also injective.*
(ii) *Assume that $M$ and $M''$ are flat. Then $M'$ is also flat.*

**Proposition C.1.11.** *Let $f : Y \to X$ be a morphism of topological spaces and $\mathcal{R}$ a sheaf of rings on $X$.*

(i) *Let $M_1 \in \mathrm{Mod}(\mathcal{R}^{\mathrm{op}})$ and $M_2 \in \mathrm{Mod}(\mathcal{R})$. Then there exists an isomorphism*

$$f^{-1}M_1 \otimes_{f^{-1}\mathcal{R}} f^{-1}M_2 \simeq f^{-1}(M_1 \otimes_{\mathcal{R}} M_2) \tag{C.1.8}$$

*in* $\mathrm{Sh}(Y)$.
(ii) *Let $M \in \mathrm{Mod}(\mathcal{R})$ and $N \in \mathrm{Mod}(f^{-1}\mathcal{R})$. Then there exists an isomorphism*

$$\mathcal{H}om_{\mathcal{R}}(M, f_*N) \simeq f_*\mathcal{H}om_{f^{-1}\mathcal{R}}(f^{-1}M, N) \tag{C.1.9}$$

*in* $\mathrm{Sh}(X)$, *where $f_*N$ is a left $\mathcal{R}$-module by the natural ring homomorphism $\mathcal{R} \to f_*f^{-1}\mathcal{R}$. In particular we have an isomorphism*

$$\mathrm{Hom}_{\mathcal{R}}(M, f_*N) \simeq \mathrm{Hom}_{f^{-1}\mathcal{R}}(f^{-1}M, N). \tag{C.1.10}$$

*Namely, the functor $f_*$ is a right adjoint of $f^{-1}$.*

(iii) *Let* $M \in \mathrm{Mod}(\mathcal{R}^{\mathrm{op}})$ *and* $N \in \mathrm{Mod}(f^{-1}\mathcal{R})$. *Then there exists a natural morphism*

$$M \otimes_{\mathcal{R}} f_!N \longrightarrow f_!(f^{-1}M \otimes_{f^{-1}\mathcal{R}} N) \tag{C.1.11}$$

*in* $\mathrm{Sh}(X)$. *Moreover, this morphism is an isomorphism if* $M$ *is a flat* $\mathcal{R}^{\mathrm{op}}$*-module.*

**Corollary C.1.12.** *Let* $f : Y \to X$ *be a morphism of topological spaces,* $\mathcal{R}$ *a sheaf of rings on* $X$ *and* $N \in \mathrm{Mod}(f^{-1}\mathcal{R})$ *an injective* $f^{-1}\mathcal{R}$*-module. Then the direct image* $f_*N$ *is an injective* $\mathcal{R}$*-module.*

**Lemma C.1.13.** *Let* $Z$ *be a locally closed subset of* $X$, $\mathcal{R}$ *a sheaf of rings on* $X$ *and* $M, N \in \mathrm{Mod}(\mathcal{R})$. *Then we have natural isomorphisms*

$$\Gamma_Z\mathcal{H}om_{\mathcal{R}}(M, N) \simeq \mathcal{H}om_{\mathcal{R}}(M, \Gamma_Z N) \simeq \mathcal{H}om_{\mathcal{R}}(M_Z, N). \tag{C.1.12}$$

**Corollary C.1.14.** *Let* $\mathcal{R}$ *be a sheaf of rings on* $X$, $Z$ *a locally closed subset of* $X$ *and* $M, N \in \mathrm{Mod}(\mathcal{R})$. *Assume that* $N$ *is an injective* $\mathcal{R}$*-module. Then the sheaf* $\mathcal{H}om_{\mathcal{R}}(M, N)$ (*resp.* $\Gamma_Z N$) *is flabby* (*resp. an injective* $\mathcal{R}$*-module*). *In particular, any injective* $\mathcal{R}$*-module is flabby.*

## C.2 Functors in derived categories of sheaves

Applying the results in Appendix B to functors of operations of sheaves, we can introduce various functors in derived categories of sheaves as follows.

Let $X$ be a topological space and $\mathcal{R}$ a sheaf of rings on $X$. Since the category $\mathrm{Mod}(\mathcal{R})$ is abelian, we obtain a derived category $D(\mathrm{Mod}(\mathcal{R}))$ of complexes in $\mathrm{Mod}(\mathcal{R})$ and its full subcategories $D^{\#}(\mathrm{Mod}(\mathcal{R}))$ ($\# = +, -, b$). In this book for $\# = \emptyset, +, -, b$ we sometimes denote $D^{\#}(\mathrm{Mod}(\mathcal{R}))$ by $D^{\#}(\mathcal{R})$ for the sake of simplicity. For example, we set $D^+(\mathbb{Z}_X) = D^+(\mathrm{Sh}(X))$. Now let $Z$ be a locally closed subset of $X$, and $f : Y \to X$ a morphism of topological spaces. Consider the following left exact functors:

$$\begin{cases} \Gamma(X, \bullet), \Gamma_c(X, \bullet), \Gamma_Z(X, \bullet) : \mathrm{Mod}(\mathcal{R}) \longrightarrow \mathcal{A}b, \\ \Gamma_Z(\bullet) : \mathrm{Mod}(\mathcal{R}) \longrightarrow \mathrm{Mod}(\mathcal{R}), \\ f_*, f_! : \mathrm{Mod}(f^{-1}\mathcal{R}) \longrightarrow \mathrm{Mod}(\mathcal{R}). \end{cases} \tag{C.2.1}$$

Since the categories $\mathrm{Mod}(\mathcal{R})$ and $\mathrm{Mod}(f^{-1}\mathcal{R})$ have enough injectives we obtain their derived functors

$$\begin{cases} \mathrm{R}\Gamma(X, \bullet), \mathrm{R}\Gamma_c(X, \bullet), \mathrm{R}\Gamma_Z(X, \bullet) : D^+(\mathcal{R}) \longrightarrow D^+(\mathcal{A}b), \\ \mathrm{R}\Gamma_Z(\bullet) : D^+(\mathcal{R}) \longrightarrow D^+(\mathcal{R}), \\ Rf_*, Rf_! : D^+(f^{-1}\mathcal{R}) \longrightarrow D^+(\mathcal{R}). \end{cases} \tag{C.2.2}$$

For example, for $M^\cdot \in D^+(\mathcal{R})$ the object $\mathrm{R}\Gamma(X, F^\cdot) \in D^+(\mathcal{A}b)$ is calculated as follows. First take a quasi-isomorphism $M^\cdot \xrightarrow{\sim} I^\cdot$ in $C^+(\mathcal{R})$ such that $I^k$ is an injective $\mathcal{R}$-module for any $k \in \mathbb{Z}$. Then we have $\mathrm{R}\Gamma(X, F^\cdot) \simeq \Gamma(X, I^\cdot)$.

Since by Lemma C.1.5 the full subcategory $\mathcal{J}$ of $\mathrm{Mod}(\mathcal{R})$ consisting of flabby sheaves is $\Gamma(X, \bullet)$-injective in the sense of Definition B.5.4, we can also take a quasi-isomorphism $M^\cdot \xrightarrow{\sim} J^\cdot$ in $C^+(\mathcal{R})$ such that $J^k \in \mathcal{J}$ for any $k \in \mathbb{Z}$ and show that $\mathrm{R}\Gamma(X, F^\cdot) \simeq \Gamma(X, J^\cdot)$. Let us apply Proposition B.5.7 to the identity $\Gamma_Z(X, \bullet) = \Gamma(X, \bullet) \circ \Gamma_Z(\bullet) : \mathrm{Mod}(\mathcal{R}) \to \mathcal{A}b$. Then by the fact that the functor $\Gamma_Z(\bullet) \simeq \mathcal{H}om_{\mathcal{R}}(\mathcal{R}_Z, \bullet)$ sends injective sheaves to injective sheaves (Corollary C.1.14), we obtain an isomorphism

$$\mathrm{R}\Gamma(X, \mathrm{R}\Gamma_Z(M^\cdot)) \simeq \mathrm{R}\Gamma_Z(X, M^\cdot) \tag{C.2.3}$$

in $D^+(\mathcal{A}b)$ for any $M^\cdot \in D^+(\mathcal{R})$. Similarly by Corollary C.1.12 we obtain an isomorphism

$$\mathrm{R}\Gamma(X, Rf_*(N^\cdot)) \simeq \mathrm{R}\Gamma(Y, N^\cdot) \tag{C.2.4}$$

in $D^+(\mathcal{A}b)$ for any $N^\cdot \in D^+(f^{-1}\mathcal{R})$ (also the similar formula $\mathrm{R}\Gamma_c(X, Rf_!(N^\cdot)) \simeq \mathrm{R}\Gamma_c(Y, N^\cdot)$ can be proved). For $M^\cdot \in D^+(\mathcal{R})$ and $i \in \mathbb{Z}$ we sometimes denote $H^i\mathrm{R}\Gamma(X, M^\cdot)$, $H^i\mathrm{R}\Gamma_Z(X, M^\cdot)$, $H^i\mathrm{R}\Gamma_Z(M^\cdot)$ simply by $H^i(X, M^\cdot)$, $H^i_Z(X, M^\cdot)$, $H^i_Z(M^\cdot)$, respectively. Now let us consider the functors

$$\begin{cases} f^{-1} : \mathrm{Mod}(\mathcal{R}) \longrightarrow \mathrm{Mod}(f^{-1}\mathcal{R}), \\ (\bullet)_Z : \mathrm{Mod}(\mathcal{R}) \longrightarrow \mathrm{Mod}(\mathcal{R}), \\ (i_Z)_! : \mathrm{Sh}(Z) \to \mathrm{Sh}(X). \end{cases} \tag{C.2.5}$$

Since these functors are exact, they extend naturally to the following functors in derived categories:

$$\begin{cases} f^{-1} : D^{\#}(\mathcal{R}) \longrightarrow D^{\#}(f^{-1}\mathcal{R}), \\ (\bullet)_Z : D^{\#}(\mathcal{R}) \longrightarrow D^{\#}(\mathcal{R}), \\ (i_Z)_! : D^{\#}(\mathrm{Sh}(Z)) \to D^{\#}(\mathrm{Sh}(X)) \end{cases} \tag{C.2.6}$$

for $\# = \emptyset, +, -, b$. Let $Z'$ be a closed subset of $Z$ and $Z_1$, $Z_2$ (resp. $U_1$, $U_2$) closed (resp. open) subsets of $X$. Then by Lemma C.1.5 and Lemma B.4.5, for $M^\cdot \in D^+(\mathcal{R})$ we obtain the following distinguished triangles in $D^+(\mathcal{R})$:

$$M_{Z\setminus Z'}{}^\cdot \longrightarrow M_Z{}^\cdot \longrightarrow M_{Z'}{}^\cdot \xrightarrow{+1}, \tag{C.2.7}$$

$$\mathrm{R}\Gamma_{Z'}(M^\cdot) \longrightarrow \mathrm{R}\Gamma_Z(M^\cdot) \longrightarrow \mathrm{R}\Gamma_{Z\setminus Z'}(M^\cdot) \xrightarrow{+1}, \tag{C.2.8}$$

$$\mathrm{R}\Gamma_{Z_1\cap Z_2}(M^\cdot) \longrightarrow \mathrm{R}\Gamma_{Z_1}(M^\cdot) \oplus \mathrm{R}\Gamma_{Z_2}(M^\cdot) \longrightarrow \mathrm{R}\Gamma_{Z_1\cup Z_2}(M^\cdot) \xrightarrow{+1}, \tag{C.2.9}$$

$$\mathrm{R}\Gamma_{U_1\cup U_2}(M^\cdot) \longrightarrow \mathrm{R}\Gamma_{U_1}(M^\cdot) \oplus \mathrm{R}\Gamma_{U_2}(M^\cdot) \longrightarrow \mathrm{R}\Gamma_{U_1\cap U_2}(M^\cdot) \xrightarrow{+1}. \tag{C.2.10}$$

The following result is also well known.

**Proposition C.2.1.** *Let*

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array}$$

*be a cartesian square of topological spaces. Then there exists an isomorphism of functors* $g^{-1} \circ Rf_! \simeq Rf'_! \circ g'^{-1} : D^+(\mathrm{Sh}(X)) \to D^+(\mathrm{Sh}(Y'))$.

For the proof see [KS2, Proposition 2.6.7]. From now on, let us introduce bifunctors in derived categories of sheaves. Let $\mathcal{R}$ be a sheaf of rings on a topological space $X$. Then by applying the construction in Section B.6 to $\mathcal{C} = \mathrm{Mod}(\mathcal{R})$ we obtain a bifunctor

$$\mathrm{RHom}_{\mathcal{R}}(\bullet, \bullet) : D^-(\mathcal{R})^{\mathrm{op}} \times D^+(\mathcal{R}) \longrightarrow D^+(\mathcal{A}b). \tag{C.2.11}$$

Similarly we obtain a bifunctor

$$R\mathcal{H}om_{\mathcal{R}}(\bullet, \bullet) : D^-(\mathcal{R})^{\mathrm{op}} \times D^+(\mathcal{R}) \longrightarrow D^+(\mathrm{Sh}(X)). \tag{C.2.12}$$

For $M^\cdot \in D^-(\mathcal{R})$ and $N^\cdot \in D^+(\mathcal{R})$ the objects $\mathrm{RHom}_{\mathcal{R}}(M^\cdot, N^\cdot) \in D^+(\mathcal{A}b)$ and $R\mathcal{H}om_{\mathcal{R}}(M^\cdot, N^\cdot) \in D^+(\mathrm{Sh}(X))$ are more explicitly calculated as follows. Take a quasi-isomorphism $N^\cdot \xrightarrow{\sim} I^\cdot$ such that $I^k$ is an injective $\mathcal{R}$-module for any $k \in \mathbb{Z}$ and consider the simple complex $\mathrm{Hom}^\cdot_{\mathcal{R}}(M^\cdot, I^\cdot) \in C^+(\mathcal{A}b)$ (resp. $\mathcal{H}om^\cdot_{\mathcal{R}}(M^\cdot, I^\cdot) \in C^+(\mathrm{Sh}(X))$) associated to the double complex $\mathrm{Hom}_{\mathcal{R}}(M^\cdot, I^\cdot)$ (resp. $\mathcal{H}om_{\mathcal{R}}(M^\cdot, I^\cdot)$) as in Section B.6. Then we have isomorphisms $\mathrm{RHom}_{\mathcal{R}}(M^\cdot, N^\cdot) \simeq \mathrm{Hom}^\cdot_{\mathcal{R}}(M^\cdot, I^\cdot)$ and $R\mathcal{H}om_{\mathcal{R}}(M^\cdot, N^\cdot) \simeq \mathcal{H}om^\cdot_{\mathcal{R}}(M^\cdot, I^\cdot)$. For $M^\cdot \in D^-(\mathcal{R})$, $N^\cdot \in D^+(\mathcal{R})$ and $i \in \mathbb{Z}$ we sometimes denote $H^i\mathrm{RHom}_{\mathcal{R}}(M^\cdot, N^\cdot)$, $H^i R\mathcal{H}om_{\mathcal{R}}(M^\cdot, N^\cdot)$ simply by $Ext_{\mathcal{R}}(M^\cdot, N^\cdot)$, $\mathcal{E}xt_{\mathcal{R}}(M^\cdot, N^\cdot)$, respectively. Since the full subcategory of $\mathrm{Sh}(X)$ consisting of flabby sheaves is $\Gamma(X, \bullet)$-injective, from Corollary C.1.14 and the obvious identity $\Gamma(X, \mathcal{H}om_{\mathcal{R}}(\bullet, \bullet)) = \mathrm{Hom}_{\mathcal{R}}(\bullet, \bullet)$, we obtain an isomorphism

$$\mathrm{R}\Gamma(X, R\mathcal{H}om_{\mathcal{R}}(M^\cdot, N^\cdot)) \simeq \mathrm{RHom}_{\mathcal{R}}(M^\cdot, N^\cdot) \tag{C.2.13}$$

in $D^+(\mathcal{A}b)$ for any $M^\cdot \in D^-(\mathcal{R})$ and $N^\cdot \in D^+(\mathcal{R})$. Let us apply the same argument to the identities in Lemma C.1.13. Then by Lemma C.1.5 and Corollary C.1.14 we obtain the following.

**Proposition C.2.2.** *Let $Z$ be a locally closed subset of $X$, $\mathcal{R}$ a sheaf of rings on $X$, $M^\cdot \in D^-(\mathcal{R})$ and $N^\cdot \in D^+(\mathcal{R})$. Then we have isomorphisms*

$$\mathrm{R}\Gamma_Z R\mathcal{H}om_{\mathcal{R}}(M^\cdot, N^\cdot) \simeq R\mathcal{H}om_{\mathcal{R}}(M^\cdot, \mathrm{R}\Gamma_Z N^\cdot) \simeq R\mathcal{H}om_{\mathcal{R}}(M_Z{}^\cdot, N^\cdot). \tag{C.2.14}$$

Similarly, from Proposition C.1.11 (ii) we obtain the following.

**Proposition C.2.3 (Adjunction formula).** *Let $f : Y \to X$ be a morphism of topological spaces, $\mathcal{R}$ a sheaf of rings on $X$. Let $M^\cdot \in D^-(\mathcal{R})$ and $N^\cdot \in D^+(f^{-1}\mathcal{R})$. Then there exists an isomorphism*

$$R\mathcal{H}om_{\mathcal{R}}(M^{\cdot}, Rf_*N^{\cdot}) \simeq Rf_*R\mathcal{H}om_{f^{-1}\mathcal{R}}(f^{-1}M^{\cdot}, N^{\cdot}) \tag{C.2.15}$$

*in* $D^+(\mathrm{Sh}(X))$. *Moreover, we have an isomorphism*

$$\mathrm{RHom}_{\mathcal{R}}(M^{\cdot}, Rf_*N^{\cdot}) \simeq \mathrm{RHom}_{f^{-1}\mathcal{R}}(f^{-1}M^{\cdot}, N^{\cdot}) \tag{C.2.16}$$

*in* $D^+(\mathcal{A}b)$.

By Proposition B.6.2 and the same argument as above, we also obtain the following.

**Proposition C.2.4.** *In the situation of Proposition* C.2.3, *for any* $L^{\cdot} \in D^+(\mathcal{R})$ *and* $N^{\cdot} \in D^+(f^{-1}\mathcal{R})$ *there exists an isomorphism*

$$\mathrm{Hom}_{D^+(\mathcal{R})}(L^{\cdot}, Rf_*N^{\cdot}) \simeq \mathrm{Hom}_{D^+(f^{-1}\mathcal{R})}(f^{-1}L^{\cdot}, N^{\cdot}). \tag{C.2.17}$$

*Namely, the functor* $f^{-1} : D^+(\mathcal{R}) \to D^+(f^{-1}\mathcal{R})$ *is a left adjoint of* $Rf_* : D^+(f^{-1}\mathcal{R}) \to D^+(\mathcal{R})$.

Next we shall introduce the derived functor of the bifunctor of tensor products. Let $X$ be a topological space and $\mathcal{R}$ a sheaf of rings on $X$. Then by the results in Section B.6, there exists a right exact bifunctor of tensor products

$$\bullet \otimes_{\mathcal{R}} \bullet : \mathrm{Mod}(\mathcal{R}^{\mathrm{op}}) \times \mathrm{Mod}(\mathcal{R}) \longrightarrow \mathrm{Sh}(X), \tag{C.2.18}$$

and its derived functor

$$\bullet \otimes^L_{\mathcal{R}} \bullet : D^-(\mathcal{R}^{\mathrm{op}}) \times D^-(\mathcal{R}) \longrightarrow D^-(\mathrm{Sh}(X)). \tag{C.2.19}$$

From now on, let us assume, moreover, that $\mathcal{R}$ has *finite weak global dimension*, i.e., there exists an integer $d > 0$ such that the weak global dimension of the ring $\mathcal{R}_x$ is less than or equal to $d$ for any $x \in X$. Then for any $M^{\cdot} \in C^+(\mathrm{Mod}(\mathcal{R}))$ (resp. $C^b(\mathrm{Mod}(\mathcal{R}))$) we can construct a quasi-isomorphism $P^{\cdot} \to M^{\cdot}$ for $P^{\cdot} \in C^+(\mathrm{Mod}(\mathcal{R}))$ (resp. $C^b(\mathrm{Mod}(\mathcal{R}))$) such that $P^k$ is a flat $\mathcal{R}$-module for any $k \in \mathbb{Z}$. Hence we obtain also bifunctors

$$\bullet \otimes^L_{\mathcal{R}} \bullet : D^{\#}(\mathcal{R}^{\mathrm{op}}) \times D^{\#}(\mathcal{R}) \longrightarrow D^{\#}(\mathrm{Sh}(X)) \tag{C.2.20}$$

for $\# = +, b$. By definition, we immediately obtain the following.

**Proposition C.2.5.** *Let* $f : Y \to X$ *be a morphism of topological spaces and* $\mathcal{R}$ *a sheaf of rings on* $X$. *Let* $M^{\cdot} \in D^-(\mathcal{R}^{\mathrm{op}})$ *and* $N^{\cdot} \in D^-(\mathcal{R})$. *Then there exists an isomorphism*

$$f^{-1}M^{\cdot} \otimes^L_{f^{-1}\mathcal{R}} f^{-1}N^{\cdot} \simeq f^{-1}(M^{\cdot} \otimes^L_{\mathcal{R}} N^{\cdot}) \tag{C.2.21}$$

*in* $D^-(\mathrm{Sh}(Y))$.

The following result is also well known.

**Proposition C.2.6 (Projection formula).** *Let $f : Y \to X$ be a morphism of topological spaces and $\mathcal{R}$ a sheaf of rings on $X$. Assume that $\mathcal{R}$ has finite weak global dimension. Let $M^\cdot \in D^+(\mathcal{R}^{\mathrm{op}})$ and $N^\cdot \in D^+(f^{-1}\mathcal{R})$. Then there exists an isomorphism*

$$M^\cdot \otimes^L_{\mathcal{R}} Rf_! N^\cdot \xrightarrow{\sim} Rf_!(f^{-1}M^\cdot \otimes^L_{f^{-1}\mathcal{R}} N^\cdot) \tag{C.2.22}$$

*in $D^+(\mathrm{Sh}(X))$.*

For the proof see [KS2, Proposition 2.6.6].

Finally, let us explain the *Poincaré–Verdier duality*. Now let $f : X \to Y$ be a continuous map of locally compact and Hausdorff topological spaces. Let $A$ be a commutative ring with finite global dimension, e.g., a field $k$. In what follows, we always assume the following condition for $f$.

**Definition C.2.7.** We say that the functor $f_! : \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$ has *finite cohomological dimension* if there exists an integer $d > 0$ such that for any sheaf $F$ on $X$ we have $H^k Rf_!(F) = 0$ for any $k > d$.

**Theorem C.2.8 (Poincaré–Verdier duality theorem).** *In the situation as above, there exists a functor of triangulated categories $f^! : D^+(A_Y) \to D^+(A_X)$ such that for any $M^\cdot \in D^b(A_X)$ and $N^\cdot \in D^+(A_Y)$ we have isomorphisms*

$$Rf_* R\mathcal{H}om_{A_X}(M^\cdot, f^! N^\cdot) \simeq R\mathcal{H}om_{A_Y}(Rf_! M^\cdot, N^\cdot), \tag{C.2.23}$$

$$\mathrm{RHom}_{A_X}(M^\cdot, f^! N^\cdot) \simeq \mathrm{RHom}_{A_Y}(Rf_! M^\cdot, N^\cdot) \tag{C.2.24}$$

*in $D^+(A_Y)$ and $D^+(\mathrm{Mod}(A))$, respectively.*

We call the functor $f^! : D^+(A_Y) \to D^+(A_X)$ the *twisted inverse image* functor by $f$. Since the construction of this functor $f^!(\bullet)$ is a little bit complicated, we do not explain it here. For the details see Kashiwara–Schapira [KS2, Chapter III]. Let us give basic properties of twisted inverse images. First, for a morphism $g : Y \to Z$ of topological spaces satisfying the same assumption as $f$, we have an isomorphism $(g \circ f)^! \simeq f^! \circ g^!$ of functors.

**Theorem C.2.9.** *Let $f : X \to Y$ be as above. Then for any $M^\cdot \in D^+(A_X)$ and $N^\cdot \in D^+(A_Y)$ we have an isomorphism*

$$\mathrm{Hom}_{D^+(A_X)}(M^\cdot, f^! N^\cdot) \simeq \mathrm{Hom}_{D^+(A_Y)}(Rf_! M^\cdot, N^\cdot).$$

*Namely, the functor $f^! : D^+(A_Y) \to D^+(A_X)$ is a right adjoint of $Rf_! : D^+(A_X) \to D^+(A_Y)$.*

**Proposition C.2.10.** *Let $f : X \to Y$ be as above. Then for any ${N_1}^\cdot \in D^b(A_Y)$ and ${N_2}^\cdot \in D^+(A_Y)$ we have an isomorphism*

$$f^! R\mathcal{H}om_{A_Y}({N_1}^\cdot, {N_2}^\cdot) \simeq R\mathcal{H}om_{A_X}(f^{-1}{N_1}^\cdot, f^! {N_2}^\cdot).$$

**Proposition C.2.11.** *Assume that $X$ is a locally closed subset of $Y$ and let $f = i_X : X \hookrightarrow Y$ be the embedding. Then we have an isomorphism*

$$f^!(N^\cdot) \simeq f^{-1}(\mathrm{R}\Gamma_{f(X)}(N^\cdot)) = (\mathrm{R}\Gamma_{f(X)}(N^\cdot))|_X \tag{C.2.25}$$

*in $D^+(A_X)$ for any $N^\cdot \in D^+(A_Y)$.*

**Proposition C.2.12.** *Assume that $X$ and $Y$ are real $C^1$-manifolds and $f : X \to Y$ is a $C^1$-submersion. Set $d = \dim X - \dim Y$. Then*

(i) *$H^j(f^!(A_Y)) = 0$ for any $j \neq -d$ and $H^{-d}(f^!(A_Y)) \in \mathrm{Mod}(A_X)$ is a locally constant sheaf of rank one over $A_X$.*
(ii) *For any $N^\cdot \in D^+(A_Y)$ there exists an isomorphism*

$$f^!(A_Y) \otimes^L_{A_X} f^{-1}(N^\cdot) \xrightarrow{\sim} f^!(N^\cdot). \tag{C.2.26}$$

**Definition C.2.13.** In the situation of Proposition C.2.12 we set $\mathrm{or}_{X/Y} = H^{-d}(f^!(A_Y)) \in \mathrm{Mod}(A_X)$ and call it the *relative orientation sheaf* of $f : X \to Y$. If, moreover, $Y$ is the space $\{\mathrm{pt}\}$ consisting of one point, we set $\mathrm{or}_X = \mathrm{or}_{X/Y} \in \mathrm{Mod}(A_X)$ and call it the *orientation sheaf* of $X$.

In the situation of Proposition C.2.12 above we thus have an isomorphism $f^!(A_Y) \simeq \mathrm{or}_{X/Y}[\dim X - \dim Y]$ and for any $N^\cdot \in D^+(A_Y)$ there exists an isomorphism

$$f^!(N^\cdot) \simeq \mathrm{or}_{X/Y} \otimes_{A_X} f^{-1}(N^\cdot)[\dim X - \dim Y]. \tag{C.2.27}$$

Note that in the above isomorphism we wrote $\otimes_{A_X}$ instead of $\otimes^L_{A_X}$ because $\mathrm{or}_{X/Y}$ is flat over $A_X$.

**Definition C.2.14.** Let $f : X \to Y$ be as above. Assume, moreover, that $Y$ is the space $\{\mathrm{pt}\}$ consisting of one point and the morphism $f$ is $X \to \{\mathrm{pt}\}$. Then we set $\omega_X{}^\cdot = f^!(A_{\{\mathrm{pt}\}}) \in D^+(A_X)$ and call it the *dualizing complex* of $X$. We sometimes denote $\omega_X{}^\cdot$ simply by $\omega_X$.

To define the dualizing complex $\omega_X{}^\cdot \in D^+(A_X)$ of $X$, we assumed that the functor $f_! : \mathrm{Sh}(X) \to \mathrm{Sh}(\{\mathrm{pt}\}) = \mathcal{A}b$ for $f : X \to \{\mathrm{pt}\}$ has finite cohomological dimension. This assumption is satisfied if $X$ is a topological manifold or a real analytic space. In what follows we assume that all topological spaces that we treat satisfy this assumption.

**Definition C.2.15.** For $M^\cdot \in D^b(A_X)$ we set

$$\mathbf{D}_X(M^\cdot) = R\mathcal{H}om_{A_X}(M^\cdot, \omega_X{}^\cdot) \in D^+(A_X)$$

and call it the *Verdier dual* of $M^\cdot$.

Since for the morphism $f : X \to Y$ of topological spaces we have $f^!\omega_Y{}^\cdot \simeq \omega_X{}^\cdot$, from Proposition C.2.10 we obtain an isomorphism

$$(f^! \circ \mathbf{D}_Y)(N^\cdot) \simeq (\mathbf{D}_X \circ f^{-1})(N^\cdot) \tag{C.2.28}$$

for any $N^\cdot \in D^b(A_Y)$. Similarly, from Theorem C.2.8 we obtain an isomorphism

$$(Rf_* \circ \mathbf{D}_X)(M^\cdot) \simeq (\mathbf{D}_Y \circ Rf_!)(M^\cdot) \tag{C.2.29}$$

for any $M^\cdot \in D^b(A_X)$.

**Example C.2.16.** In the situation as above, assume, moreover, that $A$ is a field $k$, $X$ is an orientable $C^1$-manifold of dimension $n$, and $Y = \{\mathrm{pt}\}$. In this case there exist isomorphisms $\omega_X^\cdot \simeq \mathrm{or}_X[n] \simeq k_X[n]$. Let $M^\cdot \in D^b(k_X)$ and set $\mathbf{D}'_X(M^\cdot) = R\mathcal{H}om_{k_X}(M^\cdot, k_X)$. Then by the isomorphism (C.2.29) we obtain an isomorphism $H^{n-i}(X, \mathbf{D}'_X(M^\cdot)) \simeq [H^i_c(X, M^\cdot)]^*$ for any $i \in \mathbb{Z}$, where we set $H^i_c(X, \bullet) = H^i \mathrm{R}\Gamma_c(X, \bullet)$. In the very special case where $M^\cdot = k_X$ we thus obtain the famous Poincaré duality theorem: $H^{n-i}(X, k_X) \simeq [H^i_c(X, k_X)]^*$.

## C.3 Non-characteristic deformation lemma

In this section, we prove the *non-characteristic deformation lemma* (due to Kashiwara), which plays a powerful role in deriving results on global cohomology groups of complexes of sheaves from their local properties. First, we introduce some basic results on projective systems of abelian groups. Recall that a pair $M = (M_n, \rho_{n,m})$ of a family of abelian groups $M_n$ $(n \in \mathbb{N})$ and that of group homomorphisms $\rho_{n,m} : M_m \to M_n$ $(m \geq n)$ is called a *projective system* of abelian groups (indexed by $\mathbb{N}$) if it satisfies the conditions: $\rho_{n,n} = \mathrm{id}_{M_n}$ for any $n \in \mathbb{Z}$ and $\rho_{n,m} \circ \rho_{m,l} = \rho_{n,l}$ for any $n \leq m \leq l$. If $M = (M_n, \rho_{n,m})$ is a projective system of abelian groups, we denote its projective limit by $\varprojlim M$ for short. We define morphisms of projective systems of abelian groups in an obvious way. Then the category of projective systems of abelian groups is abelian. However, the functor $\varprojlim(*)$ from this category to that of abelian groups is not exact. It is only left exact. As a remedy for this problem we introduce the following notion.

**Definition C.3.1.** Let $M = (M_n, \rho_{n,m})$ be a projective system of abelian groups. We say that $M$ satisfies the *Mittag-Leffler condition* (or M-L condition) if for any $n \in \mathbb{N}$ decreasing subgroups $\rho_{n,m}(M_m)$ $(m \geq n)$ of $M_n$ is stationary.

Let us state basic results on projective systems satisfying the M-L condition. Since the proofs of the following lemmas are straightforward, we leave them to the reader.

**Lemma C.3.2.** *Let*

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

*be an exact sequence of projective systems of abelian groups.*

(i) *Assume $L$ and $N$ satisfy the M-L condition. Then $M$ satisfies the M-L condition.*
(ii) *Assume $M$ satisfies the M-L condition. Then $N$ satisfies the M-L condition.*

**Lemma C.3.3.** *Let*

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

*be an exact sequence of projective systems of abelian groups. Assume that L satisfies the M-L condition. Then the sequence*

$$0 \longrightarrow \varprojlim L \longrightarrow \varprojlim M \longrightarrow \varprojlim N \longrightarrow 0$$

*is exact.*

Now let $X$ be a topological space and $F^\cdot \in D^b(\mathbb{Z}_X)$. Namely, $F^\cdot$ is a bounded complex of sheaves of abelian groups on $X$.

**Proposition C.3.4.** *Let $\{U_n\}_{n\in\mathbb{N}}$ be an increasing sequence of open subsets of $X$ and set $U = \cup_{n\in\mathbb{N}} U_n$. Then*

(i) *The natural morphism $\phi_i : H^i(U, F^\cdot) \to \varprojlim_n H^i(U_n, F^\cdot)$ is surjective for any $i \in \mathbb{Z}$.*

(ii) *Assume that for an integer $i \in \mathbb{Z}$ the projective system $\{H^{i-1}(U_n, F^\cdot)\}_{n\in\mathbb{N}}$ satisfies the M-L condition. Then $\phi_i : H^i(U, F^\cdot) \to \varprojlim_n H^i(U_n, F^\cdot)$ is bijective.*

*Proof.* We may assume that each term $F^i$ of $F^\cdot$ is a flabby sheaf. Then we have $H^i(U, F^\cdot) = H^i(F^\cdot(U)) = H^i(\varprojlim_n F^\cdot(U_n))$. Hence the morphism $\phi_i$ is

$$H^i(\varprojlim_n F^\cdot(U_n)) \to \varprojlim_n H^i(F^\cdot(U_n)).$$

Note that for any $i \in \mathbb{Z}$ the projective system $\{F^i(U_n)\}_{n\in\mathbb{N}}$ satisfies the M-L condition by the flabbiness of $F^i$. Set $Z^i_n = \mathrm{Ker}[F^i(U_n) \to F^{i+1}(U_n)]$ and $B^i_n = \mathrm{Im}[F^{i-1}(U_n) \to F^i(U_n)]$. Then we have exact sequences

$$0 \longrightarrow Z^i_n \longrightarrow F^i(U_n) \longrightarrow B^{i+1}_n \longrightarrow 0 \tag{C.3.1}$$

and by Lemma C.3.2 (ii) the projective systems $\{B^i_n\}_{n\in\mathbb{N}}$ satisfy the M-L condition. Therefore, applying Lemma C.3.3 to the exact sequences

$$0 \longrightarrow B^i_n \longrightarrow Z^i_n \longrightarrow H^i(U_n, F^\cdot) \longrightarrow 0 \tag{C.3.2}$$

we get an exact sequence

$$0 \longrightarrow \varprojlim_n B^i_n \longrightarrow \varprojlim_n Z^i_n \longrightarrow \varprojlim_n H^i(U_n, F^\cdot) \longrightarrow 0. \tag{C.3.3}$$

Since the functor $\varprojlim(*)$ is left exact, we also have isomorphisms

$$\varprojlim_n Z^i_n \simeq \mathrm{Ker}[\varprojlim_n F^i(U_n) \to \varprojlim_n F^{i+1}(U_n)] = \mathrm{Ker}[F^i(U) \to F^{i+1}(U)].$$

Now let us consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
 & & & & F^{i-1}(U) & \longrightarrow & \mathrm{Ker}[F^i(U) \to F^{i+1}(U)] & \longrightarrow & H^i(U; F^\cdot) & \longrightarrow & 0 \\
 & & & & \downarrow & & \wr\downarrow & & \downarrow{\scriptstyle \phi_i} & & \\
0 & \longrightarrow & \varprojlim_n B_n^i & \longrightarrow & & & \varprojlim_n Z_n^i & \longrightarrow & \varprojlim_n H^i(U_n, F^\cdot) & \longrightarrow & 0
\end{array}$$

Then we see that $\phi_i$ is surjective. The assertion (i) was proved. Let us prove (ii). Assume that the projective system $\{H^{i-1}(U_n, F^\cdot)\}_{n\in\mathbb{N}}$ satisfies the M-L condition. Then applying Lemma C.3.2 (i) to the exact sequence (C.3.2) we see that the projective system $\{Z_n^{i-1}\}_{n\in\mathbb{N}}$ satisfies the M-L condition. Hence by Lemma C.3.3 and (C.3.1) we get an exact sequence

$$0 \longrightarrow \varprojlim_n Z_n^{i-1} \longrightarrow F^{i-1}(U) \longrightarrow \varprojlim_n B_n^i \longrightarrow 0,$$

which shows that the left vertical arrow in the above diagram is surjective. Hence $\phi_i$ is bijective. This completes the proof. □

We also require the following.

**Lemma C.3.5.** *Let $\{M_t, \rho_{t,s}\}$ be a projective system of abelian groups indexed by $\mathbb{R}$. Assume that for any $t \in \mathbb{R}$ the natural morphisms*

$$\begin{cases} \alpha_t : M_t \longrightarrow \varprojlim_{s<t} M_s, \\ \beta_t : \varinjlim_{s>t} M_s \longrightarrow M_t \end{cases}$$

*are injective (resp. surjective). Then for any pair $t_1 \le t_2$ the morphism $\rho_{t_1,t_2} : M_{t_2} \to M_{t_1}$ is injective (resp. surjective).*

*Proof.* Since the proof of injectivity is easy, we only prove surjectivity. Let $t_1 \le t_2$ and $m_1 \in M_{t_1}$. Denote by $S$ the set of all pairs $(t, m)$ of $t_1 \le t \le t_2$ and $m \in M_t$ satisfying $\rho_{t_1,t}(m) = m_1$. Let us order this set $S$ in the following way: $(t, m) \le (t', m') \iff t \le t'$ and $\rho_{t,t'}(m') = m$. Then by the surjectivity of $\alpha_s$ for any $s$ we can easily prove that $S$ is an inductively ordered set. Therefore, by *Zorn's lemma* there exists a maximal element $(t, m)$ of $S$. If $t = t_2$, then $\rho_{t_1,t_2}(m) = m_1$. If $t < t_2$ then by the surjectivity of $\beta_s$'s for any $s$, there exist $t_3$ with $t < t_3 \le t_2$ and $m_3 \in M_{t_3}$ such that $\rho_{t,t_3}(m_3) = m$. This contradicts the maximality of the element $(t, m)$. □

Now let us introduce the non-characteristic deformation lemma (due to Kashiwara). This result is very useful to derive global results from the local properties of $F^\cdot \in D^b(\mathbb{C}_X)$. Here we introduce only its weak form, which is enough for the applications in this book (see also [KS2, Proposition 2.7.2]).

**Theorem C.3.6 (Non-characteristic deformation lemma).** *Let $X$ be a $C^\infty$-manifold, $\{\Omega_t\}_{t\in\mathbb{R}}$ a family of relatively compact open subsets of $X$, and $F^\cdot \in D^b(\mathbb{C}_X)$. Assume the following conditions:*

(i) *For any pair* $s < t$ *of real numbers,* $\Omega_s \subset \Omega_t$.
(ii) *For any* $t \in \mathbb{R}$, $\Omega_t = \bigcup_{s<t} \Omega_s$.
(iii) *For* $^\forall t \in \mathbb{R}$, $\bigcap_{s>t}(\Omega_s \setminus \Omega_t) = \partial\Omega_t$ *and for* $^\forall x \in \partial\Omega_t$, *we have*

$$\left[\mathrm{R}\Gamma_{X\setminus\Omega_t}(F^\cdot)\right]_x \simeq 0.$$

*Then we have an isomorphism*

$$\mathrm{R}\Gamma\left(\bigcup_{s\in\mathbb{R}} \Omega_s, F^\cdot\right) \xrightarrow{\sim} \mathrm{R}\Gamma(\Omega_t, F^\cdot)$$

*for any* $t \in \mathbb{R}$.

*Proof.* We prove the theorem by using Lemma C.3.5. First let us prove that for any $t \in \mathbb{R}$ and $i \in \mathbb{Z}$ the canonical morphism

$$\varinjlim_{s>t} H^i(\Omega_s, F^\cdot) \longrightarrow H^i(\Omega_t, F^\cdot)$$

is an isomorphism. Since we have $\mathrm{R}\Gamma_{X\setminus\Omega_t}(F^\cdot)|_{\partial\Omega_t} \simeq 0$ by the assumption (iii), we obtain

$$\mathrm{R}\Gamma(\overline{\Omega_t}, \mathrm{R}\Gamma_{X\setminus\Omega_t}(F^\cdot)) \simeq \mathrm{R}\Gamma(\partial\Omega_t, \mathrm{R}\Gamma_{X\setminus\Omega_t}(F^\cdot)) \simeq 0.$$

Then by the distinguished triangle

$$\mathrm{R}\Gamma_{X\setminus\Omega_t}(F^\cdot) \longrightarrow F^\cdot \longrightarrow \mathrm{R}\Gamma_{\Omega_t}(F^\cdot) \xrightarrow{+1}$$

we get an isomorphism

$$\mathrm{R}\Gamma(\overline{\Omega_t}, F^\cdot) \simeq \mathrm{R}\Gamma(\Omega_t, F^\cdot).$$

Taking the cohomology groups of both sides we finally obtain the desired isomorphisms

$$\varinjlim_{s>t} H^i(\Omega_s, F^\cdot) \simeq H^i(\Omega_t, F^\cdot). \tag{C.3.4}$$

Now consider the following assertions:

$$(A)_i^t : \varprojlim_{s<t} H^i(\Omega_s, F^\cdot) \simeq H^i(\Omega_t, F^\cdot)$$

for $i \in \mathbb{Z}$ and $t \in \mathbb{R}$. Assume that for an integer $j$ the assertion $(A)_i^t$ is proved for any $i < j$ and $t \in \mathbb{R}$. Then by Lemma C.3.5 we get an isomorphism $H^i(\Omega_s, F^\cdot) \simeq H^i(\Omega_t, F^\cdot)$ for any $i < j$ and any pair $s > t$. This implies that for each $t \in \mathbb{R}$ the projective system $\{H^{j-1}(\Omega_{t-\frac{1}{n}}, F^\cdot)\}_{n\in\mathbb{N}}$ satisfies the M-L condition. Hence by Proposition C.3.4 the assertion $(A)_j^t$ is proved for any $t \in \mathbb{R}$. Repeating this argument, we can finally prove $(A)_i^t$ for all $i \in \mathbb{Z}$ and all $t \in \mathbb{R}$. Together with the isomorphisms (C.3.4), we obtain by Lemma C.3.5 an isomorphism $\mathrm{R}\Gamma(\Omega_s, F^\cdot) \simeq \mathrm{R}\Gamma(\Omega_t, F^\cdot)$ for any pair $s > t$. This completes the proof. □

# D

# Filtered Rings

## D.1 Good filtration

Let $A$ be a ring. Assume that we are given a family $F = \{F_l A\}_{l\in\mathbb{Z}}$ of additive subgroups of $A$ satisfying

(a) $F_l A = 0$ for $l < 0$,
(b) $1 \in F_0 A$,
(c) $F_l A \subset F_{l+1} A$,
(d) $(F_l A)(F_m A) \subset F_{l+m} A$,
(e) $A = \bigcup_{l\in\mathbb{Z}} F_l A$.

Then we call $(A, F)$ a *filtered ring*. For a filtered ring $(A, F)$ we set

$$\mathrm{gr}^F A = \bigoplus_{l\in\mathbb{Z}} \mathrm{gr}_l^F A, \qquad \mathrm{gr}_l^F A = F_l A / F_{l-1} A.$$

The canonical map $F_l A \to \mathrm{gr}_l^F A$ is denoted by $\sigma_l$. The additive group $\mathrm{gr}^F A$ is endowed with a structure of a ring by

$$\sigma_l(a)\sigma_m(b) = \sigma_{l+m}(ab).$$

We call the ring $\mathrm{gr}^F A$ the *associated graded ring*.

Let $(A, F)$ be a filtered ring. Let $M$ be a (left) $A$-module $M$, and assume that we are given a family $F = \{F_p M\}_{p\in\mathbb{Z}}$ of additive subgroups of $M$ satisfying

(a) $F_p M = 0$ for $p \ll 0$,
(b) $F_p M \subset F_{p+1} M$,
(c) $(F_l A)(F_p M) \subset F_{l+p} M$,
(d) $M = \bigcup_{p\in\mathbb{Z}} F_p M$.

Then $F$ is called a filtration of $M$ and $(M, F)$ is called a filtered (left) $A$-module. For a filtered $A$-module $(M, F)$ we set

$$\mathrm{gr}^F M = \bigoplus_{p\in\mathbb{Z}} \mathrm{gr}_p^F M, \qquad \mathrm{gr}_p^F M = F_pM/F_{p-1}M.$$

Denote the canonical map $F_pM \to \mathrm{gr}_p^F M$ by $\tau_p$. The additive group $\mathrm{gr}^F M$ is endowed with a structure of a $\mathrm{gr}^F A$-module by

$$\sigma_l(a)\tau_p(m) = \tau_{l+p}(am).$$

We call the $\mathrm{gr}^F A$-module $\mathrm{gr}^F M$ the *associated graded module*.

We can also define the notion of a filtration of a right $A$-module and the associated graded module of a right filtered $A$-module. We will only deal with left $A$-modules in the following; however, parallel facts also hold for right modules.

**Proposition D.1.1.** *Let $M$ be an $A$-module.*

(i) *Let $F$ be a filtration of $M$ such that $\mathrm{gr}^F M$ is finitely generated over $\mathrm{gr}^F A$. Then there exist finitely many integers $p_k$ $(k = 1, \ldots, r)$ and $m_k \in F_{p_k}M$ such that for any $p$ we have $F_pM = \sum_{p\geq p_k}(F_{p-p_k}A)m_k$. In particular, the $A$-module $M$ is generated by finitely many elements $m_1, \ldots, m_k$.*

(ii) *Let $M$ an $A$-module generated by finitely many elements $m_1, \ldots, m_k$. For $p_k \in \mathbb{Z}$ $(k = 1, \ldots, r)$ set $F_pM = \sum_{p\geq p_k}(F_{p-p_k}A)m_k$. Then $F$ is a filtration of $M$ such that $\mathrm{gr}^F M$ is a finitely generated $\mathrm{gr}^F A$-module.*

*Proof.* (i) We take integers $p_k$ $(k = 1, \ldots, r)$ and $m_k \in F_{p_k}M$ so that $\{\tau_{p_k}(m_k)\}_{1\leq k\leq r}$ generates the $\mathrm{gr}^F A$-module $\mathrm{gr}^F M$. Then we can show $F_pM = \sum_{p\geq p_k}(F_{p-p_k}A)m_k$ by induction on $p$. (ii) is obvious. □

**Corollary D.1.2.** *The following conditions on an $A$-module $M$ are equivalent:*

(i) *$M$ is a finitely generated $A$-module,*

(ii) *there exists a filtration $F$ of $M$ such that $\mathrm{gr}^F M$ is a finitely generated $\mathrm{gr}^F A$-module.*

Let $(M, F)$ be a filtered $A$-module. If $\mathrm{gr}^F M$ is a finitely generated $\mathrm{gr}^F A$-module, then $F$ is called a *good filtration* of $M$, and $(M, F)$ is called a good filtered $A$-module.

**Proposition D.1.3.** *Let $M$ be a finitely generated $A$-module and let $F$, $G$ be filtrations of $M$. If $F$ is good, then there exist an integers $a$ such that for any $p \in \mathbb{Z}$ we have*

$$F_pM \subset G_{p+a}M.$$

*In particular, if $G$ is also good, then for $a \gg 0$ we have*

$$F_{p-a}M \subset G_pM \subset F_{p+a}M \qquad (\forall p).$$

*Proof.* By Proposition D.1.1 we can take elements $m_k$ $(1 \leq k \leq r)$ of $M$ and integers $p_k$ $(1 \leq k \leq r)$ such that $F_pM = \sum_{p\geq p_k}(F_{p-p_k}A)m_k$. Take $q_k \in \mathbb{Z}$ such that $m_k \in G_{q_k}M$ and denote the maximal value of $q_k - p_k$ by $a$. Then we have

$$F_pM = \sum_{p\geq p_k}(F_{p-p_k}A)m_k \subset \sum_{p\geq p_k}(F_{p-p_k}A)G_{q_k}M$$
$$\subset \sum_{p\geq p_k} G_{p+(q_k-p_k)}M \subset G_{p+a}M.$$

The proof is complete. □

Let $(M, F)$ be a filtered $A$-module, and let

$$0 \to L \to M \to N \to 0$$

be an exact sequence of $A$-modules. Then we have the induced filtrations of $L$ and $N$ defined by

$$F_pL = F_pM \cap L, \qquad F_pN = \mathrm{Im}(F_pM \to N),$$

for which we have the exact sequence

$$0 \to \mathrm{gr}^F L \to \mathrm{gr}^F M \to \mathrm{gr}^F N \to 0.$$

Hence, if $(M, F)$ is a good filtered $A$-module, then so is $(N, F)$. If, moreover, $\mathrm{gr}^F A$ is a left noetherian ring, then $(L, F)$ is also a good filtered $A$-module.

**Proposition D.1.4.** *Let $(A, F)$ be a filtered ring. If $\mathrm{gr}^F A$ is a left (or right) noetherian ring, then so is $A$.*

*Proof.* In order to show that $A$ is a left noetherian ring it is sufficient to show that any left ideal $I$ of $A$ is finitely generated. Define a filtration $F$ of a left $A$-module $I$ by $F_pI = I \cap F_pA$. Then $\mathrm{gr}^F I$ is a left ideal of $\mathrm{gr}^F A$. Since $\mathrm{gr}^F A$ is a left noetherian ring, $\mathrm{gr}^F I$ is finitely generated over $\mathrm{gr}^F A$. Hence $I$ is finitely generated by Corollary D.1.2. The statement for right noetherian rings is proved similarly. □

## D.2 Global dimensions

Let $(M, F)$, $(N, F)$ be filtered $A$-modules. An $A$-homomorphism $f : M \to N$ such that $f(F_pM) \subset F_pN$ for any $p$ is called a *filtered* $\boldsymbol{A}$*-homomorphism*. In this case we write $f : (M, F) \to (N, F)$. A filtered $A$-homomorphism $f : (M, F) \to (N, F)$ induces a homomorphism $\mathrm{gr}\, f : \mathrm{gr}^F M \to \mathrm{gr}^F N$ of $\mathrm{gr}^F A$-modules. A filtered $A$-homomorphism $f : (M, F) \to (N, F)$ is called *strict* if it satisfies $f(F_pM) = \mathrm{Im}\, f \cap F_pN$. The following fact is easily proved.

**Lemma D.2.1.** *Let $f : (L, F) \to (M, F)$, $g : (M, F) \to (N, F)$ be strict filtered $A$-homomorphisms such that $L \to M \to N$ is exact. Then $\mathrm{gr}^F L \to \mathrm{gr}^F M \to \mathrm{gr}^F N$ is exact.*

Let $W$ be a free $A$-module of rank $r < \infty$ with basis $\{w_k\}_{1\leqq k\leqq r}$. For integers $p_k$ $(1 \leqq k \leqq r)$ we can define a filtration $F$ of $W$ by $F^pW = \sum_k (F_{p-p_k}A)w_k$. This type of filtered $A$-module $(W, F)$ is called a filtered free $A$-module of rank $r$. We can easily show the following.

**Lemma D.2.2.** *Assume that $A$ is left noetherian. For a good filtered $A$-module $(M, F)$ we can take filtered free $A$-modules $(W_i, F)$ $(i \in \mathbb{N})$ of finite ranks and strict filtered $A$-homomorphisms $(W_{i+1}, F) \to (W_i, F)$ $(i \in \mathbb{N})$ and $(W_0, F) \to (M, F)$ such that*

$$\cdots \to W_1 \to W_0 \to M \to 0$$

*is an exact sequence of $A$-modules.*

For filtered $A$-modules $(M, F)$, $(N, F)$ and $p \in \mathbb{Z}$ set

$$F^p \operatorname{Hom}_A(M, N) = \{f \in \operatorname{Hom}_A(M, N) \mid f(F_q M) \subset F_{q+p} N \ (\forall q \in \mathbb{Z})\}.$$

This defines an increasing filtration of the abelian group $\operatorname{Hom}_A(M, N)$. Set

$$\operatorname{gr}_p^F \operatorname{Hom}_A(M, N) = F_p \operatorname{Hom}_A(M, N)/F_{p-1} \operatorname{Hom}_A(M, N),$$
$$\operatorname{gr}^F \operatorname{Hom}_A(M, N) = \bigoplus_p \operatorname{gr}_p^F \operatorname{Hom}_A(M, N).$$

Then we have a canonical homomorphism

$$\operatorname{gr}^F \operatorname{Hom}_A(M, N) \to \operatorname{Hom}_{\operatorname{gr}^F A}(\operatorname{gr}^F M, \operatorname{gr}^F N)$$

of abelian groups. The following is easily proved.

**Lemma D.2.3.** *Let $(M, F)$ be a good filtered $A$-module and $(N, F)$ a filtered $A$-module.*

(i) $\operatorname{Hom}_A(M, N) = \bigcup_{p \in \mathbb{Z}} F_p \operatorname{Hom}_A(M, N)$.
(ii) $F_p \operatorname{Hom}_A(M, N) = 0$ *for* $p \ll 0$.
(iii) *The canonical homomorphism* $\operatorname{gr}^F \operatorname{Hom}_A(M, N) \to \operatorname{Hom}_{\operatorname{gr}^F A}(\operatorname{gr}^F M, \operatorname{gr}^F N)$ *is injective. Moreover, it is surjective if $(M, F)$ is a filtered free $A$-module of finite rank.*

**Lemma D.2.4.** *Let $(A, F)$ be a filtered ring such that $\operatorname{gr}^F A$ is left noetherian. Let $(M, F)$ be a good filtered $A$-module and $(N, F)$ a filtered $A$-module. Then there exists an increasing filtration $F$ of the abelian group $\operatorname{Ext}_A^i(M, N)$ such that*

(i) $\operatorname{Ext}_A^i(M, N) = \bigcup_{p \in \mathbb{Z}} F_p \operatorname{Ext}_A^i(M, N)$,
(ii) $F_p \operatorname{Ext}_A^i(M, N) = 0$ *for* $p \ll 0$,
(iii) $\operatorname{gr}^F \operatorname{Ext}_A^i(M, N)$ *is isomorphic to a subquotient of* $\operatorname{Ext}^i_{\operatorname{gr}^F A}(\operatorname{gr}^F M, \operatorname{gr}^F N)$.

*Proof.* Take $(W_i, F)$ $(i \in \mathbb{N})$, $(W_{i+1}, F) \to (W_i, F)$ $(i \in \mathbb{N})$ and $(W_0, F) \to (M, F)$ as in Lemma D.2.2. Then we have $\operatorname{Ext}_A^i(M, N) = H^i(K^\cdot)$ with $K^\cdot = \operatorname{Hom}_A(W_\cdot, N)$. Note that each term $K^i = \operatorname{Hom}_A(W_i, N)$ of $K^\cdot$ is equipped with increasing filtration $F$ satisfying $d^i(F_p K^i) \subset F_p K^{i+1}$, where $d^i : K^i \to K^{i+1}$ denotes the boundary homomorphism.

Consider the complex $\operatorname{gr}^F K^\cdot$ with $i$th term $\operatorname{gr}^F K^i$. We have $\operatorname{gr}^F K^\cdot \simeq \operatorname{Hom}_{\operatorname{gr}^F A}(\operatorname{gr}^F W_\cdot, \operatorname{gr}^F N)$ by Lemma D.2.3. Hence by the exact sequence

$$\cdots \to \mathrm{gr}^F W_1 \to \mathrm{gr}^F W_0 \to \mathrm{gr}^F M \to 0,$$

(see Lemma D.2.1) we obtain

$$\mathrm{Ext}^i_{\mathrm{gr}^F A}(\mathrm{gr}^F M, \mathrm{gr}^F N) = H^i(\mathrm{Hom}_{\mathrm{gr}^F A}(\mathrm{gr}^F W_\cdot, \mathrm{gr}^F N) \simeq H^i(\mathrm{gr}^F K^\cdot).$$

Now define an increasing filtration of $H^i(K^\cdot) = \mathrm{Ext}^i_A(M, N)$ by

$$F_p H^i(K^\cdot) = \mathrm{Im}(H^i(F^p K^\cdot) \to H^i(K^\cdot)).$$

For each $i \in \mathbb{N}$ we have

$$K^i = \bigcup_p F_p K^i, \qquad F_p K^i = 0 \ (p \ll 0).$$

by Lemma D.2.3. From this we easily see that

$$H^i(K^\cdot) = \bigcup_p F_p H^i(K^\cdot), \qquad F_p H^i(K^\cdot) = 0 \ (p \ll 0).$$

It remains to show that $\mathrm{gr}^F H^i(K^\cdot)$ is a subquotient of $H^i(\mathrm{gr}^F K^\cdot)$. By definition we have

$$\mathrm{gr}^F_p H^i(K^\cdot) = (F_p K^i \cap \mathrm{Ker}\, d^i + \mathrm{Im}\, d^{i-1})/(F_{p-1} K^i \cap \mathrm{Ker}\, d^i + \mathrm{Im}\, d^{i-1}),$$
$$H^i(\mathrm{gr}^F_p K^\cdot) = \mathrm{Ker}(F_p K^i \to \mathrm{gr}^F_p K^{i+1})/(F_{p-1} K^i + d^{i-1}(F_p K^{i-1})).$$

Set $L = F_p K^i \cap \mathrm{Ker}\, d^i/(F_{p-1} K^i \cap \mathrm{Ker}\, d^i + d^{i-1}(F_p K^{i-1}))$. Then we can easily check that $L$ is isomorphic to a submodule of $H^i(\mathrm{gr}^F_p K^\cdot)$ and that $\mathrm{gr}^F_p H^i(K^\cdot)$ is a quotient of $L$. □

Let us consider the situation where $N = A$ (with canonical filtration $F$) in Lemma D.2.3 and Lemma D.2.4. Let $(A, F)$ be a filtered ring and let $(M, F)$ be a good filtered $A$-module. We easily see that the filtration $F$ of the right $A$-module $\mathrm{Hom}_A(M, A)$ is a good filtration and the canonical homomorphism $\mathrm{gr}^F \mathrm{Hom}_A(M, N) \to \mathrm{Hom}_{\mathrm{gr}^F A}(\mathrm{gr}^F M, \mathrm{gr}^F N)$ preserves the $\mathrm{gr}^F A$-modules structure. Hence (the proof of) Lemma D.2.4 implies the following.

**Lemma D.2.5.** *Let $(A, F)$ be a filtered ring such that $\mathrm{gr}^F A$ is left noetherian, and let $(M, F)$ be a good filtered $A$-module. Then there exists a good filtration $F$ of the right $A$-module $\mathrm{Ext}^i_A(M, A)$ such that $\mathrm{gr}^F \mathrm{Ext}^i_A(M, A)$ is isomorphic to a subquotient of $\mathrm{Ext}^i_{\mathrm{gr}^F A}(\mathrm{gr}^F M, \mathrm{gr}^F A)$ as a right $\mathrm{gr}^F A$-module.*

**Theorem D.2.6.** *Let $(A, F)$ be a filtered ring such that $\mathrm{gr}^F A$ is left (resp. right) noetherian. Then the left (resp. right) global dimension of the ring $A$ is smaller than or equal to that of $\mathrm{gr}^F A$.*

*Proof.* We will only show the statement for left global dimensions. Denote the left global dimension of $\mathrm{gr}^F A$ by $n$. If $n = \infty$, there is nothing to prove. Assume that $n < \infty$. We need to show $\mathrm{Ext}^i_A(M, N) = 0$ $(i > n)$ for arbitrary $A$-modules $M$, $N$. Since $A$ is left noetherian, we may assume that $M$ is finitely generated. Choose a good filtration $F$ of $M$ and a filtration $F$ of $N$. Then we have $\mathrm{Ext}^i_{\mathrm{gr}^F A}(\mathrm{gr}^F M, \mathrm{gr}^F N) = 0$ $(i > n)$. Hence the assertion follows from Lemma D.2.4. □

## D.3 Singular supports

Let $R$ be a commutative noetherian ring and let $M$ be a finitely generated $R$-module. We denote by $\mathrm{supp}(M)$ the set of prime ideals $\mathfrak{p}$ of $R$ satisfying $M_{\mathfrak{p}} \neq 0$, and by $\mathrm{supp}_0(M)$ the set of minimal elements of $\mathrm{supp}(M)$. We have $\mathfrak{p} \in \mathrm{supp}(M)$ if and only if $\mathfrak{p}$ contains the annihilating ideal

$$\mathrm{Ann}_R(M) = \{r \in R \mid rM = 0\}.$$

In fact, we have

$$\sqrt{\mathrm{Ann}_R(M)} = \bigcap_{\mathfrak{p} \in \mathrm{supp}(M)} \mathfrak{p}.$$

For $\mathfrak{p} \in \mathrm{supp}_0(M)$ we denote the *length* of the artinian $R_{\mathfrak{p}}$-module $M_{\mathfrak{p}}$ by $\ell_{\mathfrak{p}}(M)$. We set $\ell_{\mathfrak{q}}(M) = 0$ for a prime ideal $\mathfrak{q} \notin \mathrm{supp}(M)$. For an exact sequence

$$0 \to L \to M \to N \to 0$$

of finitely generated $R$-modules we have

$$\mathrm{supp}(M) = \mathrm{supp}(L) \cup \mathrm{supp}(N).$$

Moreover, for $\mathfrak{p} \in \mathrm{supp}_0(M)$ we have

$$\ell_{\mathfrak{p}}(M) = \ell_{\mathfrak{p}}(L) + \ell_{\mathfrak{p}}(N).$$

In the rest of this section $(A, F)$ denotes a filtered ring such that $\mathrm{gr}^F A$ is a commutative noetherian ring. Let $M$ be a finitely generated $A$-module. By choosing a good filtration $F$ we can consider $\mathrm{supp}(\mathrm{gr}^F M)$ and $\ell_{\mathfrak{p}}(\mathrm{gr}^F M)$ for $\mathfrak{p} \in \mathrm{supp}_0(M)$.

**Lemma D.3.1.** $\mathrm{supp}(\mathrm{gr}^F M)$ *and* $\ell_{\mathfrak{p}}(\mathrm{gr}^F M)$ *for* $\mathfrak{p} \in \mathrm{supp}_0(M)$ *do not depend on the choice of a good filtration* $F$.

*Proof.* We say two good filtrations $F$ and $G$ are "*adjacent*" if they satisfy the condition

$$F_i M \subset G_i M \subset F_{i+1} M \quad ({}^{\forall} i \in \mathbb{Z}).$$

We first show the assertion in this case. Consider the natural homomorphism $\varphi_i : F_i M / F_{i-1} M \to G_i M / G_{i-1} M$. Then we have $\mathrm{Ker}\, \varphi_i \simeq G_{i-1} M / F_{i-1} M \simeq \mathrm{Coker}\, \varphi_{i-1}$. Therefore, the morphism $\varphi : \mathrm{gr}^F M \to \mathrm{gr}^G M$ entails an isomorphism $\mathrm{Ker}\, \varphi \simeq \mathrm{Coker}\, \varphi$. Consider the exact sequence

$$0 \to \mathrm{Ker}\, \varphi \to \mathrm{gr}^F M \xrightarrow{\varphi} \mathrm{gr}^G M \to \mathrm{Coker}\, \varphi \to 0$$

of finitely generated $\mathrm{gr}^F A$-modules. From this we obtain

$$\mathrm{supp}(\mathrm{gr}^F M) = \mathrm{supp}(\mathrm{Ker}\, \varphi) \cup \mathrm{supp}(\mathrm{Im}\, \varphi),$$
$$\mathrm{supp}(\mathrm{gr}^G M) = \mathrm{supp}(\mathrm{Im}\, \varphi) \cup \mathrm{supp}(\mathrm{Coker}\, \varphi).$$

Hence $\operatorname{Ker}\varphi \simeq \operatorname{Coker}\varphi$ implies $\operatorname{supp}(\operatorname{gr}^F M) = \operatorname{supp}(\operatorname{gr}^G M)$. Moreover, for $\mathfrak{p} \in \operatorname{supp}_0(\operatorname{gr}^F M) = \operatorname{supp}_0(\operatorname{gr}^G M)$ we have

$$\ell_{\mathfrak{p}}(\operatorname{gr}^F M) = \ell_{\mathfrak{p}}(\operatorname{Ker}\varphi) + \ell_{\mathfrak{p}}(\operatorname{Im}\varphi) = \ell_{\mathfrak{p}}(\operatorname{gr}^G M).$$

The assertion is proved for adjacent good filtrations.

Let us consider the general case. Namely, assume that $F$ and $G$ are arbitrary good filtrations of $M$. For $k \in \mathbb{Z}$ set

$$F_i^{(k)} M = F_i M + G_{i+k} M \quad (i \in \mathbb{Z}).$$

By Proposition D.1.3 $F^{(k)}$ is a good filtration of $M$ satisfying the conditions

$$\begin{cases} F^{(k)} = F & (k \ll 0), \\ F^{(k)} = G[k] & (k \gg 0), \\ F^{(k)} \text{ and } F^{(k+1)} \text{ are adjacent,} \end{cases}$$

where $G[k]$ is a filtration obtained form $G$ by the degree shift $[k]$. Therefore, our assertion follows form the adjacent case. □

**Definition D.3.2.** For a finitely generated $A$-module $M$ we set

$$\begin{aligned} \operatorname{SS}(M) &= \operatorname{supp}(\operatorname{gr}^F M), \\ \operatorname{SS}_0(M) &= \operatorname{supp}_0(\operatorname{gr}^F M), \\ J_M &= \sqrt{\operatorname{Ann}_{\operatorname{gr}^F A}(\operatorname{gr}^F M)} = \bigcap_{\mathfrak{p} \in \operatorname{SS}_0(M)} \mathfrak{p}, \\ d(M) &= \operatorname{Krull\,dim}\left(\operatorname{gr}^F A / J_M\right), \\ m_{\mathfrak{p}}(M) &= \ell_{\mathfrak{p}}(\operatorname{gr}^F M) \qquad (\mathfrak{p} \in \operatorname{SS}_0(M) \text{ or } \mathfrak{p} \notin \operatorname{SS}(M)), \end{aligned}$$

where $F$ is a good filtration of $M$. $\operatorname{SS}(M)$ and $J_M$ are called the *singular support* and the *characteristic ideal* of $M$, respectively.

**Lemma D.3.3.** *For an exact sequence*

$$0 \to L \to M \to N \to 0$$

*of finitely generated $A$-modules we have*

$$\begin{aligned} \operatorname{SS}(M) &= \operatorname{SS}(L) \cup \operatorname{SS}(N), \\ d(M) &= \max\{d(L), d(N)\}, \\ m_{\mathfrak{p}}(M) &= m_{\mathfrak{p}}(L) + m_{\mathfrak{p}}(N) \qquad (\mathfrak{p} \in \operatorname{SS}_0(M)). \end{aligned}$$

*Proof.* Take a good filtration $F$ of $M$. With respect to the induced filtrations of $L$ and $N$ we have a short exact sequence

$$0 \longrightarrow \operatorname{gr}^F M \longrightarrow \operatorname{gr}^F N \longrightarrow \operatorname{gr}^F L \longrightarrow 0.$$

Hence the assertions for SS and $\ell_{\mathfrak{p}}$ are obvious. The assertion for $d$ follows from the one for SS. □

Since $\mathrm{gr}^F A$ is commutative, we have $[F_pA, F_qA] \subset F_{p+q-1}A$. Here, for $a, b \in A$ we set $[a, b] = ab - ba$. Hence we obtain a bi-additive product

$$\{\ ,\ \} : \mathrm{gr}_p^F A \times \mathrm{gr}_q^F A \to \mathrm{gr}_{p+q-1}^F A, \quad (\{\sigma_p(a), \sigma_q(b)\} = \sigma_{p+q-1}([a, b])).$$

Its bi-additive extension

$$\{\ ,\ \} : \mathrm{gr}^F A \times \mathrm{gr}^F A \to \mathrm{gr}^F A$$

is called the *Poisson bracket*. It satisfies the following properties:

(i) $\{a, b\} + \{b, a\} = 0$,
(ii) $\{\{a, b\}, c\} + \{\{b, c\}, a\} + \{\{c, a\}, b\} = 0$,
(iii) $\{a, bc\} = \{a, b\}c + b\{a, c\}$.

We say that an ideal $I$ of $\mathrm{gr}^F A$ is *involutive* if it satisfies $\{I, I\} \subset I$.

We state the following deep result of Gabber [Ga] without proof.

**Theorem D.3.4.** *Assume that $(A, F)$ is a filtered ring such that the center of $A$ contains a subring isomorphic to $\mathbb{Q}$ and that $\mathrm{gr}^F A$ is a commutative noetherian ring. Let $M$ be a finitely generated $A$-module. Then any $\mathfrak{p} \in \mathrm{SS}_0(M)$ is involutive. In particular, $J_M$ is involutive.*

## D.4 Duality

In this section $(A, F)$ is a filtered ring such that $\mathrm{gr}^F A$ is a regular commutative ring of pure dimension $m$ (a commutative ring $R$ is called a regular ring of pure dimension $m$ if its localization at any maximal ideal is a regular local ring of dimension $m$). In particular, $\mathrm{gr}^F A$ is a noetherian ring whose global dimension and Krull dimension are equal to $m$. Hence $A$ is a left and right noetherian ring with global dimension $\leqq m$ by Proposition D.1.4 and Theorem D.2.6. We will consider properties of the Ext-groups $\mathrm{Ext}_A^i(M, A)$ for finitely generated $A$-modules $M$.

Note first that for any (left) $A$-module $M$ the Ext-groups $\mathrm{Ext}_A^i(M, A)$ are endowed with a right $A$-module structure (i.e., a left $A^{\mathrm{op}}$-module structure, where $A^{\mathrm{op}}$ denotes the opposite ring) by the right multiplication of $A$ on $A$. Since $A$ has global dimension $\leqq m$, we have $\mathrm{Ext}_A^i(M, A) = 0$ for $i > m$. Moreover, if $M$ is finitely generated, then $\mathrm{Ext}_A^i(M, A)$ are also finitely generated since $A$ is left noetherian.

Let us give a formulation in terms of the derived category. Let $\mathrm{Mod}(A)$ and $\mathrm{Mod}_f(A)$ denote the category of (left) $A$-modules and its full subcategory consisting of finitely generated $A$-modules, respectively. Denote by $D^b(A)$ and $D_f^b(A)$ the bounded derived category of $\mathrm{Mod}(A)$ and its full subcategory consisting of complexes whose cohomology groups belong to $\mathrm{Mod}_f(A)$. Our objectives are the functors

$$\mathbb{D} = R\,\mathrm{Hom}_A(\bullet, A) : D_f^b(A) \to D_f^b(A^{\mathrm{op}})^{\mathrm{op}},$$
$$\mathbb{D}' = R\,\mathrm{Hom}_{A^{\mathrm{op}}}(\bullet, A^{\mathrm{op}}) : D_f^b(A^{\mathrm{op}}) \to D_f^b(A)^{\mathrm{op}},$$

where $D_f^b(A^{\mathrm{op}})$ is defined similarly.

**Proposition D.4.1.** *We have* $\mathbb{D}' \circ \mathbb{D} \simeq \mathrm{Id}$ *and* $\mathbb{D} \circ \mathbb{D}' \simeq \mathrm{Id}$.

*Proof.* By symmetry we have only to show $\mathbb{D}' \circ \mathbb{D} \simeq \mathrm{Id}$. We first construct a canonical morphism $M^{\cdot} \to \mathbb{D}'\mathbb{D}M^{\cdot}$ for $M^{\cdot} \in D^b_f(A)$. Set $H^{\cdot} = \mathbb{D}M^{\cdot} = R\,\mathrm{Hom}_A(M^{\cdot}, A)$. By

$$R\,\mathrm{Hom}_{A\otimes_{\mathbb{Z}}A^{\mathrm{op}}}(M^{\cdot} \otimes_{\mathbb{Z}} H^{\cdot}, A) \simeq R\,\mathrm{Hom}_A(M^{\cdot}, R\,\mathrm{Hom}_{A^{\mathrm{op}}}(H^{\cdot}, A^{\mathrm{op}}))$$

we have

$$\mathrm{Hom}_{A\otimes_{\mathbb{Z}}A^{\mathrm{op}}}(M^{\cdot} \otimes_{\mathbb{Z}} H^{\cdot}, A) \simeq \mathrm{Hom}_A(M^{\cdot}, R\,\mathrm{Hom}_{A^{\mathrm{op}}}(H^{\cdot}, A^{\mathrm{op}}))$$

Hence the canonical morphism $M^{\cdot} \otimes_{\mathbb{Z}} H^{\cdot} (= M^{\cdot} \otimes_{\mathbb{Z}} R\,\mathrm{Hom}_A(M^{\cdot}, A)) \to A$ in $D^b(A \otimes_{\mathbb{Z}} A^{\mathrm{op}})$ gives rise to a canonical morphism

$$M^{\cdot} \to R\,\mathrm{Hom}_{A^{\mathrm{op}}}(H^{\cdot}, A^{\mathrm{op}}) (= \mathbb{D}'\mathbb{D}M)$$

in $D^b(A)$. It remains to show that $M^{\cdot} \to \mathbb{D}'\mathbb{D}M^{\cdot}$ is an isomorphism. By taking a free resolution of $M^{\cdot}$ we may replace $M^{\cdot}$ with $A$. In this case the assertion is clear. □

For $M^{\cdot} \in D^b_f(A)$ (or $D^b_f(A^{\mathrm{op}})$) we set

$$\mathrm{SS}(M^{\cdot}) = \bigcup_i \mathrm{SS}(H^i(M^{\cdot})).$$

We easily see by Lemma D.3.3 that for a distinguished triangle

$$L^{\cdot} \longrightarrow M^{\cdot} \longrightarrow N^{\cdot} \xrightarrow{+1}$$

we have $\mathrm{SS}(M^{\cdot}) \subset \mathrm{SS}(L^{\cdot}) \cup \mathrm{SS}(N^{\cdot})$.

**Proposition D.4.2.** *For* $M^{\cdot} \in D^b_f(A)$ (*resp.* $D^b_f(A^{\mathrm{op}})$) *we have* $\mathrm{SS}(\mathbb{D}M^{\cdot}) = \mathrm{SS}(M^{\cdot})$ (*resp.* $\mathrm{SS}(\mathbb{D}'M^{\cdot}) = \mathrm{SS}(M^{\cdot})$).

*Proof.* By Proposition D.4.1 and symmetry we have only to show $\mathrm{SS}(\mathbb{D}M^{\cdot}) \subset \mathrm{SS}(M^{\cdot})$ for $M^{\cdot} \in D^b_f(A)$. We use induction on the cohomological length of $M^{\cdot}$. We first consider the case where $M^{\cdot} = M \in \mathrm{Mod}_f(A)$. Take a good filtration $F$ of $M$ and consider a good filtration $F$ of $\mathrm{Ext}^i_A(M, A)$ as in Lemma D.2.5. By Lemma D.2.5 we have

$$\begin{aligned}\mathrm{SS}(\mathrm{Ext}^i_A(M, A)) &= \mathrm{supp}(\mathrm{gr}^F \mathrm{Ext}^i_A(M, A)) \subset \mathrm{supp}(\mathrm{Ext}^i_{\mathrm{gr}^F A}(\mathrm{gr}^F M, \mathrm{gr}^F A))\\ &\subset \mathrm{supp}(\mathrm{gr}^F M) = \mathrm{SS}(M).\end{aligned}$$

The assertion is proved in the case where $M^{\cdot} = M \in \mathrm{Mod}_f(A)$. Now we consider the general case. Set $k = \min\{i \mid H^i(M^{\cdot}) \neq 0\}$. Then we have a distinguished triangle

$$H^k(M^{\cdot})[-k] \longrightarrow M^{\cdot} \longrightarrow N^{\cdot} \xrightarrow{+1},$$

where $N^{\cdot} = \tau^{\geqslant k+1} M^{\cdot}$. By

$$H^i(N^\cdot) = \begin{cases} H^i(M^\cdot) & (i \neq k) \\ 0 & (i = k) \end{cases}$$

we obtain $\mathrm{SS}(M^\cdot) = \mathrm{SS}(N^\cdot) \cup \mathrm{SS}(H^k(M^\cdot))$. Moreover, by the hypothesis of induction we have $\mathrm{SS}(\mathbb{D}N^\cdot) \subset \mathrm{SS}(N^\cdot)$ and $\mathrm{SS}(\mathbb{D}H^k(M^\cdot)) \subset \mathrm{SS}(H^k(M^\cdot))$. Hence by the distinguished triangle

$$\mathbb{D}N^\cdot \longrightarrow \mathbb{D}M^\cdot \longrightarrow (\mathbb{D}H^k(M^\cdot))[k] \xrightarrow{+1}$$

we obtain

$$\mathrm{SS}(\mathbb{D}M^\cdot) \subset \mathrm{SS}(\mathbb{D}H^k(M^\cdot)) \cup \mathrm{SS}(\mathbb{D}N^\cdot) \subset \mathrm{SS}(H^k(M^\cdot)) \cup \mathrm{SS}(N^\cdot) = \mathrm{SS}(M^\cdot).$$

The proof is complete. □

For a finitely generated $A$-module $M$ set

$$j(M) := \min\{i \mid \mathrm{Ext}^i_A(M,\ A) \neq 0\}.$$

**Theorem D.4.3.** *Let $M$ be a finitely generated $A$-module.*

(i) $j(M) + d(M) = m$,
(ii) $d(\mathrm{Ext}^i_A(M,\ A)) \leq m - i \quad (i \in \mathbb{Z})$,
(iii) $d(\mathrm{Ext}^{j(M)}_A(M,\ A)) = d(M)$.

(*Recall that $m$ denotes the global dimension of* $\mathrm{gr}^F A$.)

The following corresponding fact for regular commutative rings is well known (see [Ser2], [Bj1]).

**Theorem D.4.4.** *Let $R$ be a regular commutative ring of dimension $m'$. For a finitely generated $R$-module $N$ we set $d(N) := \mathrm{Krull\ dim}(R/\mathrm{Ann}_R N)$ and $j(N) := \min\{i \mid \mathrm{Ext}^i_R(N,\ R) \neq 0\}$.*

(i) $d(N) + j(N) = m'$,
(ii) $d(\mathrm{Ext}^i_R(N,\ R)) \leq m' - i \quad (i \in \mathbb{Z})$,
(iii) $d(\mathrm{Ext}^{j(N)}_R(N,\ R)) = d(N)$.

*Proof of Theorem* D.4.3. We apply Theorem D.4.4 to the case $R = \mathrm{gr}^F A$. Fix a good filtration $F$ of $M$. By Lemma D.2.4 (iii) we have $\mathrm{SS}(\mathrm{Ext}^i_A(M, A)) \subset \mathrm{supp}(\mathrm{Ext}^i_{\mathrm{gr}^F A}(\mathrm{gr}^F M, \mathrm{gr}^F A))$ and hence

$$d(\mathrm{Ext}^i_A(M,\ A)) \leqq d(\mathrm{Ext}^i_{\mathrm{gr}^F A}(\mathrm{gr}^F M, \mathrm{gr}^F A)).$$

Thus (ii) follows from the corresponding fact for $\mathrm{gr}^F A$. Moreover, we have $\mathrm{Ext}^i_A(M, A) = 0$ for $i < j(\mathrm{gr}^F M)$. Hence in order to show (i) and (iii) it is sufficient to verify

$$d(\mathrm{Ext}_A^{j(\mathrm{gr}^F M)}(M, A)) = d(M).$$

By Proposition D.4.2 we have

$$d(M) = \max_{i \geqq j(\mathrm{gr}^F M)} d(\mathrm{Ext}_A^i(M, A)).$$

For $i > j(\mathrm{gr}^F M)$ we have

$$d(\mathrm{Ext}_A^i(M, A)) \leqq m - i < m - j(\mathrm{gr}^F M) = d(\mathrm{gr}^F M) = d(M),$$

and hence we must have $d(\mathrm{Ext}_A^{j(\mathrm{gr}^F M)}(M, A)) = d(M)$. □

**Corollary D.4.5.** *For an exact sequence*

$$0 \to L \to M \to N \to 0$$

*of finitely generated A-modules we have*

$$j(M) = \min\{j(L), j(N)\}.$$

## D.5 Codimension filtration

In this section $(A, F)$ is a filtered ring such that $\mathrm{gr}^F A$ is a regular commutative ring of pure dimension $m$. For a finitely generated $A$-module $M$ and $s \geq 0$ we denote by $C^s(M)$ the sum of all submodules $N$ of $M$ satisfying $j(N) \geq s$. Since $C^s(M)$ is finitely generated, we have $j(C^s(M)) \geq s$ by Corollary D.4.5 and hence $C^s(M)$ is the largest submodule $N$ of $M$ satisfying $j(N) \geq s$. By definition we have a decreasing filtration

$$0 = C^{m+1}(M) \subset C^m(M) \subset \cdots \subset C^1(M) \subset C^0(M) = M.$$

We say that a finitely generated $A$-module $M$ is purely $s$-codimensional if $C^s(M) = M$ and $C^{s+1}(M) = 0$.

**Lemma D.5.1.** *For any finitely generated A-module M $C^s(M)/C^{s+1}(M)$ is purely s-codimensional.*

*Proof.* Set $N = C^s(M)/C^{s+1}(M)$. Then we have $j(N) \geq j(C^s(M)) \geq s$ and hence $C^s(N) = N$. Set $K = \mathrm{Ker}\,(C^s(M) \to N/C^{s+1}(N))$. Then by the exact sequence

$$0 \to C^{s+1}(M) \to K \to C^{s+1}(N) \to 0$$

we have $j(K) = \min\{j(C^{s+1}(M)), j(C^{s+1}(N))\} \geq s + 1$. By the maximality of $C^{s+1}(M)$ we obtain $K = C^{s+1}(M)$, i.e., $C^{s+1}(N) = 0$. □

We will give a cohomological interpretation of this filtration.
For a finitely generated $A$-module $M$ and $s \geq 0$ we set

$$T^s(M) = \mathrm{Ext}^0_{A^{\mathrm{op}}}(\tau^{\geqslant s} R\,\mathrm{Hom}_A(M, A), A^{\mathrm{op}}).$$

By Proposition D.4.1 we have $T^0(M) = M$. By $j(\mathrm{Ext}^s(M, A)) \geq s$ we have $\mathrm{Ext}^{s-1}_{A^{\mathrm{op}}}(\mathrm{Ext}^s(M, A), A^{\mathrm{op}}) = 0$. Hence by applying $R\,\mathrm{Hom}_{A^{\mathrm{op}}}(\bullet, A^{\mathrm{op}})$ to the distinguished triangle

$$\mathrm{Ext}^s_A(M, A)[-s] \longrightarrow \tau^{\geqslant s} R\,\mathrm{Hom}_A(M, A) \longrightarrow \tau^{\geqslant s+1} R\,\mathrm{Hom}_A(M, A) \xrightarrow{+1}$$

and taking the cohomology groups we obtain an exact sequence

$$0 \to T^{s+1}(M) \to T^s(M) \to \mathrm{Ext}^s_{A^{\mathrm{op}}}(\mathrm{Ext}^s_A(M, A), A^{\mathrm{op}}).$$

Hence we obtain a decreasing filtration

$$0 = T^{m+1}(M) \subset T^m(M) \subset \cdots \subset T^1(M) \subset T^0(M) = M.$$

**Proposition D.5.2.** *For any $s$ we have $C^s(M) = T^s(M)$.*

*Proof.* By $j(\mathrm{Ext}^s_{A^{\mathrm{op}}}(\mathrm{Ext}^s(M, A), A^{\mathrm{op}})) \geq s$ we see from the exact sequence

$$0 \to T^{s+1}(M) \to T^s(M) \to \mathrm{Ext}^s_{A^{\mathrm{op}}}(\mathrm{Ext}^s_A(M, A), A^{\mathrm{op}})$$

using the backward induction on $s$ that $j(T^s(M)) \geq s$. Hence $T^s(M) \subset C^s(M)$. It remains to show the opposite inclusion. Set $N = C^s(M)$. By $j(N) \geq s$ we have

$$\tau^{\geqslant s} R\,\mathrm{Hom}_A(N, A) = R\,\mathrm{Hom}_A(N, A),$$

and hence $N = T^s(N)$. By the functoriality of $T^s$ we have a commutative diagram

$$\begin{array}{ccc} T^s(N) & = & N \\ \downarrow & & \downarrow \\ T^s(M) & \longrightarrow & M, \end{array}$$

which implies $N \subset T^s(M)$. □

**Theorem D.5.3.** *Let $M$ be a finitely generated $A$-module which is purely $s$-codimensional. Then for any $\mathfrak{p} \in \mathrm{SS}_0(M)$ we have*

$$\mathrm{Krull\ dim}((\mathrm{gr}^F A)/\mathfrak{p}) = m - s.$$

*Proof.* The assertion being trivial for $M = 0$ we assume that $M \neq 0$. In this case we have $j(M) = s$. Let $F$ be a good filtration of $M$. Then there exists a good filtration $F$ of $N = \mathrm{Ext}^s_A(M, A)$ such that $\mathrm{gr}^F N$ is a subquotient of $\mathrm{Ext}^s_{\mathrm{gr}^F A}(\mathrm{gr}^F M, \mathrm{gr}^F A)$. Hence

$$\begin{aligned}\mathrm{SS}(N) &= \mathrm{supp}(\mathrm{gr}^F N) \subset \mathrm{supp}(\mathrm{Ext}^s_{\mathrm{gr}^F A}(\mathrm{gr}^F M, \mathrm{gr}^F A))\\ &\subset \mathrm{supp}(\mathrm{gr}^F M) = \mathrm{SS}(M).\end{aligned}$$

On the other hand since $M$ is purely $s$-codimensional, we have

$$M = T^s(M)/T^{s+1}(M) \subset \mathrm{Ext}^s_{A^{\mathrm{op}}}(\mathrm{Ext}^s_A(M, A), A^{\mathrm{op}}) = \mathrm{Ext}^s_{A^{\mathrm{op}}}(N, A^{\mathrm{op}})$$

and hence $\mathrm{SS}(M) \subset \mathrm{SS}(\mathrm{Ext}^s_{A^{\mathrm{op}}}(N, A^{\mathrm{op}})) \subset \mathrm{SS}(N)$. Therefore, we have

$$\mathrm{SS}(M) = \mathrm{supp}(\mathrm{gr}^F M) = \mathrm{supp}(\mathrm{Ext}^s_{\mathrm{gr}^F A}(\mathrm{gr}^F M, \mathrm{gr}^F A)).$$

By $j(\mathrm{gr}^F M) = j(M) = s$ we have $j((\mathrm{gr}^F A)/\mathfrak{p}) \geq s$ for any $\mathfrak{p} \in \mathrm{supp}_0(\mathrm{gr}^F M)$. Set $\Lambda = \{\mathfrak{p} \in \mathrm{supp}_0(\mathrm{gr}^F M) \mid j((\mathrm{gr}^F A)/\mathfrak{p}) = s)\}$. By a well-known fact in commutative algebra there exists a submodule $L$ of $\mathrm{gr}^F M$ such that $j(L) > s$ and $\mathrm{supp}_0(\mathrm{gr}^F M/L) = \Lambda$. We need to show $\mathrm{supp}(\mathrm{gr}^F M) = \mathrm{supp}(\mathrm{gr}^F M/L)$. We have obviously $\mathrm{supp}(\mathrm{gr}^F M) \supset \mathrm{supp}(\mathrm{gr}^F M/L)$. On the other hand by $\mathrm{Ext}^s(L, \mathrm{gr}^F A) = 0$ we have an injection $\mathrm{Ext}^s(\mathrm{gr}^F M, \mathrm{gr}^F A) \to \mathrm{Ext}^s(\mathrm{gr}^F M/L, \mathrm{gr}^F A)$, and hence

$$\begin{aligned}\mathrm{supp}(\mathrm{gr}^F M) &= \mathrm{supp}(\mathrm{Ext}^s(\mathrm{gr}^F M, \mathrm{gr}^F A)) \subset \mathrm{supp}(\mathrm{Ext}^s(\mathrm{gr}^F M/L, \mathrm{gr}^F A))\\ &\subset \mathrm{supp}(\mathrm{gr}^F M/L).\end{aligned}$$

The proof is complete. □

# E

# Symplectic Geometry

In this chapter we first present basic results in symplectic geometry laying special emphasis on cotangent bundles of complex manifolds. Most of the results are well known and we refer the reader to Abraham–Mardsen [AM] and Duistermaat [Dui] for details. Next we will precisely study conic Lagrangian analytic subsets in the cotangent bundles of complex manifolds. We prove that such a Lagrangian subset is contained in the union of the conormal bundles of strata in a Whitney stratification of the base manifold (Kashiwara's theorem in [Kas3], [Kas8]).

## E.1 Symplectic vector spaces

Let $V$ be a finite-dimensional vector space over a field $k$. A *symplectic form* $\sigma$ on $V$ is a non-degenerate anti-symmetric bilinear form on $V$. If a vector space $V$ is endowed with a symplectic form $\sigma$, we call the pair $(V, \sigma)$ a *symplectic vector space*. The dimension of a symplectic vector space is even. Let $(V, \sigma)$ be a symplectic vector space. Denote by $V^*$ the dual of $V$. Then for any $\theta \in V^*$ there exists a unique $H_\theta \in V$ such that

$$< \theta, v >= \sigma(v, H_\theta) \quad (v \in V)$$

by the non-degeneracy of $\sigma$. The correspondence $\theta \mapsto H_\theta$ defines the *Hamiltonian isomorphism* $H : V^* \simeq V$. For a linear subspace $W$ of $V$ consider its orthogonal complement $W^\perp = \{v \in V | \sigma(v, W) = 0\}$ with respect to $\sigma$. Then again by the non-degeneracy of $\sigma$ we obtain $\dim W + \dim W^\perp = \dim V$. Now let us introduce the following important linear subspaces of $V$.

**Definition E.1.1.** A linear subspace $W$ of $V$ is called *isotropic* (resp. *Lagrangian*, resp. *involutive*) if it satisfies $W \subset W^\perp$ (resp. $W = W^\perp$, resp. $W \supset W^\perp$).

Note that if a linear subspace $W \subset V$ is isotropic (resp. Lagrangian, resp. involutive) then $\dim W \leq \frac{1}{2} \dim V$ (resp. $\dim W = \frac{1}{2} \dim V$, resp. $\dim W \geq \frac{1}{2} \dim V$). Moreover, a one-dimensional subspace (resp. a hyperplane) of $V$ is always isotropic (resp. involutive).

**Example E.1.2.** Let $W$ be a finite-dimensional vector space and $W^*$ its dual. Set $V = W \oplus W^*$ and define a bilinear form $\sigma$ on $V$ by

$$\sigma((x,\xi),(x',\xi')) = <x',\xi> - <x,\xi'> \quad ((x,\xi),(x',\xi') \in V = W \oplus W^*).$$

Then $(V,\sigma)$ is a symplectic vector space. Moreover, $W$ and $W^*$ are Lagrangian subspaces of $V$.

## E.2 Symplectic structures on cotangent bundles

A complex manifold $X$ is called a (holomorphic) *symplectic manifold* if there exists a holomorphic 2-form $\sigma$ globally defined on $X$ which induces a symplectic form on the tangent space $T_xX$ of $X$ at each $x \in X$. The dimension of a symplectic manifold is necessarily even. As one of the most important examples of symplectic manifolds, we treat here cotangent bundles of complex manifolds.

Now let $X$ be a complex manifold and $TX$ (resp. $T^*X$) its tangent (resp. cotangent) bundle. We denote by $\pi : T^*X \to X$ the canonical projection. By differentiating $\pi$ we obtain the tangent map $\pi' : T(T^*X) \to (T^*X) \times_X (TX)$ and its dual $\rho_\pi : (T^*X) \times_X (T^*X) \to T^*(T^*X)$. If we restrict $\rho_\pi$ to the diagonal $T^*X$ of $(T^*X) \times_X (T^*X)$ then we get a map $T^*X \to T^*(T^*X)$. Since this map is a holomorphic section of the bundle $T^*(T^*X) \to T^*X$, it corresponds to a (globally defined) holomorphic 1-form $\alpha_X$ on $T^*X$. We call $\alpha_X$ the *canonical* **1***-form*. If we take a local coordinate $(x_1, x_2, \ldots, x_n)$ of $X$ on an open subset $U \subset X$, then any point $p$ of $T^*U \subset T^*X$ can be written uniquely as $p = (x_1, x_2, \ldots, x_n; \xi_1 dx_1 + \xi_2 dx_2 + \cdot + \xi_n dx_n)$ where $\xi_i \in \mathbb{C}$. We call $(x_1, x_2, \ldots, x_n; \xi_1, \xi_2, \ldots, \xi_n)$ the local coordinate system of $T^*X$ associated to $(x_1, x_2, \ldots, x_n)$. In this local coordinate of $T^*X$ the canonical 1-form $\alpha_X$ is written as $\alpha_X = \sum_{i=1}^n \xi_i dx_i$. Set $\sigma_X = d\alpha_X = \sum_{i=1}^n d\xi_i \wedge dx_i$. Then we see that the holomorphic 2-form $\sigma_X$ defines a symplectic structure on $T_p(T^*X)$ at each point $p \in T^*X$. Namely, the cotangent bundle $T^*X$ is endowed with a structure of a symplectic manifold by $\sigma_X$. We call $\sigma_X$ the (canonical) *symplectic form* of $T^*X$. Since there exists the Hamiltonian isomorphism $H : T_p^*(T^*X) \simeq T_p(T^*X)$ at each $p \in T^*X$, we obtain the global isomorphism $H : T^*(T^*X) \simeq T(T^*X)$. For a holomorphic function $f$ on $T^*X$ we define a holomorphic vector field $H_f$ to be the image of the 1-form $df$ by $H : T^*(T^*X) \simeq T(T^*X)$. The vector field $H_f$ is called the *Hamiltonian vector field* of $f$. Define the *Poisson bracket* of two holomorphic functions $f, g$ on $T^*X$ by $\{f,g\} = H_f(g) = \sigma_X(H_f, H_g)$. In the local coordinate $(x_1, x_2, \ldots, x_n; \xi_1, \xi_2, \ldots, \xi_n)$ of $T^*X$ we have the explicit formula

$$H_f = \sum_{i=1}^n \left( \frac{\partial f}{\partial \xi_i} \frac{\partial}{\partial x_i} - \frac{\partial f}{\partial x_i} \frac{\partial}{\partial \xi_i} \right).$$

We can also easily verify the following:

$$\begin{cases} \{f,g\} = -\{g,f\}, \\ \{f,hg\} = h\{f,g\} + g\{f,h\}, \\ \{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\} = 0. \end{cases}$$

Moreover, we have $[H_f, H_g] = H_{\{f,g\}}$, where $[H_f, H_g]$ is the Lie bracket of $H_f$ and $H_g$.

**Definition E.2.1.** An analytic subset $V$ of $T^*X$ is called isotropic (resp. Lagrangian, resp. involutive) if for any smooth point $p \in V_{\mathrm{reg}}$ of $V$ the tangent space $T_pV$ at $p$ is a isotropic (resp. a Lagrangian, resp. an involutive) subspace in $T_p(T^*X)$.

By definition the dimension of a Lagrangian analytic subset of $T^*X$ is equal to $\dim X$.

**Example E.2.2.**

(i) Let $Y \subset X$ be a complex submanifold of $X$. Then the *conormal bundle* $T^*_Y X$ of $Y$ in $X$ is a Lagrangian submanifold of $T^*X$.

(ii) Let $f$ be a holomorphic function on $X$. Set $\Lambda_f = \{(x, \operatorname{grad} f(x) | x \in X\}$. Then $\Lambda_f$ is a Lagrangian submanifold of $T^*X$.

For an analytic subset $V$ of $T^*X$ denote by $\mathcal{I}_V$ the subsheaf of $\mathcal{O}_{T^*X}$ consisting of holomorphic functions vanishing on $V$.

**Lemma E.2.3.** *For an analytic subset $V$ of $T^*X$ the following conditions are equivalent:*

(i) *$V$ is involutive.*

(ii) $\{\mathcal{I}_V, \mathcal{I}_V\} \subset \mathcal{I}_V$.

*Proof.* By the definition of Hamiltonian isomorphisms, for each smooth point $p \in V_{\mathrm{reg}}$ of $V$ the orthogonal complement $(T_pV)^\perp$ of $T_pV$ in the symplectic vector space $T_p(T^*X)$ is spanned by the Hamiltonian vector fields $H_f$ of $f \in \mathcal{I}_V$. Assume that $V$ is involutive. If $f, g \in \mathcal{I}_V$ then the Hamiltonian vector field $H_f$ is tangent to $V_{\mathrm{reg}}$ and hence $\{f, g\} = H_f(g) = 0$ on $V_{\mathrm{reg}}$. Since $\{f, g\}$ is holomorphic and $V_{\mathrm{reg}}$ is dense in $V$, $\{f, g\} = 0$ on the whole $V$, i.e., $\{f, g\} \in \mathcal{I}_V$. The part (i) $\Longrightarrow$ (ii) was proved. The converse can be proved more easily. □

## E.3 Lagrangian subsets of cotangent bundles

Let $X$ be a complex manifold of dimension $n$. Since the fibers of the cotangent bundle $T^*X$ are complex vector spaces, there exists a natural action of the multiplicative group $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$ on $T^*X$. We say that an analytic subset $V$ of $T^*X$ is *conic* if $V$ is stable by this action of $\mathbb{C}^\times$. In this subsection we focus our attention on conic Lagrangian analytic subsets of $T^*X$.

First let us examine the image $H(\alpha_X)$ of the canonical 1-form $\alpha_X$ by the Hamiltonian isomorphism $H : T^*(T^*X) \simeq T(T^*X)$. In a local coordinate $(x_1, x_2, \ldots, x_n; \xi_1, \xi_2, \ldots, \xi_n)$ of $T^*X$ the holomorphic vector field $H(\alpha_X)$ thus obtained has the form

$$H(\alpha_X) = -\sum_{i=1}^{n} \xi_i \frac{\partial}{\partial \xi_i}.$$

It follows that $-H(\alpha_X)$ is the infinitesimal generator of the action of $\mathbb{C}^\times$ on $T^*X$. We call this vector field the *Euler vector field*.

**Lemma E.3.1.** *Let $V$ be a conic complex submanifold of $T^*X$. Then $V$ is isotropic if and only if the pull-back $\alpha_X|_V$ of $\alpha_X$ to $V$ is identically zero.*

*Proof.* Assume that $\alpha_X|_V$ is identically zero on $V$. Then also the pull-back of the symplectic 2-form $\sigma_X = d\alpha_X$ to $V$ vanishes. Hence $V$ is isotropic. Let us prove the converse. Assume that $V$ is isotropic. Then for any local section $\delta$ of the tangent bundle $TV \to V$ we have

$$\langle \alpha_X, \delta \rangle = \sigma_X(\delta, H(\alpha_X)) = 0$$

on $V$ because the Euler vector field $-H(\alpha_X)$ is tangent to $V$ by the conicness of $V$. This means that $\alpha_X|_V$ is identically zero on $V$. □

**Corollary E.3.2.** *Let $\Lambda$ be a conic Lagrangian analytic subset of $T^*X$. Then the pull-back of $\alpha_X$ to the regular part $\Lambda_{\mathrm{reg}}$ of $\Lambda$ is identically zero.*

Let $Z$ be an analytic subset of the base space $X$ and denote by $\overline{T^*_{Z_{\mathrm{reg}}}X}$ the closure of the conormal bundle $T^*_{Z_{\mathrm{reg}}}X$ of $Z_{\mathrm{reg}}$ in $T^*X$. Since the closure is taken with respect to the classical topology of $T^*X$, it is not clear if $\overline{T^*_{Z_{\mathrm{reg}}}X}$ is an analytic subset of $T^*X$ or not. In Proposition E.3.5 below, we will prove the analyticity of $\overline{T^*_{Z_{\mathrm{reg}}}X}$. For this purpose, recall the following definitions.

**Definition E.3.3.** Let $S$ be an analytic space. A locally finite partition $S = \bigsqcup_{\alpha\in A} S_\alpha$ of $S$ by locally closed complex manifolds $S_\alpha$'s is called a *stratification* of $S$ if for each $S_\alpha$ the closure $\overline{S}_\alpha$ and the boundary $\partial S_\alpha = \overline{S}_\alpha \setminus S_\alpha$ are analytic and unions of $S_\beta$'s. A complex manifold $S_\alpha$ in it is called a *stratum* of the stratification $S = \bigsqcup_{\alpha\in A} S_\alpha$.

**Definition E.3.4.** Let $S$ be an analytic space. Then we say that a subset $S'$ of $S$ is *constructible* if there exists stratification $S = \bigsqcup_{\alpha\in A} S_\alpha$ of $S$ such that $S'$ is a union of some strata in it.

For an analytic space $S$ the family of constructible subsets of $S$ is closed under various set-theoretical operations. Note also that by definition the closure of a constructible subset is analytic. Moreover, if $f : S \to S'$ is a morphism (resp. proper morphism) of analytic spaces, then the inverse (resp. direct) image of a constructible subset of $S'$ (resp. $S$) by $f$ is again constructible. Now we are ready to prove the following.

**Proposition E.3.5.** *$\overline{T^*_{Z_{\mathrm{reg}}}X}$ is an analytic subset of $T^*X$.*

*Proof.* We may assume that $Z$ is irreducible. By Hironaka's theorem there exists a proper holomorphic map $f : Y \to X$ from a complex manifold $Y$ and an analytic subset $Z' \neq Z$ of $Z$ such that $f(Y) = Z$, $Z_0 := Z \setminus Z'$ is smooth, and the restriction of $f$ to $Y_0 := f^{-1}(Z_0)$ induces a biholomorphic map $f|_{Y_0} : Y_0 \simeq Z_0$. From $f : Y \to X$ we obtain the canonical morphisms

$$T^*Y \xleftarrow{\rho_f} Y \times_X T^*X \xrightarrow{\varpi_f} T^*X.$$

We easily see that $T^*_{Z_0}X = \varpi_f \rho_f^{-1}(T^*_{Y_0}Y_0)$, where $T^*_{Y_0}Y_0 \simeq Y_0$ is the zero-section of $T^*Y_0 \subset T^*Y$. Since $T^*_{Y_0}Y_0$ is a constructible subset of $T^*Y$ and $\varpi_f$ is proper, $T^*_{Z_0}X$ is a constructible subset of $T^*X$. Hence the closure $\overline{T^*_{Z_0}X} = \overline{T^*_{Z_{\mathrm{reg}}}X}$ is an analytic subset of $T^*X$. □

By Example E.2.2 and Proposition E.3.5, for an irreducible analytic subset $Z$ of $X$ we conclude that the closure $\overline{T^*_{Z_{\mathrm{reg}}}X}$ is an irreducible conic Lagrangian analytic subset of $T^*X$. The following result, which was first proved by Kashiwara [Kas3], [Kas8], shows that any irreducible conic Lagrangian analytic subset of $T^*X$ is obtained in this way.

**Theorem E.3.6 (Kashiwara [Kas3], [Kas8]).** *Let $\Lambda$ be a conic Lagrangian analytic subset of $T^*X$. Assume that $\Lambda$ is irreducible. Then $Z = \pi(\Lambda)$ is an irreducible analytic subset of $X$ and $\Lambda = \overline{T^*_{Z_{\mathrm{reg}}}X}$.*

*Proof.* Since $\Lambda$ is conic, $Z = \pi(\Lambda) = (T^*_X X) \cap \Lambda$ is an analytic subset of $X$. Moreover, by definition we easily see that $Z$ is irreducible. Denote by $\Lambda_0$ the open subset of $\pi^{-1}(Z_{\mathrm{reg}}) \cap \Lambda_{\mathrm{reg}}$ consisting of points where the map $\pi|_{\Lambda_{\mathrm{reg}}}$ has the maximal rank. Then $\Lambda_0$ is open dense in $\pi^{-1}(Z_{\mathrm{reg}}) \cap \Lambda$ and the maximal rank is equal to $\dim Z$. Now let $p$ be a point in $\Lambda_0$. Taking a local coordinate $(x_1, x_2, \ldots, x_n)$ of $X$ around the point $\pi(p) \in Z_{\mathrm{reg}}$, we may assume that $Z = \{x_1 = x_2 = \cdots = x_d = 0\}$ where $d = n - \dim Z$. Let us choose a local section $s : Z_{\mathrm{reg}} \hookrightarrow \Lambda_{\mathrm{reg}}$ of $\pi|_{\Lambda_{\mathrm{reg}}}$ such that $s(\pi(p)) = p$. Let $i_{\Lambda_{\mathrm{reg}}} : \Lambda_{\mathrm{reg}} \hookrightarrow T^*X$ be the embedding. Then by Corollary E.3.2 the pull-back of the canonical 1-form $\alpha_X$ to $Z_{\mathrm{reg}}$ by $i_{\Lambda_{\mathrm{reg}}} \circ s$ is zero. On the other hand, this 1-form on $Z_{\mathrm{reg}}$ has the form $\xi_{d+1}(x')dx_{d+1} + \cdots + \xi_n(x')dx_n$, where we set $x' = (x_{d+1}, \ldots, x_n)$. Therefore, the point $p$ should be contained in $\{\xi_{d+1} = \cdots = \xi_n = 0\}$. We proved that $\Lambda_0 \subset T^*_{Z_{\mathrm{reg}}}X$. Since $\dim \Lambda_0 = \dim T^*_{Z_{\mathrm{reg}}}X = n$ we obtain $\overline{T^*_{Z_{\mathrm{reg}}}X} = \overline{\Lambda_0} \subset \Lambda$. Then the result follows from the irreducibility of $\Lambda$. □

To treat general conic Lagrangian analytic subsets of $T^*X$ let us briefly explain Whitney stratifications.

**Definition E.3.7.** Let $S$ be an analytic subset of a complex manifold $M$. A stratification $S = \bigsqcup_{\alpha \in A} S_\alpha$ of $S$ is called a *Whitney stratification* if it satisfies the following Whitney conditions (a) and (b):

(a) Assume that a sequence $x_i \in S_\alpha$ of points converges to a point $y \in S_\beta$ $(\alpha \neq \beta)$ and the limit $T$ of the tangent spaces $T_{x_i}S_\alpha$ exists. Then we have $T_y S_\beta \subset T$.
(b) Let $x_i \in S_\alpha$ and $y_i \in S_\beta$ be two sequences of points which converge to the same point $y \in S_\beta$ $(\alpha \neq \beta)$. Assume further that the limit $l$ (resp. $T$) of the lines $l_i$ jointing $x_i$ and $y_i$ (resp. of the tangent spaces $T_{x_i}S_\alpha$) exists. Then we have $l \subset T$.

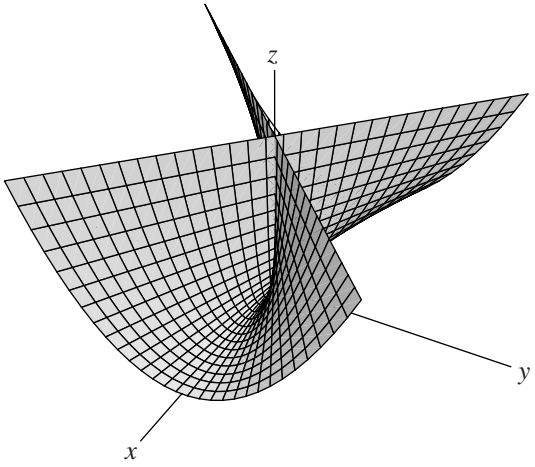
It is well known that any stratification of an analytic set can be refined to satisfy the Whitney conditions. Intuitively, the Whitney conditions means that the geometrical normal structure of the stratification $S = \bigsqcup_{\alpha \in A} S_\alpha$ is locally constant along each stratum $S_\alpha$ as is illustrated in the example below.

**Example E.3.8 (Whitney's umbrella).** Consider the analytic set $S = \{(x, y, z) \in \mathbb{C}^3 \mid y^2 = zx^2\}$ in $\mathbb{C}^3$ and the following two stratifications $S = \bigsqcup_{i=1}^{2} S'_i$ and $S = \bigsqcup_{i=1}^{3} S_i$ of $S$:

$$\begin{cases} S'_1 = \{(x, y, z) \in \mathbb{C}^3 \mid x = y = 0\} \\ S'_2 = S \setminus S'_1 \end{cases}$$

$$\begin{cases} S_1 = \{0\} \\ S_2 = \{(x, y, z) \in \mathbb{C}^3 \mid x = y = 0\} \setminus S_1 \\ S_3 = S \setminus (S_1 \sqcup S_2) \end{cases}$$

Then the stratification $S = \bigsqcup_{i=1}^{3} S_i$ satisfies the Whitney conditions (a) and (b), but the stratification $S = \bigsqcup_{i=1}^{2} S'_i$ does not.




We see that along each stratum $S_i$ $(i = 1, 2, 3)$, the geometrical normal structure of $S = \bigsqcup_{i=1}^{3} S_i$ is constant.

Now consider a Whitney stratification $X = \bigsqcup_{\alpha \in A} X_\alpha$ of a complex manifold $X$. Then it is a good exercise to prove that for each point $x \in X$ there exists a sufficiently small sphere centered at $x$ which is transversal to all the strata $X_\alpha$'s. This result follows easily from the Whitney condition (b). For the details see [Kas8], [Schu]. Moreover, by the Whitney conditions we can prove easily that the union $\bigsqcup_{\alpha \in A} T^*_{X_\alpha} X$ of the conormal bundles $T^*_{X_\alpha} X$ is a closed analytic subset of $T^*X$ (the analyticity follows from Proposition E.3.5). The following theorem was proved by Kashiwara [Kas3], [Kas8] and plays a crucial role in proving the constructibility of the solutions to holonomic $D$-modules.

**Theorem E.3.9.** *Let $X$ be a complex manifold and $\Lambda$ a conic Lagrangian analytic subset of $T^*X$. Then there exists a Whitney stratification $X = \bigsqcup_{\alpha \in A} X_\alpha$ of $X$ such that*

$$\Lambda \subset \bigsqcup_{\alpha \in A} T^*_{X_\alpha} X.$$

*Proof.* Let $\Lambda = \cup_{i \in I} \Lambda_i$ be the irreducible decomposition of $\Lambda$ and set $Z_i = \pi(\Lambda_i)$. Then we can take a Whitney stratification $X = \bigsqcup_{\alpha \in A} X_\alpha$ of $X$ such that $Z_i$ is a union of strata in it for any $i \in I$. Note that for each $i \in I$ there exists a (unique) stratum $X_{\alpha_i} \subset (Z_i)_{\mathrm{reg}}$ which is open dense in $Z_i$. Hence we have $\Lambda_i = \overline{T^*_{(Z_i)_{\mathrm{reg}}} X} = \overline{T^*_{X_{\alpha_i}} X}$ by Theorem E.3.6 and $\Lambda = \cup_{i \in I} \Lambda_i \subset \sqcup_{\alpha \in A} T^*_{X_\alpha} X$. □

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# List of Notation

- $\mathbb{N} = \mathbb{Z}_{\geq 0} = \{0, 1, 2, \dots\}$
- $\mathbb{N}^+ = \mathbb{Z}_{>0} = \{1, 2, 3, \dots\}$

### Chapter 1

- $\mathcal{O}_X$ 15
- $\Theta_X$ 15
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## Progress in Mathematics

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